



Optimal control for linear and nonlinear semistabilization[☆]

Andrea L'Afflitto^a, Wassim M. Haddad^{a,*}, Qing Hui^b

^a*School of Aerospace Engineering, Georgia Institute of Technology, Atlanta, GA 30332-0150, USA*

^b*Department of Mechanical Engineering, Texas Tech University, Lubbock, TX 79409-1021, USA*

Received 21 February 2014; received in revised form 2 October 2014; accepted 20 November 2014

Available online 4 December 2014

Abstract

The state feedback linear-quadratic optimal control problem for asymptotic stabilization has been extensively studied in the literature. In this paper, the optimal linear and nonlinear control problem is extended to address a weaker version of closed-loop stability, namely, semistability, which involves convergent trajectories and Lyapunov stable equilibria and which is of paramount importance for consensus control of network dynamical systems. Specifically, we show that the optimal semistable state-feedback controller can be solved using a form of the Hamilton–Jacobi–Bellman conditions that does not require the cost-to-go function to be sign definite. This result is then used to solve the optimal linear-quadratic regulator problem using a Riccati equation approach. Finally, two numerical examples are presented to demonstrate the efficacy of the proposed linear and nonlinear semistabilization framework.

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1. Introduction

A form of stability that lies between Lyapunov stability and asymptotic stability is *semistability* [1,2], that is, the property whereby every trajectory that starts in a neighborhood of

[☆]This work was supported in part by the Air Force Office of Scientific Research under Grant FA9550-12-1-0192, the Domenica Rea D'Onofrio Fellowship, and the Defense Threat Reduction Agency under Grants HDTRA1-10-1-0090 and HDTRA1-13-1-0048.

*Corresponding author.

E-mail addresses: a.laffitto@gatech.edu (A. L'Afflitto), wm.haddad@aerospace.gatech.edu (W.M. Haddad), qing.hui@ttu.edu (Q. Hui).

a Lyapunov stable equilibrium converges to a (possibly different) Lyapunov stable equilibrium. Semistability implies Lyapunov stability, and is implied by asymptotic stability [1–3]. This notion of stability arises naturally in systems having a continuum of equilibria and includes such systems as mechanical systems having rigid body modes, chemical reaction systems [4], compartmental systems [5,6], and isospectral matrix dynamical systems. Semistability also arises naturally in dynamical network systems [7–9], which cover a broad spectrum of applications including cooperative control of unmanned air vehicles, autonomous underwater vehicles, distributed sensor networks, air and ground transportation systems, swarms of air and space vehicle formations, and congestion control in communication networks, to cite but a few examples.

A unique feature of the closed-loop dynamics under any control algorithm that achieves consensus in dynamic networks is the existence of a continuum of equilibria representing a desired state of consensus [8,9]. Under such dynamics, the desired limiting state is not determined completely by the closed-loop system dynamics, but depends on the initial system state as well [8–11]. From a practical viewpoint, it is not sufficient to only guarantee that a network converges to a state of consensus since steady-state convergence is not sufficient to guarantee that small perturbations from the limiting state will lead to only small transient excursions from the state of consensus. It is also necessary to guarantee that the equilibrium states representing consensus are Lyapunov stable, and consequentially, semistable.

In [10,11], the authors develop \mathcal{H}_2 optimal semistable control theory for linear dynamical systems. Specifically, unlike the standard \mathcal{H}_2 optimal control problem, it is shown in [10,11] that a complicating factor of the \mathcal{H}_2 optimal semistable stabilization problem is that the closed-loop Lyapunov equation guaranteeing semistability can admit multiple solutions. In addition, the authors show that the \mathcal{H}_2 optimal solution is given by a least squares solution to the closed-loop Lyapunov equation over all possible semistabilizing solutions. Moreover, it is shown that this least squares solution can be characterized by a linear matrix inequality minimization problem.

In this paper, we address the problem of finding a state-feedback nonlinear control law $u = \phi(x)$ that minimizes the performance measure

$$J(x_0, u(\cdot)) \triangleq \int_0^{\infty} L(x(t), u(t)) dt \quad (1)$$

and guarantees semistability of the nonlinear dynamical system

$$\dot{x}(t) = F(x(t), u(t)), \quad x(0) = x_0, \quad t \geq 0, \quad (2)$$

$$y(t) = H(x(t), u(t)), \quad (3)$$

where, for every $t \geq 0$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, \mathcal{D} is an open set, $u(t) \in U \subseteq \mathbb{R}^m$, $y(t) \in Y \subseteq \mathbb{R}^l$, $L: \mathcal{D} \times U \rightarrow \mathbb{R}$, $F: \mathcal{D} \times U \rightarrow \mathbb{R}^n$ is Lipschitz continuous in x and u on $\mathcal{D} \times U$, and $H: \mathcal{D} \times U \rightarrow Y$. Specifically, our approach focuses on the role of the Lyapunov function guaranteeing semistability of Eq. (2) with a feedback control law $u = \phi(x)$, and we provide sufficient conditions for optimality in a form that corresponds to a steady-state version of a Hamilton–Jacobi–Bellman-type equation.

In addition, we provide sufficient conditions for the existence of a feedback gain $K \in \mathbb{R}^{m \times n}$ such that the state feedback control law $u = Kx$ minimizes the quadratic performance measure

$$J(x_0, u(\cdot)) = \int_0^{\infty} [(x(t) - x_e)^T C^T C (x(t) - x_e) + (u(t) - u_e)^T R_2 (u(t) - u_e)] dt \quad (4)$$

and guarantees semistability of the linear dynamical system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad t \geq 0, \tag{5}$$

$$y(t) = Cx(t), \tag{6}$$

where $u_e \triangleq Kx_e$, $x_e \triangleq \lim_{t \rightarrow \infty} x(t)$, R_2 is positive definite, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and $C \in \mathbb{R}^{l \times n}$. The proposed Riccati equation-based framework for optimal linear semistable stabilization presented in this paper is different from the framework presented in [10,11] using linear matrix inequalities.

The contents of the paper are as follows. In Section 2, we establish notation, definitions, and develop some key results on semistability, semicontrollability, semiobservability, and semistabilization. In Section 3, we consider a nonlinear system with a performance functional evaluated over the infinite horizon. The performance functional is then evaluated in terms of a Lyapunov function that guarantees semistability. This result is then specialized to the linear-quadratic case. We then, in Section 4, state an optimal control problem and provide sufficient conditions for characterizing an optimal nonlinear feedback controller guaranteeing semistable stabilization. Section 5 presents two application design examples of optimal semistable control involving optimal consensus control for multiagent systems and a nonlinear mechanical system involving an eccentric rotational inertia on a translational oscillator. Finally, in Section 6 we draw conclusions and highlight recommendations for future research.

2. Notation, definitions, and mathematical preliminaries

The notation used in this paper is fairly standard. Specifically, \mathbb{R} denotes the set of real numbers, \mathbb{C} denotes the set of complex numbers, \mathbb{R}_+ denotes the set of positive real numbers, \mathbb{R}^n (resp., \mathbb{C}^n) denotes the set of $n \times 1$ real (resp., complex) column vectors, and $\mathbb{R}^{n \times m}$ denotes the set of $n \times m$ real matrices. Furthermore, we write $V'(x) \triangleq \partial V(x)/\partial x$ for the Fréchet derivative of V at x , $\|\cdot\|$ for the Euclidean vector norm, $\|\cdot\|_F$ for the Frobenius matrix norm, \mathcal{S}^\perp for the orthogonal complement of a set \mathcal{S} , $\text{span } \mathcal{S}$ for the span of the set \mathcal{S} , $\mathcal{R}(A)$ and $\mathcal{N}(A)$ for the range space and the null space of a matrix A , respectively, $\text{spec}(A)$ for the spectrum of the square matrix A including multiplicity, $\text{rank } A$ for the rank of the matrix A , and $(\cdot)^\#$ for the group generalized inverse. Finally, \otimes denotes the Kronecker product, \oplus denotes the Kronecker sum, $\text{vec}(\cdot)$ denotes the column stacking operator, and $\text{vec}^{-1}(\cdot)$ denotes the inverse vec operator.

Consider the nonlinear dynamical system given by

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \quad t \geq 0, \tag{7}$$

where, for every $t \geq 0$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$ and $f : \mathcal{D} \rightarrow \mathbb{R}^n$ is locally Lipschitz continuous on \mathcal{D} . The solution of Eq. (7) with initial condition $x(0) = x$ defined on $[0, \infty)$ is denoted by $s(\cdot, x)$. The above assumptions imply that the map $s : [0, \infty) \times \mathcal{D} \rightarrow \mathcal{D}$ is continuous [12, Theorem 2.1], satisfies the consistency property $s(0, x) = x$, and possesses the semigroup property $s(t, s(\tau, x)) = s(t + \tau, x)$ for all $t, \tau \geq 0$ and $x \in \mathcal{D}$. Given $t \geq 0$ and $x \in \mathcal{D}$, we denote the map $s(t, \cdot) : \mathcal{D} \rightarrow \mathcal{D}$ by s_t and the map $s(\cdot, x) : [0, \infty) \rightarrow \mathcal{D}$ by s^x .

The orbit \mathcal{O}_x of a point $x \in \mathcal{D}$ is the set $s^x([0, \infty))$. A set $\mathcal{D}_p \subseteq \mathcal{D}$ is positively invariant relative to Eq. (7) if $s_t(\mathcal{D}_p) \subseteq \mathcal{D}_p$ for all $t \geq 0$ or, equivalently, \mathcal{D}_p contains the orbits of all its points. The set \mathcal{D}_p is invariant relative to Eq. (7) if $s_t(\mathcal{D}_p) = \mathcal{D}_p$ for all $t \geq 0$. The positive limit set of $x \in \mathbb{R}^n$ is the set $\omega(x)$ of all subsequential limits of sequences of the form $\{s(t_i, x)\}_{i=0}^\infty$, where $\{t_i\}_{i=0}^\infty$ is an increasing divergent sequence in $[0, \infty)$. Recall that, for every $x \in \mathbb{R}^n$ that has bounded orbits, $\omega(x)$ is nonempty and compact, and, for every neighborhood \mathcal{N} of $\omega(x)$, there exists $T > 0$ such that $s_t(x) \in \mathcal{N}$ for every $t > T$ [3, Chapter 2]. If $\mathcal{D}_p \subset \mathcal{D}$ is positively invariant and closed, then

$\omega(x) \subseteq \mathcal{D}_p$ for all $x \in \mathcal{D}_p$. In addition, $\lim_{t \rightarrow \infty} s(t, x)$ exists if and only if $\omega(x)$ is a singleton. Finally, the set of equilibrium points of Eq. (7) is denoted by $f^{-1}(0) \triangleq \{x \in \mathcal{D} : f(x) = 0\}$.

The following definition is needed.

Definition 2.1 (Haddad and Chellaboina [3]). Let $\mathcal{D} \subseteq \mathbb{R}^n$ be an open positively invariant set with respect to Eq. (7). An equilibrium point $x_e \in \mathcal{D}$ of Eq. (7) is *semistable* with respect to \mathcal{D} if x_e is Lyapunov stable and there exists an open subset \mathcal{D}_0 of \mathcal{D} containing x_e such that, for all initial conditions in \mathcal{D}_0 , the solutions of Eq. (7) converge to a Lyapunov stable equilibrium point. The system (7) is *semistable* with respect to \mathcal{D} if every solution with initial condition in \mathcal{D} converges to a Lyapunov stable equilibrium. Finally, Eq. (7) is said to be *globally semistable* if Eq. (7) is semistable with respect to \mathbb{R}^n .

Note that if, for $\varepsilon > 0$, $\mathcal{B}_\varepsilon(x_e) \cap f^{-1}(0) = \{x_e\}$ is a singleton, where $\mathcal{B}_\varepsilon(x_e)$ denotes the *open ball* centered at x_e with radius ε , then Definition 2.1 reduces to the definitions of local and global asymptotic stability. Recall that for $B=0$, Eq. (5) is semistable if and only if $\text{spec}(A) \subset \{s \in \mathbb{C} : \text{Re } s < 0\} \cup \{0\}$ and, if $0 \in \text{spec}(A)$, then 0 is semisimple [13, Definition 11.8.1]. In this case, we say that A is *semistable*. Furthermore, if A is semistable, then the index of A is zero or one, and hence, A is group invertible. The group inverse $A^\#$ of A is a special case of the Drazin inverse A^D in the case where A has index zero or one [13, p. 369]. In this case, for every $x_0 \in \mathbb{R}^n$, $x_e = \lim_{t \rightarrow \infty} x(t) = (I_n - AA^\#)x_0$ or, equivalently, $\lim_{t \rightarrow \infty} e^{At} = I_n - AA^\#$ [13, Proposition 11.8.1].

Lemma 2.1 (Haddad and Chellaboina [3, Proposition 4.7]). Consider the nonlinear dynamical system (7) and let $x \in \mathbb{R}^n$. If the positive limit set of Eq. (7) contains a Lyapunov stable equilibrium point y with respect to \mathcal{D} , then $y = \lim_{t \rightarrow \infty} s(t, x)$, that is, $\omega(x) = \{y\}$.

Next, we introduce the definitions of semicontrollability and semiobservability for linear systems.

Definition 2.2 (Hui and Liu [14]). Consider the system given by Eq. (5). The pair (A, B) is *semicontrollable* if

$$\sum_{i=1}^n \mathcal{R}(A^{i-1}B) = \mathcal{R}(A), \tag{8}$$

where $A^0 \triangleq I_n$ and, for the given sets \mathcal{S}_1 and \mathcal{S}_2 , $\mathcal{S}_1 + \mathcal{S}_2 \triangleq \{x + y : x \in \mathcal{S}_1, y \in \mathcal{S}_2\}$ denotes the Minkowski sum.

The following lemma is needed to connect semicontrollability to the classical notion of controllability involving the existence of a continuous control $u : [0, t_f] \rightarrow \mathbb{R}^m$ such that the solution $x(\cdot)$ of Eq. (5) with $x(0) = x_0$ satisfies $x(t_f) = 0$.

Lemma 2.2. Let $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$. Then

$$\sum_{i=1}^n \mathcal{R}(A^{i-1}B) \subseteq \mathcal{R}(A) \cup \mathcal{R}(B). \tag{9}$$

Proof. It follows from Fact 2.9.16 of [13, p. 121] that

$$\left(\sum_{i=1}^n \mathcal{R}(A^{i-1}B) \right)^\perp = \bigcap_{i=1}^n \mathcal{R}(A^{i-1}B)^\perp. \tag{10}$$

Moreover, it follows from Theorem 2.4.3 of [13, p. 103] that $\mathcal{R}(A^{i-1}B)^\perp = \mathcal{N}(B^T(A^T)^{i-1})$ for every $i \in \{1, \dots, n\}$. Since $\mathcal{N}(A^T) \subseteq \mathcal{N}(B^T(A^T)^i)$ for every $i \in \{1, \dots, n-1\}$, it follows that

$$\mathcal{N}(A^T) \cap \mathcal{N}(B^T) \subseteq \bigcap_{i=1}^n \mathcal{N}(B^T(A^T)^{i-1}). \tag{11}$$

Now, by Theorem 2.4.3 of [13, p. 103], $\mathcal{R}(A)^\perp = \mathcal{N}(A^T)$, and hence, it follows from Eqs. (10) and (11) that

$$\mathcal{R}(A)^\perp \cap \mathcal{R}(B)^\perp \subseteq \bigcap_{i=1}^n \mathcal{R}(A^{i-1}B)^\perp = \left(\sum_{i=1}^n \mathcal{R}(A^{i-1}B) \right)^\perp. \tag{12}$$

Next, it follows from Fact 2.9.16 of [13, p. 121] that $\mathcal{R}(A)^\perp \cap \mathcal{R}(B)^\perp = (\mathcal{R}(A) + \mathcal{R}(B))^\perp$, and hence, by Eq. (12) and Fact 2.9.14 of [13, p. 121],

$$\sum_{i=1}^n \mathcal{R}(A^{i-1}B) \subseteq \mathcal{R}(A) + \mathcal{R}(B). \tag{13}$$

Finally, it follows from Fact 2.9.11 of [13, p. 121] that $\mathcal{R}(A) + \mathcal{R}(B) = \mathcal{R}(A) \cup \mathcal{R}(B)$, which proves Eq. (9). \square

Recall that the *controllable subspace* $\mathcal{C}_{t_f}(A, B)$ at time $t_f > 0$ is the subspace

$$\mathcal{C}_{t_f}(A, B) = \{x_f \in \mathbb{R}^n : \text{there exists a continuous control } u : [0, t_f] \rightarrow \mathbb{R}^m \text{ such that the solution } x(\cdot) \text{ of 5 with } x(0) = x_0 \text{ satisfies } x(t_f) = x_f\}.$$

Furthermore, recall that $\mathcal{C}_{t_f}(A, B)$ is independent of t_f , and hence, we write $\mathcal{C}(A, B)$ for $\mathcal{C}_{t_f}(A, B)$, and call $\mathcal{C}(A, B)$ the *controllable subspace* of (A, B) [13].

The next result characterizes semicontrollability in several equivalent ways.

Theorem 2.1. *The following statements are equivalent:*

- (i) (A, B) is semicontrollable.
- (ii) $\mathcal{R}(\int_0^{t_f} e^{At} B B^T e^{A^T t} dt) = \mathcal{R}(A)$.
- (iii) $\mathcal{C}(A, B) = \mathcal{R}(A)$.
- (iv) $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ and, for every $x_0 \in \mathcal{R}(A)$, there exists a continuous control $u : [0, t_f] \rightarrow \mathbb{R}^m$ such that the solution $x(t)$ of Eq. (5) with $x(0) = x_0$ satisfies $x(t_f) = 0$.

Proof. First, note that it follows from Lemma 12.6.2 of [13, p. 808] that

$$\mathcal{C}(A, B) = \mathcal{R}\left(\int_0^{t_f} e^{At} B B^T e^{A^T t} dt\right) = \sum_{i=1}^n \mathcal{R}(A^{i-1}B). \tag{14}$$

To show the equivalence of (i) and (ii), note that if (A, B) is semicontrollable, then it follows from Eqs. (8) and (14) that $\mathcal{R}(\int_0^{t_f} e^{At} B B^T e^{A^T t} dt) = \mathcal{R}(A)$ holds. Conversely, if $\mathcal{R}(\int_0^{t_f} e^{At} B \cdot B^T e^{A^T t} dt) = \mathcal{R}(A)$ holds, then it follows from Eq. (14) that Eq. (8) holds. Hence, by definition, (A, B) is semicontrollable.

Next, it follows from Eq. (14) that (ii) holds if and only if (iii) holds, which shows the equivalence of (ii) and (iii).

To show that (iv) implies (i) note that, for every $x_0 \in \mathcal{R}(A)$, Eq. (5) satisfies

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}Bu(s) ds, \quad t \geq 0. \tag{15}$$

Furthermore, note that e^{At} is nonsingular for every $t \geq 0$. Now, since $x(t_f) = 0$, it follows from Eq. (15) that, for every $x_0 \in \mathcal{R}(A)$,

$$x_0 = - \int_0^{t_f} e^{-As}Bu(s) ds. \tag{16}$$

Using the Cayley–Hamilton theorem, it follows that there exist $\alpha_j(t) \in \mathbb{R}$, $j = 0, \dots, n - 1$, such that

$$e^{-At} = \sum_{j=0}^{n-1} \alpha_j(t)A^j, \quad t \geq 0. \tag{17}$$

Next, define z_j , $j = 0, \dots, n - 1$, by

$$z_j \triangleq - \int_0^{t_f} \alpha_j(s)u(s) ds.$$

Substituting z_j and Eq. (17) into Eq. (16) yields

$$x_0 = - \int_0^{t_f} e^{-As}Bu(s) ds = - \int_0^{t_f} \sum_{j=0}^{n-1} \alpha_j(s)A^jBu(s) ds = \sum_{j=0}^{n-1} A^jBz_j \in \sum_{j=0}^{n-1} \mathcal{R}(A^jB)$$

for every $x_0 \in \mathcal{R}(A)$. Hence, $\mathcal{R}(A) \subseteq \sum_{j=0}^{n-1} \mathcal{R}(A^jB)$. Now, it follows from Lemma 2.2 and $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ that $\sum_{j=0}^{n-1} \mathcal{R}(A^jB) \subseteq \mathcal{R}(A) \cup \mathcal{R}(B) = \mathcal{R}(A)$. Consequently, $\sum_{j=0}^{n-1} \mathcal{R}(A^jB) = \mathcal{R}(A)$, and hence, by definition, (A, B) is semicontrollable.

To show (i) implies (iv), assume that (A, B) is semicontrollable. Then it follows from (8) that $\mathcal{R}(B) \subseteq \mathcal{R}(A)$. Let $x_0 \in \mathcal{R}(A)$ so that there exists $y \in \mathbb{R}^n$ such that $x_0 = Ay$. Next, construct the continuous control $u : [0, t_f] \rightarrow \mathbb{R}^m$ as

$$u(t) = -B^T e^{A^T(t_f-t)}W_c^+ e^{At_f}x_0, \tag{18}$$

where $W_c \triangleq \int_0^{t_f} e^{At}BB^T e^{A^T t} dt$ and X^+ denotes the Moore–Penrose generalized inverse of X . Then the solution $x(t)$ of Eq. (5) with $x(0) = x_0$ satisfies

$$\begin{aligned} x(t_f) &= e^{At_f}x_0 + \int_0^{t_f} e^{A(t_f-s)}Bu(s) ds \\ &= e^{At_f}x_0 - \int_0^{t_f} e^{A(t_f-s)}BB^T e^{A^T(t_f-s)}W_c^+ e^{At_f}x_0 ds \\ &= e^{At_f}x_0 - \int_0^{t_f} e^{At}BB^T e^{A^T t} dt W_c^+ e^{At_f}x_0 \\ &= e^{At_f}x_0 - W_c W_c^+ e^{At_f}x_0 \\ &= (I_n - W_c W_c^+)e^{At_f}x_0 \\ &= (I_n - W_c W_c^+)e^{At_f}Ay \\ &= (I_n - W_c W_c^+)Ae^{At_f}y, \end{aligned} \tag{19}$$

where we used the fact that $Ae^{At_f} = e^{At_f}A$. Now, it follows from (vi) of Proposition 6.1.6 of [13, p. 399] that $\mathcal{R}(W_c) = \mathcal{N}(I_n - W_c W_c^+)$. In addition, it follows from (i) \Rightarrow (ii) that

$\mathcal{R}(A) = \mathcal{R}(W_c) = \mathcal{N}(I_n - W_c W_c^+)$, which implies $Ae^{At_f}y \in \mathcal{R}(A)$. Consequently, $(I_n - W_c W_c^+)Ae^{At_f}y = 0$, and hence, by Eq. (19), $x(t_f) = 0$. \square

It follows from Theorem 2.1 that semicontrollability of the linear controlled system (5) implies the existence of a continuous control input such that the solution $x(\cdot)$ of Eq. (5) can be driven to the origin in finite-time for every initial condition in the range space of system matrix A .

The following proposition is needed for some of the key results in this section.

Proposition 2.1. Consider the dynamical system given by Eq. (5). Then Eq. (8) holds if and only if

$$\left[\bigcap_{k=1}^n \mathcal{N}(B^T(A^{k-1})^T) \right]^\perp = [\mathcal{N}(A^T)]^\perp. \tag{20}$$

Furthermore, Eq. (20) is equivalent to

$$\text{span} \left\{ \bigcup_{k=1}^n \mathcal{R}(A^{k-1}B) \right\} = \mathcal{R}(A). \tag{21}$$

Proof. First we show that $\sum_{i=1}^n \mathcal{R}(A^{i-1}B) = \text{span}(\bigcup_{i=1}^n \mathcal{R}(A^{i-1}B))$. Note that, for every $i \in \{1, \dots, n\}$, $\mathcal{R}(A^{i-1}B)$ is a subspace of \mathbb{R}^n , and hence, by Fact 2.9.13 of [13, p. 121], the above equality holds. Now, it follows from Eq. (8) that (A, B) is semicontrollable if and only if Eq. (21) holds. Finally, to show that Eq. (8) is equivalent to Eq. (20), note that it follows from Equation (2.4.14) of [13, p. 103] that $[\mathcal{N}(A^T)]^\perp = \mathcal{R}(A)$. Hence, by Fact 2.9.16 of [13, p. 121], Eq. (20) holds if and only if $\sum_{i=1}^n [\mathcal{N}(B^T(A^{i-1})^T)]^\perp = \sum_{i=1}^n \mathcal{R}(A^{i-1}B) = [\mathcal{N}(A^T)]^\perp = \mathcal{R}(A)$. Consequently, Eq. (21) is equivalent to Eq. (20). \square

Definition 2.3 (Haddad et al. [10]). Consider the system given by Eqs. (5) and (6) with $B=0$. The pair (A, C) is *semiobservable* if

$$\bigcap_{k=1}^n \mathcal{N}(CA^{k-1}) = \mathcal{N}(A). \tag{22}$$

Next, recall that the *unobservable subspace* $\mathcal{U}_{t_f}(A, C)$ at time $t_f > 0$ is the subspace

$$\mathcal{U}_{t_f}(A, C) = \{x_0 \in \mathbb{R}^n : y(t) = 0 \text{ for all } t \in [0, t_f]\}.$$

As in the controllable subspace case, $\mathcal{U}_{t_f}(A, C)$ is independent of t_f , and hence, we write $\mathcal{U}(A, C)$ for $\mathcal{U}_{t_f}(A, C)$, and call $\mathcal{U}(A, C)$ the *unobservable subspace* of (A, C) [13].

The next result characterizes semiobservability in several equivalent ways.

Theorem 2.2. The following statements are equivalent:

- (i) (A, C) is semiobservable.
- (ii) $\mathcal{N}(\int_0^{t_f} e^{A^T t} C^T C e^{A t} dt) = \mathcal{N}(A)$.
- (iii) $\mathcal{U}(A, C) = \mathcal{N}(A)$.
- (iv) $\mathcal{N}(A) \subseteq \mathcal{N}(C)$ and, for every $x_0 \in \mathcal{R}(A^T)$, the initial state $x(0) = x_0$ can be uniquely determined from $y(t)$ on $[0, t_f]$.

Proof. First, note that it follows from Lemma 12.3.2 of [13, p. 800] that

$$\mathcal{U}(A, C) = \mathcal{N}\left(\int_0^{t_f} e^{A^T t} C^T C e^{A t} dt\right) = \bigcap_{i=1}^n \mathcal{N}(CA^{i-1}). \tag{23}$$

To show the equivalence of (i) and (ii), note that if (A, C) is semiobservable, then it follows from Eq. (23) that $\mathcal{N}(\int_0^{t_f} e^{A^T t} C^T C e^{A t} dt) = \mathcal{N}(A)$ holds. Conversely, if $\mathcal{N}(\int_0^{t_f} e^{A^T t} C^T C e^{A t} dt) = \mathcal{N}(A)$ holds, then it follows from Eq. (23) that Eq. (22) holds. Hence, by definition, (A, C) is semiobservable.

Next, it follows from Eq. (23) that (ii) holds if and only if (iii) holds, which shows the equivalence of (ii) and (iii).

To show (i) implies (iv), assume that (A, C) is semiobservable and note that it follows from Eq. (22) that $\mathcal{N}(A) = \bigcap_{i=1}^n \mathcal{N}(CA^{i-1}) \subseteq \mathcal{N}(C)$. Moreover, it follows from $i) \Rightarrow iii)$ that $\mathcal{U}(A, C) = \mathcal{N}(A)$. Hence, $\mathcal{U}(A, C)^\perp = \mathcal{N}(A)^\perp = \mathcal{R}(A^T)$. Thus, for every $x_0 \in \mathcal{R}(A^T)$, $x_0 \in \mathcal{U}(A, C)^\perp$. Now, it follows from Lemma 12.3.6 of [13, p. 802] that, for every $x_0 \in \mathcal{U}(A, C)^\perp$,

$$x_0 = W_o^+ \int_0^{t_f} e^{A^T t} C^T y(t) dt, \tag{24}$$

where $W_o \triangleq \int_0^{t_f} e^{A^T t} C^T C e^{A t} dt$, and hence, it follows from Eq. (24) that x_0 can be uniquely determined from $y(t)$ on $[0, t_f]$.

To show that (iv) implies (i), note that $\mathcal{N}(A) \subseteq \mathcal{N}(CA^i)$ for every $i \in \{1, \dots, n-1\}$. Furthermore, it follows from $\mathcal{N}(A) \subseteq \mathcal{N}(C)$ and Eq. (23) that $\mathcal{N}(A) = \mathcal{N}(A) \cap \mathcal{N}(C) \subseteq \bigcap_{i=1}^n \mathcal{N}(CA^{i-1}) = \mathcal{U}(A, C)$.

Let $x_0 \in \mathcal{U}(A, C)$. Since $y(t) = 0$ for all $t \in [0, t_f]$ is the free response corresponding to $x_0 = 0$ (since $y(t) = C e^{A t} x_0$ for all $t \geq 0$), it follows that $0 \in \mathcal{U}(A, C)$. Now, suppose that there exists a nonzero vector $x_0 \in \mathcal{U}(A, C)$. In this case, it follows from Eq. (23) that $x_0 \in \bigcap_{i=1}^n \mathcal{N}(CA^{i-1}) = \mathcal{N}(W_o)$. Then, with $x(0) = x_0$, the free response is given by $y(t) = 0$ for all $t \in [0, t_f]$, and hence, x_0 cannot be uniquely determined from the knowledge of $y(t)$ for all $t \in [0, t_f]$.

We claim that $x_0 \in \mathcal{N}(A)$. Suppose, *ad absurdum*, that $x_0 \notin \mathcal{N}(A)$. Since $\mathcal{N}(A)$ is closed, it follows that $\mathcal{N}(A) \oplus \mathcal{N}(A)^\perp = \mathbb{R}^n$, where \oplus denotes the direct sum. Hence, $x_0 \in \mathcal{N}(A)^\perp = \mathcal{R}(A^T)$. However, by assumption, x_0 can be uniquely determined from $y(t)$ on $[0, t_f]$. This implies that $x_0 \notin \mathcal{U}(A, C)$, which contradicts the fact that $x_0 \in \mathcal{U}(A, C)$. Hence, $\mathcal{U}(A, C) \subseteq \mathcal{N}(A)$. Consequently, $\mathcal{U}(A, C) = \mathcal{N}(A)$. Now, it follows from (iii) \Rightarrow (i) that (A, C) is semiobservable. \square

It follows from Theorem 2.2 that semiobservability of the linear dynamical system (5) and (6) implies that given the system output y , the state x belonging to range space of A^T can be reconstructed uniquely. Thus, semicontrollability and semiobservability are extensions of controllability and observability. In particular, semicontrollability is an extension of null controllability to *nonisolated equilibrium controllability*, whereas semiobservability is an extension of zero-state observability to *nonisolated equilibrium observability*.

The following result gives a necessary and sufficient conditions for semistability of Eqs. (5) and (6).

Theorem 2.3 (Haddad et al. [10]). *Consider the dynamical system \mathcal{G} given by Eq. (5) with $B=0$ and output given by Eq. (6). Then \mathcal{G} is semistable if and only if for every semiobservable pair (A, C) there exists a $n \times n$ matrix $P = P^T \geq 0$ such that*

$$0 = A^T P + P A + C^T C \tag{25}$$

is satisfied. Furthermore, if (A, C) is semiobservable and P satisfies Eq. (25), then

$$P = \int_0^\infty e^{A^T t} C^T C e^{A t} dt + P_0, \tag{26}$$

for some $P_0 = P_0^T \in \mathbb{R}^{n \times n}$ satisfying

$$0 = A^T P_0 + P_0 A \tag{27}$$

and

$$P_0 \geq - \int_0^\infty e^{A^T t} C^T C e^{A t} dt. \tag{28}$$

In addition, $\min_{P \in \mathcal{P}} \|P\|_F$ has a unique least squares solution P given by

$$P_{LS} = \int_0^\infty e^{A^T t} C^T C e^{A t} dt, \tag{29}$$

where \mathcal{P} denotes the set of all P satisfying Eq. (25).

Next, we introduce the notions of *semistabilizability* and *semidetectability* [15] as generalizations of stabilizability and detectability.

Definition 2.4 (Hui and Liu [15]). Consider the dynamical system given by Eqs. (5) and (6). The pair (A, B) is *semistabilizable* if

$$\text{rank}[B \ j\omega I_n - A] = n \tag{30}$$

for every nonzero $\omega \in \mathbb{R}$. The pair (A, C) is *semidetectable* if

$$\text{rank} \begin{bmatrix} C \\ j\omega I_n - A \end{bmatrix} = n \tag{31}$$

for every nonzero $\omega \in \mathbb{R}$.

Note that (A, C) is semidetectable if and only if (A^T, C^T) is semistabilizable. Furthermore, it is important to note that semistabilizability and semidetectability are *different* notions from the standard notions of stabilizability and detectability used in linear system theory. Recall that (A, B) is stabilizable if and only if $\text{rank}[B \ \lambda I_n - A] = n$ for every $\lambda \in \mathbb{C}$ in the closed right-half plane, and (A, C) is detectable if and only if $\text{rank} \begin{bmatrix} C \\ \lambda I_n - A \end{bmatrix} = n$ for every $\lambda \in \mathbb{C}$ in the closed right-half plane. Hence, if (A, C) is detectable, then (A, C) is semidetectable; however, the converse is not true. A similar remark holds for the notions of controllability and observability. Namely, if (A, C) (resp., (A, B)) is observable (resp., controllable), then (A, C) (resp., (A, B)) is semidetectable (resp., semicontrollable); however, the converse is not true. Hence, semidetectability (resp., semistabilizability) is a weaker notion than both observability and detectability (resp., controllability and stabilizability). Since Eqs. (30) and (31) only concern stabilizability and detectability of the pairs (A, B) and (A, C) on the imaginary axis, we refer to these notions as semistabilizability and semidetectability.

Remark 2.1. It follows from Facts 2.11.1 to 2.11.3 of [13, pp. 130–131] that Eqs. (30) and (31) are equivalent to

$$\dim[\mathcal{R}(j\omega I_n - A) + \mathcal{R}(B)] = n \tag{32}$$

and

$$\mathcal{N}(j\omega I_n - A) \cap \mathcal{N}(C) = \{0\}, \tag{33}$$

respectively, where $\dim(\cdot)$ denotes the dimension of a set.

Example 2.1. Consider $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Clearly, (A, B) is not stabilizable. However, it can be verified using Eq. (30) that (A, B) is semistabilizable.

As in the case of controllability and stabilizability, state feedback control does not destroy semistabilizability and semicontrollability. This is shown in the next lemma.

Lemma 2.3. *Let $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and $K \in \mathbb{R}^{m \times n}$. If (A, B) is semistabilizable (resp., semicontrollable), then $(A + BK, B)$ is semistabilizable (resp., semicontrollable).*

Proof. Since (A, B) is semistabilizable, it follows that $\text{rank}[B \ j\omega I_n - A] = n$ for all nonzero $\omega \in \mathbb{R}$. Hence, using Sylvester’s inequality, it follows that

$$\begin{aligned} n &= n + (m + n) - (m + n) \\ &= \text{rank}[B \ j\omega I_n - A] + \text{rank} \begin{bmatrix} I_m & -K \\ 0 & I_n \end{bmatrix} - (m + n) \\ &\leq \text{rank} \left(\begin{bmatrix} B \ j\omega I_n - A \\ 0 & I_n \end{bmatrix} \begin{bmatrix} I_m & -K \\ 0 & I_n \end{bmatrix} \right) \\ &\leq \text{rank}[B \ j\omega I_n - A] \\ &= n \end{aligned} \tag{34}$$

for all nonzero $\omega \in \mathbb{R}$. Now, since

$$[B \ j\omega I_n - A - BK] = [B \ j\omega I_n - A] \begin{bmatrix} I_m & -K \\ 0 & I_n \end{bmatrix},$$

it follows from Eq. (34) that

$$\text{rank}[B \ j\omega I_n - A - BK] = n$$

for all nonzero $\omega \in \mathbb{R}$. Thus, $(A + BK, B)$ is semistabilizable. The proof for semicontrollability follows similarly as in the proof of Proposition 2.1 in [10]. \square

Next, using the notions of semistabilizability and semidetectability, we provide a generalization of Theorem 2.3. First, however, the following lemmas are needed.

Lemma 2.4. *Let $A \in \mathbb{R}^{n \times n}$. Then A is semistable if and only if $\mathcal{N}(A) \cap \mathcal{R}(A) = \{0\}$ and $\text{spec}(A) \subseteq \{\lambda \in \mathbb{C} : \lambda + \lambda^* < 0\} \cup \{0\}$, where λ^* denotes the complex conjugate of λ .*

Proof. If A is semistable, then it follows from Definition 11.8.1 of [13, p. 727] that $\text{spec}(A) \subseteq \{\lambda \in \mathbb{C} : \lambda + \lambda^* < 0\} \cup \{0\}$ and either A is Hurwitz or there exists an invertible matrix $S \in \mathbb{R}^{n \times n}$ such that $A = S \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} S^{-1}$, where $J \in \mathbb{R}^{r \times r}$, $r = \text{rank } A$, and J is Hurwitz. If A is Hurwitz, then $\mathcal{N}(A) = \{0\} = \mathcal{N}(A) \cap \mathcal{R}(A)$. Alternatively, if A is not Hurwitz, then $\mathcal{N}(A) = \{S[0_{1 \times r}, y_2^T]^T : y_2 \in \mathbb{R}^{n-r}\}$. In this case, for every $S[0_{1 \times r}, x_2^T]^T \in \mathcal{N}(A) \cap \mathcal{R}(A)$, there exists

$z \in \mathbb{R}^n$ such that $S[0_{1 \times r}, x_2^T]^T = Az$. Hence,

$$S[0_{1 \times r}, x_2^T]^T = S \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} S^{-1}z,$$

that is,

$$\begin{bmatrix} 0 \\ x_2 \end{bmatrix} = \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} S^{-1}z,$$

which implies that $x_2 = 0$. Thus, $\mathcal{N}(A) \cap \mathcal{R}(A) = \{0\}$.

Conversely, assume that $\mathcal{N}(A) \cap \mathcal{R}(A) = \{0\}$ and $\text{spec}(A) \subseteq \{\lambda \in \mathbb{C} : \lambda + \lambda^* < 0\} \cup \{0\}$. If A is nonsingular, then A is Hurwitz, and hence, A is semistable. Next, we consider the case where A is singular. Let $x \in \mathcal{N}(A^2)$ and note that it follows from $A^2x = AAx = 0$ that $Ax \in \mathcal{N}(A)$. Now, noting that $Ax \in \mathcal{R}(A)$, it follows from $\mathcal{N}(A) \cap \mathcal{R}(A) = \{0\}$ that $Ax = 0$, that is, $x \in \mathcal{N}(A)$. Hence, $\mathcal{N}(A^2) \subseteq \mathcal{N}(A)$. However, since $\mathcal{N}(A) \subseteq \mathcal{N}(A^2)$, it follows that $\mathcal{N}(A) = \mathcal{N}(A^2)$. Thus, by Proposition 5.5.8 of [13, p. 323], $0 \in \text{spec}(A)$ is semisimple, and hence, by Definition 11.8.1 of [13, p. 727], A is semistable. \square

Lemma 2.5. Let $A \in \mathbb{R}^{n \times n}$ and $C \in \mathbb{R}^{l \times n}$. If A is semistable and $\mathcal{N}(A) \subseteq \mathcal{N}(C)$, then $CL = 0$, where L is given by

$$L \triangleq I_n - AA^\# \tag{35}$$

Proof. It follows from the semistability of A and Proposition 11.8.1 of Eq. [13] that L is well defined. Next, we show that $CLx = 0$ for every $x \in \mathbb{R}^n$. Suppose, *ad absurdum*, that there exists $x \in \mathbb{R}^n$, $x \neq 0$, such that $CLx \neq 0$. Then, $Lx \notin \mathcal{N}(C)$. Since $\mathcal{N}(A) \subseteq \mathcal{N}(C)$, it follows that $Lx \notin \mathcal{N}(A)$. However, $ALx = A(I_n - AA^\#)x = (A - AAA^\#)x = 0$, which implies that $Lx \in \mathcal{N}(A)$, which is a contradiction. Hence, $CLx = 0$ for every $x \in \mathbb{R}^n$. \square

Theorem 2.4. Consider the dynamical system \mathcal{G} given by Eq. (5) with $B=0$ and output given by Eq. (6). Then the following statements are equivalent:

- (i) \mathcal{G} is semistable.
- (ii) $\text{rank}(j\omega I_n - A) = n$ for every nonzero $\omega \in \mathbb{R}$ and there exist a positive integer p , a $p \times n$ matrix E , and a $n \times n$ matrix $P = P^T \geq 0$ such that

$$0 = A^T P + PA + E^T E. \tag{36}$$

In this case,

$$P = \int_0^\infty e^{A^T t} (E^T E + L^T E^T E L) e^{A t} dt + P_0, \tag{37}$$

where $L = I_n - AA^\#$ and P_0 satisfies Eqs. (27) and (28).

- (iii) For every matrix $C \in \mathbb{R}^{l \times n}$ such that (A, C) is semiobservable, there exists a $n \times n$ matrix $P = P^T \geq 0$ such that Eq. (25) holds.
- (iv) There exist a positive integer p , a $p \times n$ matrix E , and a $n \times n$ matrix $P = P^T \geq 0$ such that (A, E) is semiobservable and Eq. (36) holds.
- (v) There exist a positive integer p , a $p \times n$ matrix E , and a $n \times n$ matrix $P = P^T \geq 0$ such that (A, E) is semidetactable and Eq. (36) holds.

Proof. First, note that if A is semistable, then it follows from the definition of semistability that $j\omega \notin \text{spec}(A)$, $\omega \neq 0$. Hence, $\text{rank}(A - j\omega I_n) = n$ for every nonzero $\omega \in \mathbb{R}$.

To prove the existence of a nonnegative definite solution to Eq. (36), let E be such that $\mathcal{N}(A) \subseteq \mathcal{N}(E)$. For every such pair (A, E) , let

$$\hat{P} = \int_0^\infty e^{A^T t} E^T E e^{A t} dt. \tag{38}$$

Now, it follows from Proposition 2.2 of [10] that \hat{P} is well defined. Clearly, $\hat{P} = \hat{P}^T \geq 0$. Since A is semistable, it follows from Lemma 2.4 that $\mathcal{N}(A) \cap \mathcal{R}(A) = \{0\}$, and hence, A is group invertible [16, p. 119]. Hence, it follows from Eq. (35) and (38) that

$$\begin{aligned} A^T \hat{P} + \hat{P} A &= \int_0^\infty \frac{d}{dt} \left(e^{A^T t} E^T E e^{A t} \right) dt \\ &= (I_n - AA^\#)^T E^T E (I_n - AA^\#) - E^T E \\ &= L^T E^T E L - E^T E. \end{aligned} \tag{39}$$

Next, setting $\hat{P} = P - Z$, where $Z \in \mathbb{R}^{n \times n}$ and $Z = Z^T \geq 0$, it follows from Eq. (39) that

$$A^T P + P A + E^T E = A^T Z + Z A + L^T E^T E L. \tag{40}$$

Furthermore, it follows from Lemma 2.4 in [10] that $Lx = 0$ for all $x \in \mathcal{N}(A)$, and hence, the pair (A, EL) is semiobservable since $ELx = 0$ for all $x \in \mathcal{N}(A)$. Consequently, it follows from Theorem 2.3 that

$$Z = \int_0^\infty e^{A^T t} L^T E^T E L e^{A t} dt + P_0, \tag{41}$$

which is a nonnegative-definite solution of

$$0 = A^T Z + Z A + L^T E^T E L. \tag{42}$$

Thus, it follows from Eq. (40) that Eq. (37) satisfies Eq. (36), which proves that (i) implies (ii).

Let $V(x) = x^T P_1 x$, where $P_1 \triangleq \hat{P} + L^T L$. If $V(x) = 0$ for some $x \in \mathbb{R}^n$, then $\hat{P}x = 0$ and $Lx = 0$. It follows from (i) of Lemma 2.4 in [10] that $x \in \mathcal{N}(A)$, and $Lx = 0$ implies that $x \in \mathcal{R}(A)$. Now, it follows from (ii) of Lemma 2.4 in [10] that $x = 0$. Hence, P_1 is positive definite. Note that P_1 satisfies Eq. (36) since $LA = A - AA^\#A = 0$, and hence,

$$\begin{aligned} A^T P_1 + P_1 A + E^T E &= A^T \hat{P} + \hat{P} A + E^T E + A^T L^T L + L^T L A \\ &= L^T E^T E L + (LA)^T L + L^T L A \\ &= 0. \end{aligned}$$

Also note that $\dot{V}(x) = -x^T E^T E x \leq 0$, $x \in \mathbb{R}^n$, which implies that A is Lyapunov stable. Furthermore, it follows from $\text{rank}(A - j\omega I_n) = n$ for every nonzero $\omega \in \mathbb{R}$ that $j\omega \in \text{spec}(A)$, $\omega \neq 0$. Hence, A is semistable, which proves that (ii) implies (i).

The proof of the equivalence of (i) and (iii) follows from Theorem 2.2 in [10]. Next, we show that (i) is equivalent to (iv). It follows from Theorem 2.3 that (iv) implies (i). Alternatively, if (i) holds, then choose E such that $\mathcal{N}(E) = \mathcal{N}(A)$ (an obvious choice is $E = A$). Since $\mathcal{N}(EA^i) \supseteq \mathcal{N}(A)$ and $\mathcal{N}(EA^{i+1}) \supseteq \mathcal{N}(EA^i)$ for every $i \in \{0, \dots, n-1\}$, it follows that $\mathcal{N}(A) \subseteq \bigcap_{i=1}^n \mathcal{N}(EA^{i-1}) \subseteq \mathcal{N}(E) = \mathcal{N}(A)$, and hence, $\bigcap_{i=1}^n \mathcal{N}(EA^{i-1}) = \mathcal{N}(A)$. Thus, (A, E) is semiobservable. Now, using similar arguments as in the proof of the equivalence of (i) and (ii), there exists $P = P^T \geq 0$ such that Eq. (36) holds, which shows that (i) implies (iv).

Finally, we show the equivalence of *i*) and *v*). If *A* is semistable, then $j\omega \notin \text{spec}(A)$, $\omega \neq 0$, and hence, $\text{rank}(j\omega I_n - A) = n$ for every nonzero $\omega \in \mathbb{R}$. Thus, $\text{rank} \begin{bmatrix} E \\ j\omega I_n - A \end{bmatrix} = n$ for every $E \in \mathbb{R}^{p \times n}$ and every positive integer *p*. The proof of the existence of a positive-definite solution to Eq. (36) follows exactly as in the proof of (i) \Rightarrow (ii). The converse follows using similar arguments as in the proof of (ii) \Rightarrow (i) for *A* Lyapunov stable.

To show that *A* is semistable, suppose, *ad absurdum*, $j\omega \in \text{spec}(A)$, where $\omega \in \mathbb{R}$ is nonzero, and let $x \in \mathbb{C}^n$, $x \neq 0$, be an associated eigenvector of *A*. Then, it follows from Eq. (36) that

$$\begin{aligned} -x^* E^T E x &= x^* (A^T P + P A) x \\ &= x^* [(j\omega I_n - A)^* P + P (j\omega I_n - A)] x \\ &= 0. \end{aligned}$$

Hence, $E x = 0$, and thus, $\begin{bmatrix} E \\ j\omega I_n - A \end{bmatrix} x = 0$, which, since $\text{rank} \begin{bmatrix} E \\ j\omega I_n - A \end{bmatrix} = n$, implies that $x = 0$, which is a contradiction. Consequently, $j\omega \notin \text{spec}(A)$ for all nonzero $\omega \in \mathbb{R}$. Hence, $\text{spec}(A) \subset \{\lambda \in \mathbb{C} : \text{Re } \lambda < 0\} \cup \{0\}$ and, if $0 \in \text{spec}(A)$, then 0 is semisimple. Therefore, *A* is semistable. \square

Lemma 2.6. *Let $x, y \in \mathbb{R}^n$ be such that $xy^T = yx^T \geq 0$. Then $y = \alpha x$, where $\alpha \geq 0$.*

Proof. Note that for $x = 0$ or $y = 0$ the inequality is immediate. Next, if *x* and *y* are linearly dependent, then it follows from $xy^T = yx^T \geq 0$ that $y = \alpha x$, where $\alpha \geq 0$. Alternatively, assume, *ad absurdum*, that *x* and *y* are linearly independent. In this case, it follows from Proposition 7.1.8 of [13, p. 441] that $xy^T = yx^T$ if and only if $\text{vec}^{-1}(y \otimes x) = \text{vec}^{-1}(x \otimes y)$, which further implies that $y \otimes x = x \otimes y$. Let $x = [x_1, \dots, x_n]^T$ and $y = [y_1, \dots, y_n]^T$. Then it follows from $y \otimes x = x \otimes y$ that $y_i x = x_i y$ for every $i \in \{1, \dots, n\}$. Since *x* and *y* are linearly independent, it follows that $y_i x - x_i y = 0$ for every $i \in \{1, \dots, n\}$ if and only if $y_i = x_i = 0$ for every $i \in \{1, \dots, n\}$. This contradicts the assumption that *x* and *y* are linearly independent. Now, the assertion follows directly from the first case. \square

Theorem 2.5. *Consider the dynamical system \mathcal{G} given by Eq. (5) with $B = 0$ and output given by Eq. (6). Assume that there exists a $n \times n$ matrix $P = P^T \geq 0$ such that Eq. (25) holds. Then \mathcal{G} is semistable if and only if the pair (*A*, *C*) is semidetactable. Furthermore, if (*A*, *C*) is semidetactable and *P* satisfies Eq. (25), then*

$$P = \int_0^\infty e^{A^T t} C^T C e^{A t} dt + \alpha z z^T, \tag{43}$$

where $\alpha \geq 0$, $z \in \mathcal{N}(A^T)$,

$$\alpha z z^T = \int_0^\infty e^{A^T t} L^T C^T C L e^{A t} dt + P_0, \tag{44}$$

$L = I_n - A A^\#$, and P_0 satisfies Eqs. (27) and (28).

Proof. The first part of the result is a direct consequence of Theorem 2.4. To prove that *P* has the form given by Eq. (43), first note that it follows from Eq. (25) that $(A \oplus A)^T \text{vec } P = -\text{vec}(C^T C)$. Hence, $\text{vec}(C^T C) \in \mathcal{R}((A \oplus A)^T)$. Next, it follows from Lemma 3.8 of [11]

that $(A \oplus A)^T$ is semistable, and hence, by Lemma 3.9 of [11],

$$\begin{aligned} \text{vec}^{-1}(((A \oplus A)^T)^\# \text{vec}(C^T C)) &= - \int_0^\infty \text{vec}^{-1} \left(e^{(A \oplus A)^T t} \text{vec}(C^T C) \right) dt \\ &= - \int_0^\infty \text{vec}^{-1} \left(e^{A^T t} \otimes e^{A^T t} \right) \text{vec}(C^T C) dt \\ &= - \int_0^\infty e^{A^T t} C^T C e^{A t} dt, \end{aligned} \tag{45}$$

where in Eq. (45) we used the facts that $e^{X \oplus Y} = e^X \otimes e^Y$ and $\text{vec}(XYZ) = (Z^T \otimes X)\text{vec} Y$ [13]. Hence, $P = \int_0^\infty e^{A^T t} C^T C e^{A t} dt + \text{vec}^{-1}(w)$, where w satisfies $w \in \mathcal{N}((A \oplus A)^T)$ and $\text{vec}^{-1}(w) = (\text{vec}^{-1}(w))^T \geq 0$. (The nonnegative definiteness of $\text{vec}^{-1}(w)$ is guaranteed by Theorem 4.2a of [17].) Since $(A \oplus A)^T$ is semistable, it follows that a general solution to the equation $(A \oplus A)^T w = 0$ is given by $w = z \otimes y$, where $z, y \in \mathcal{N}(A^T)$. Hence, $\text{vec}^{-1}(w) = \text{vec}^{-1}(z \otimes y) = yz^T$, where we used the fact that $zy^T = \text{vec}^{-1}(y \otimes z)$. Furthermore, $zy^T = yz^T \geq 0$. Now, it follows from Lemma 2.6 that $y = \alpha z$, where $\alpha \geq 0$. Finally, Eq. (44) directly follows from Theorem 2.4 by comparing Eq. (37) with Eq. (43) for $C = E$. \square

Consider the dynamical system given by Eqs. (5) and (6) with $B = 0$. If the pair (A, C) is semiobservable, then (A, C) is semidetectable and, in this case, it follows from Theorems 2.3 and 2.5 that $\int_0^\infty e^{A^T t} L^T C^T C L e^{A t} dt = 0$.

Lemma 2.7 (Bhat and Bernstein [18]). *Consider the dynamical system \mathcal{G} given by Eq. (5) with $B = 0$ and output given by Eq. (6). If the pair (A, C) is semiobservable and there exists an $n \times n$ matrix $P = P^T \geq 0$ such that Eq. (25) is satisfied, then (i) $\mathcal{N}(P) \subseteq \mathcal{N}(A) \subseteq \mathcal{N}(C)$ and (ii) $\mathcal{R}(A) \cap \mathcal{N}(A) = \{0\}$.*

The following theorem is a direct consequence of Theorem 2.3.

Theorem 2.6 (Haddad et al. [10]). *Consider the closed-loop system \mathcal{G} given by Eqs. (5) and (6) with feedback controller $u(t) = Kx(t)$, where $K \in \mathbb{R}^{m \times n}$. Then \mathcal{G} is semistable if and only if for every semicontrollable pair (A, B) and semiobservable pair (A, C) there exists a $n \times n$ matrix $P = P^T \geq 0$ such that*

$$0 = \tilde{A}^T P + P \tilde{A} + C^T C + K^T R_2 K, \tag{46}$$

where $\tilde{A} \triangleq A + BK$. Furthermore, the least squares solution of Eq. (46) is given by

$$P_{LS} \triangleq \int_0^\infty e^{\tilde{A}^T t} (C^T C + K^T R_2 K) e^{\tilde{A} t} dt. \tag{47}$$

Finally, in this case (4) is given by

$$J(x_0, K) = x_0^T P_{LS} x_0. \tag{48}$$

Next, we give an alternative form of Theorem 2.6 using semidetectability.

Theorem 2.7. *Consider the closed-loop system \mathcal{G} given by Eqs. (5) and (6) with feedback controller $u(t) = Kx(t)$, where $K \in \mathbb{R}^{m \times n}$. Assume that there exists a $n \times n$ matrix $P = P^T \geq 0$ such that (46) holds. Then \mathcal{G} is semistable if and only if (A, C) is semidetectable. Furthermore, Eq. (4) is given by Eq. (48).*

Proof. The first assertion is a direct consequence of **Theorem 2.5**. To show that Eq. (4) is given by Eq. (48), it follows from Eq. (46) that $-x^T(\tilde{A}^T P + P\tilde{A})x = x(C^T C + K^T R_2 K)x$ for every $x \in \mathbb{R}^n$, and hence, $\mathcal{N}(\tilde{A}) \subseteq \mathcal{N}(C) \cap \mathcal{N}(R_2 K)$. Thus, for $x_e \in \mathcal{N}(\tilde{A})$, $Cx_e = 0$ and $R_2 Kx_e = 0$. Now, it follows from Eq. (4) that

$$\begin{aligned} J(x_0, K) &= \int_0^\infty x^T(t)(C^T C + K^T R_2 K)x(t) dt \\ &= x_0^T \int_0^\infty e^{\tilde{A}^T t} (C^T C + K^T R_2 K) e^{\tilde{A} t} dt x_0 \\ &= x_0^T P_{LS} x_0, \end{aligned}$$

which completes the proof. \square

Finally, the following lemma is needed.

Lemma 2.8 (Haddad et al. [10]). *Consider the linear dynamical system \mathcal{G} given by Eqs. (5) and (6) with $u \equiv 0$. If \mathcal{G} is semistable, then for every $x_0 \in \mathbb{R}^n$, the performance measure*

$$J(x_0) = \int_0^\infty [(x(t) - x_e)^T C^T C (x(t) - x_e)] dt, \tag{49}$$

where $x_e = (I - AA^\#)x_0$, is finite.

3. Semistability analysis of nonlinear systems

In this section, we provide connections between Lyapunov functions and nonquadratic cost evaluation. Specifically, we consider the problem of evaluating a nonlinear-nonquadratic cost functional that depends on the solution of the nonlinear dynamical system (7). In particular, we show that the nonlinear-nonquadratic cost functional

$$J(x_0) \triangleq \int_0^\infty L(x(t)) dt, \tag{50}$$

where $L : \mathcal{D} \rightarrow \mathbb{R}$ and $x(t)$, $t \geq 0$, satisfies Eq. (7), can be evaluated in a convenient form so long as Eq. (7) is related to an underlying Lyapunov-like function that proves semistability of Eq. (7).

Theorem 3.1. *Consider the nonlinear dynamical system \mathcal{G} given by Eq. (7) with performance functional (50), and let \mathcal{Q} be an open neighborhood of $f^{-1}(0)$. Suppose that the solution $x(t)$, $t \geq 0$, of Eq. (7) is bounded for all $x \in \mathcal{Q}$ and assume that there exists a continuously differentiable function $V : \mathcal{D} \rightarrow \mathbb{R}$ such that*

$$V'(x)f(x) \leq 0, \quad x \in \mathcal{Q}, \tag{51}$$

$$L(x) + V'(x)f(x) = 0, \quad x \in \mathcal{D}. \tag{52}$$

If every point in the largest invariant set \mathcal{M} of $\{x \in \mathcal{Q} : V'(x)f(x) = 0\}$ is Lyapunov stable, then Eq. (7) is semistable and

$$J(x_0) = V(x_0) - V(x_e), \quad x_0 \in \mathcal{Q}, \tag{53}$$

where $x_e = \lim_{t \rightarrow \infty} x(t)$.

Proof. Let $x(t)$, $t \geq 0$, satisfy Eq. (7). Then

$$\dot{V}(x(t)) \triangleq \frac{d}{dt} V(x(t)) = V'(x(t))f(x(t)), \quad t \geq 0.$$

Hence, it follows from Eq. (51) that $\dot{V}(x(t)) \leq 0$, $t \geq 0$. Since every solution of Eq. (7) is bounded, it follows from the hypothesis on $V(\cdot)$ that, for every $x \in \mathcal{Q}$, the positive limit set $\omega(x)$ of Eq. (7) is nonempty and contained in the largest invariant set \mathcal{M} of $\{x \in \mathcal{Q} : V'(x)f(x) = 0\}$. Since every point in \mathcal{M} is a Lyapunov stable equilibrium point, it follows from Lemma 2.1 that $\omega(x)$ contains a single point for every $x \in \mathcal{Q}$, and $\lim_{t \rightarrow \infty} s(t, x)$ exists for every $x \in \mathcal{Q}$. Now, since $\lim_{t \rightarrow \infty} s(t, x) \in \mathcal{M}$ is Lyapunov stable for every $x \in \mathcal{Q}$, semistability is immediate. Consequently, $x(t) \rightarrow x_e$ as $t \rightarrow \infty$ for all initial conditions $x_0 \in \mathcal{Q}$.

Next, since

$$0 = -\dot{V}(x(t)) + V'(x(t))f(x(t)), \quad t \geq 0, \tag{54}$$

it follows from Eq. (52) that

$$L(x(t)) = -\dot{V}(x(t)) + L(x(t)) + V'(x)f(x(t)) = -\dot{V}(x(t)). \tag{55}$$

Now, integrating over $[0, t]$ yields

$$\int_0^t L(x(s)) \, ds = V(x_0) - V(x(t)). \tag{56}$$

Letting $t \rightarrow \infty$ and noting that $V(x(t)) \rightarrow V(x_e)$ for all $x_0 \in \mathcal{Q}$ yields Eq. (53). \square

The following theorem uses Theorem 3.1 to develop an analogous result for linear dynamical systems without the a priori assumption of boundedness of solutions. First, however, recall that a continuous function $V : \mathcal{D} \rightarrow \mathbb{R}$ is said to be *proper relative to* $\mathcal{D}_p \subseteq \mathcal{D}$ if $V^{-1}(\mathcal{D}_c)$ is a relatively compact subset of \mathcal{D}_p for all compact subsets \mathcal{D}_c of \mathbb{R} , where $V^{-1}(\cdot)$ denotes the inverse image of \mathcal{D}_c .

Theorem 3.2. Consider the linear dynamical system \mathcal{G} given by Eqs. (5) and (6) with $B=0$ and with quadratic performance measure (49). If (A, C) is semiobservable, then \mathcal{G} is globally semistable and

$$J(x_0) = x_0^T (AA^\#)^T P A A^\# x_0, \tag{57}$$

where $P = P^T \geq 0$ is a solution of

$$\begin{bmatrix} AA^\# \\ I_n \end{bmatrix}^T \begin{bmatrix} P & 0 \\ 0 & -P + P_0 \end{bmatrix} \begin{bmatrix} AA^\# \\ I_n \end{bmatrix} = 0 \tag{58}$$

and P_0 satisfies Eqs. (27) and (28).

Proof. Let $f(x) = A(x - x_e)$, $L(x) = (x - x_e)^T C^T C(x - x_e)$, and $\mathcal{Q} = \mathbb{R}^n$, and note that with $V(x) = (x - x_e)^T P(x - x_e)$, where $P = P^T \geq 0$, Eq. (52) specializes to Eqs. (25) and (51) is satisfied for all $x \in \mathbb{R}^n$. Furthermore, note that

$$V'(x)f(x) = V'(x)A(x - x_e) = (x - x_e)^T (A^T P + PA)(x - x_e) = -(x - x_e)^T C^T C(x - x_e),$$

and hence, $\mathcal{N}(A) \subseteq \mathcal{N}(C)$. In addition, since (A, C) is semiobservable, it follows that $\mathcal{N}(C) \subseteq \mathcal{N}(A)$, and hence, $\mathcal{N}(C) = \mathcal{N}(A)$. Thus, $\mathcal{N}(A)$ is the largest invariant set of $\{x \in \mathcal{Q} : V'(x)A(x - x_e) = 0\}$.

Next, since (A, C) is semiobservable, it follows from (ii) of Lemma 2.7 that $\mathcal{R}(A) \cap \mathcal{N}(A) = \{0\}$, which implies that A is group invertible [16, p. 119]. Now, let $L = I - AA^\#$ and consider the Lyapunov function candidate $\hat{V}(\hat{x}) \triangleq \hat{x}^T(P + L^T L)\hat{x}$, where $\hat{x} \triangleq x - x_e$. If $\hat{V}(\hat{x}) = 0$ for some $\hat{x} \in \mathbb{R}^n$, then $P\hat{x} = 0$ and $L\hat{x} = 0$, and hence, $\hat{x} \in \mathcal{N}(P)$. Thus, it follows from Eq. (25) and the semiobservability of (A, C) that $\hat{x} \in \mathcal{N}(A)$. In addition, $\hat{V}(\hat{x}) = 0$ for some $\hat{x} \in \mathbb{R}^n$ implies that $\hat{x} \in \mathcal{N}(L)$, and hence, $\hat{x} \in \mathcal{R}(A)$. Thus, it follows from Lemma 2.7 that $\hat{V}(\hat{x}) = 0$ only if $\hat{x} = 0$, and hence, $\hat{V}(\cdot)$ is positive definite and proper relative to \mathbb{R}^n .

Next, note that the time derivative of $\hat{V}(\hat{x})$ along the trajectories of Eq. (5) with $B=0$ is given by

$$\hat{V}'(\hat{x}(t))A\hat{x}(t) = -\hat{x}^T(t)C^T C\hat{x}(t) + 2\hat{x}^T(t)L^T L A\hat{x}(t) = -\hat{x}^T(t)C^T C\hat{x}(t) \leq 0, \quad t \geq 0,$$

and hence, $x(t) \equiv x_e, t \geq 0$, is Lyapunov stable for every $x_e \in \mathcal{N}(A)$, which implies that every orbit of Eq. (5) with $B=0$ is bounded. Therefore, it follows from Theorem 3.1 that $x(t), t \geq 0$, is semistable and, since $V(\cdot)$ and $\hat{V}(\cdot)$ are sign definite and proper relative to \mathbb{R}^n , \mathcal{G} is globally semistable.

Since \mathcal{G} is globally semistable, it follows from Lemma 2.8 that the quadratic performance measure (49) is finite and, by Eq. (53) of Theorem 3.1, it follows that

$$J(x_0) = (x_0 - x_e)^T P(x_0 - x_e) = x_0^T (AA^\#)^T PAA^\# x_0, \tag{59}$$

which proves Eq. (57). Finally, note that the performance measure (49) can be equivalently written as

$$J(x_0) = x_0^T \int_0^\infty e^{A^T t} C^T C e^{At} dt x_0, \tag{60}$$

which, using Theorem 2.3, yields

$$J(x_0) = x_0^T (P - P_0)x_0. \tag{61}$$

Now, Eq. (58) follows from Eqs. (59) and (61). \square

Note that Eq. (58) can be written as

$$P = (AA^\#)^T PAA^\# + P_0. \tag{62}$$

Hence, since $A^\#A = AA^\#$ and $AA^\#A = A$ [13, p. 403], premultiplying and postmultiplying Eq. (62) by A^T and A , respectively, it follows that $A^T P_0 A = 0$, which is implied by Eq. (27).

Proposition 3.1. Consider the linear dynamical system \mathcal{G} given by Eqs. (5) and (6) with $B=0$ and with quadratic performance measure (49). If (A, C) is semidetactable and there exists $P = P^T \geq 0$ such that Eq. (25) holds, then \mathcal{G} is globally semistable and Eq. (57) holds. In addition, P satisfies

$$\begin{bmatrix} AA^\# \\ I_n \end{bmatrix}^T \begin{bmatrix} P & 0 \\ 0 & -P + \alpha z z^T \end{bmatrix} \begin{bmatrix} AA^\# \\ I_n \end{bmatrix} = 0, \tag{63}$$

where $\alpha \geq 0$ and $z \in \mathcal{N}(A^T)$ satisfies Eq. (44).

Proof. Global semistability of \mathcal{G} is a direct consequence of Theorem 2.4. Next, let $f(x) = A(x - x_e)$, $L(x) = (x - x_e)^T C^T C(x - x_e)$, $\mathcal{Q} = \mathbb{R}^n$, and $V(x) = (x - x_e)^T P(x - x_e)$. Since \mathcal{G} is globally semistable, it follows from Lemma 2.8 that the quadratic performance measure (49) is

finite and, by [Theorem 3.1](#), it follows that

$$J(x_0) = (x_0 - x_e)^T P(x_0 - x_e) = x_0^T (AA^\#)^T PAA^\# x_0, \tag{64}$$

which proves [Eq. \(57\)](#). Finally, note that the performance measure [\(49\)](#) can be equivalently written as

$$J(x_0) = x_0^T \int_0^\infty e^{A^T t} C^T C e^{At} dt x_0, \tag{65}$$

which, using [Theorem 2.5](#), yields

$$J(x_0) = x_0^T (P - \alpha z z^T) x_0, \tag{66}$$

where $\alpha \geq 0$ and $z \in \mathcal{N}(A^T)$ satisfies [Eq. \(44\)](#). Now, [Eq. \(63\)](#) follows from [Eqs. \(64\)](#) and [\(66\)](#). \square

4. Optimal control for semistabilization

In this section, we use the approach of [Theorem 3.1](#) to obtain a characterization of optimal feedback controllers that guarantee closed-loop semistability. Specifically, sufficient conditions for optimality are given in a form that corresponds to a steady-state version of a Hamilton–Jacobi–Bellman-type equation. To address the optimal semistabilization problem, we consider the controlled nonlinear dynamical system [\(2\)](#) with $u(\cdot)$ restricted to the class of *admissible* controls consisting of measurable functions $u(\cdot)$ such that $u(t) \in U, t \geq 0$. A measurable function $\phi : \mathcal{D} \rightarrow U$ satisfying $\phi(x_e) = u_e$, where $x_e \in \mathcal{D}$ is an equilibrium point of [Eq. \(2\)](#) for some $u_e \in U$, is called a *control law*. If $u(t) = \phi(x(t)), t \geq 0$, where $\phi(\cdot)$ is a control law and $x(t)$ satisfies [Eq. \(2\)](#), then we call $u(\cdot)$ a *feedback control law*. Note that the feedback control law is an admissible control since $\phi(\cdot)$ has values in U . Given a control law $\phi(\cdot)$ and a feedback control $u(t) = \phi(x(t)), t \geq 0$, the closed-loop system [\(2\)](#) is given by

$$\dot{x}(t) = F(x(t), \phi(x)), \quad x(0) = x_0, \quad t \geq 0. \tag{67}$$

For the statement of the main theorem of this section, define the set of set-point regulation controllers $\mathcal{S}(x_0)$ for every initial condition $x_0 \in \mathcal{D}$ by

$$\mathcal{S}(x_0) \triangleq \{u(\cdot) : u(\cdot) \text{ is admissible and } x(\cdot) \text{ given by 2 is bounded and satisfies } x(t) \rightarrow x_e \text{ as } t \rightarrow \infty\},$$

where $x_e \in \mathcal{D}$ is an equilibrium point of [Eq. \(2\)](#) for some $u_e \in U$. Note that restricting our minimization problem to $u(\cdot) \in \mathcal{S}(x_0)$, that is, control inputs corresponding to convergent solutions, can be interpreted as incorporating a semidetectability condition through the cost.

Theorem 4.1. *Consider the controlled nonlinear dynamical system [\(2\)](#) with $u(\cdot) \in \mathcal{S}(x_0)$ and performance measure [\(1\)](#), and suppose there exists a continuously differentiable function $V : \mathcal{D} \rightarrow \mathbb{R}$ and a control law $\phi : \mathcal{D} \rightarrow U$ such that*

$$\phi(x_e) = u_e, \quad (x_e, u_e) \in \mathcal{Q} \times U, \tag{68}$$

$$V'(x)F(x, \phi(x)) \leq 0, \quad x \in \mathcal{Q}, \tag{69}$$

$$L(x, \phi(x)) + V'(x)F(x, \phi(x)) = 0, \quad x \in \mathcal{D}, \tag{70}$$

$$L(x, u) + V'(x)F(x, u) \geq 0, \quad (x, u) \in \mathcal{D} \times U, \tag{71}$$

where \mathcal{Q} is an open neighborhood of $F^{-1}(0) \triangleq \{x \in \mathcal{D} : F(x, \phi(x)) = 0\}$. If every point in the largest invariant set \mathcal{M} of $\{x \in \mathcal{Q} : V'(x)F(x, \phi(x)) = 0\}$ is Lyapunov stable, then, with the feedback control $u(\cdot) = \phi(x(\cdot))$, the solution $x(t) = x_e, t \geq 0$, of the closed-loop system (67) is semistable and

$$J(x_0, \phi(x(\cdot))) = V(x_0) - V(x_e). \tag{72}$$

Furthermore, the feedback control $u(\cdot) = \phi(x(\cdot))$ minimizes $J(x_0, u(\cdot))$ in the sense that

$$J(x_0, \phi(x(\cdot))) = \min_{u(\cdot) \in \mathcal{S}(x_0)} J(x_0, u(\cdot)). \tag{73}$$

Proof. If $u(\cdot) \in \mathcal{S}(x_0)$, then the solution $x(t), t \geq 0$, of Eq. (2) is bounded for all initial conditions $x_0 \in \mathcal{Q}$. Thus, semistability is a direct consequence of Eqs. (69) and (70) by applying Theorem 3.1 to the closed-loop system (67). Furthermore, using Eq. (70), condition (72) is a restatement of Eq. (53). To prove Eq. (73), note that

$$\dot{V}(x(t)) = V'(x(t))F(x(t), u(t)), \tag{74}$$

or, equivalently,

$$0 = -\dot{V}(x(t)) + V'(x(t))F(x(t), u(t)). \tag{75}$$

Hence,

$$L(x(t), u(t)) = -\dot{V}(x(t)) + L(x(t), u(t)) + V'(x(t))F(x(t), u(t)). \tag{76}$$

Now, using Eqs. (71) and (72), and the fact that $u(\cdot) \in \mathcal{S}(x_0)$, it follows that

$$\begin{aligned} J(x_0, u(\cdot)) &= \int_0^\infty L(x(t), u(t)) \, dt \\ &= \int_0^\infty -\dot{V}(x(t)) \, dt + \int_0^\infty (L(x(t), u(t)) + V'(x)F(x(t), u(t))) \, dt \\ &= V(x_0) - V(x_e) + \int_0^\infty [L(x(t), u(t)) + V'(x)F(x(t), u(t))] \, dt \\ &\geq V(x_0) - V(x_e), \end{aligned} \tag{77}$$

which yields Eq. (73). \square

Remark 4.1. Theorem 4.1 requires that $u(\cdot) \in \mathcal{S}(x_0)$ or, equivalently, the solution of the closed-loop system is bounded for all $x \in \mathcal{Q}$. For asymptotic stabilization this is automatically satisfied since we additionally require $V(0) = 0, V(x) > 0, x \in \mathcal{D} \setminus \{0\}$, and $V'(x)F(x, \phi(x)) < 0, x \in \mathcal{D}$, in the place of Eq. (69) (see [19, Theorem 3.1] and [3, Theorem 8.2]). This guarantees asymptotic stability of the closed-loop system, and hence, all closed-loop solutions are bounded. One can replace the assumption $u(\cdot) \in \mathcal{S}(x_0)$ in Theorem 4.1 with $u(\cdot)$ being simply admissible and not invoking any assumption on the sign definiteness of $V(\cdot)$. In this case, however, the conditions of Theorem 4.1 need to be supplemented by assuming a nontangency condition of the closed-loop vector field to invariant or negatively invariant subsets of the level sets of $V(\cdot)$ containing the system equilibrium. For details, see [1].

Note that Theorem 4.1 guarantees optimality with respect to the set of admissible semistabilizing controllers $\mathcal{S}(x_0)$ with the optimal control law given by the state feedback controller

$$\phi(x) = \arg \min_{u \in \mathcal{S}(x_0)} [L(x, u) + V'(x)F(x, u)], \tag{78}$$

which invokes a steady-state Hamilton–Jacobi–Bellman-type equation and is independent of the initial condition x_0 . It is important to note that an explicit characterization of $\mathcal{S}(x_0)$ is not required.

The following result specializes [Theorem 4.1](#) to nonlinear affine dynamical systems of the form

$$\dot{x}(t) = f(x(t)) + G(x(t))u(t), \quad x(0) = x_0, \quad t \geq 0, \quad (79)$$

$$y(t) = h(x(t)), \quad (80)$$

where, for every $t \geq 0$, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^l$, $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $G: \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$ are locally Lipschitz continuous in x , $h: \mathbb{R}^n \rightarrow \mathbb{R}^l$ is continuous in x , and $0 = f(x_e) + G(x_e)u_e$ and $y_e = h(x_e)$. Furthermore, we consider performance integrands $L(x, u)$ of the form

$$L(x, u) = (h(x) - y_e)^T (h(x) - y_e) + (u - u_e)^T R_2(x)(u - u_e), \quad (81)$$

where $R_2(x) > 0$, $x \in \mathbb{R}^n$, so that [Eq. \(1\)](#) becomes

$$J(x_0, u(\cdot)) = \int_0^\infty \left[(y(t) - y_e)^T (y(t) - y_e) + (u(t) - u_e)^T R_2(x(t))(u(t) - u_e) \right] dt. \quad (82)$$

Corollary 4.1. Consider the controlled nonlinear dynamical system [\(79\)](#) and [\(80\)](#) with $u(\cdot) \in \mathcal{S}(x_0)$ and performance measure [\(82\)](#), and assume there exists a continuously differentiable function $V: \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$V'(x_e) = 0, \quad x_e \in \mathbb{R}^n, \quad (83)$$

$$\begin{aligned} & (y - y_e)^T (y - y_e) + V'(x)f(x) + V'(x)G(x)u_e \\ & - \frac{1}{4} V'(x)G(x)R_2^{-1}(x)G^T(x)V'^T(x) = 0, \quad (x, u_e) \in \mathbb{R}^n \times \mathbb{R}^m. \end{aligned} \quad (84)$$

If, with the feedback control

$$u = \phi(x) = -\frac{1}{2} R_2^{-1}(x)G^T(x)V'^T(x) + u_e, \quad (85)$$

every equilibrium point $x_e \in F^{-1}(0) = \{x \in \mathcal{D} : f(x) + G(x)\phi(x) = 0\}$ of the closed-loop system

$$\dot{x}(t) = f(x(t)) + G(x(t))\phi(x(t)), \quad x(0) = x_0, \quad t \geq 0, \quad (86)$$

is Lyapunov stable, then the solution $x(t) = x_e$, $t \geq 0$, of the closed-loop system [\(67\)](#) is semistable and

$$J(x_0, \phi(x(\cdot))) = V(x_0) - V(x_e). \quad (87)$$

Furthermore, the feedback control [\(85\)](#) minimizes $J(x_0, u(\cdot))$ in the sense that

$$J(x_0, \phi(x(\cdot))) = \min_{u(\cdot) \in \mathcal{S}(x_0)} J(x_0, u(\cdot)). \quad (88)$$

Proof. The result follows as a consequence of [Theorem 4.1](#) with $L(x, u) = (y - y_e)^T (y - y_e) + (u - u_e)^T R_2(x)(u - u_e)$, $\mathcal{D} = \mathbb{R}^n$, and $U = \mathbb{R}^m$. Specifically, the feedback control law [\(85\)](#) follows from [Eq. \(78\)](#) by setting

$$\frac{\partial}{\partial u} \left[(y - y_e)^T (y - y_e) + (u - u_e)^T R_2(x)(u - u_e) + V'(x)(f(x) + G(x)u) \right] = 0. \quad (89)$$

Now, [Eq. \(84\)](#) is equivalent to [Eq. \(70\)](#) with $\phi(x)$ given by [Eq. \(85\)](#).

Next, since $(y - y_e)^T (y - y_e) \geq 0$, $(y, y_e) \in \mathbb{R}^l \times \mathbb{R}^l$, and $V'(x)G(x)R_2^{-1}(x)G^T(x)V'^T(x) \geq 0$, $x \in \mathbb{R}^n$, Eq. (84) implies that

$$\begin{aligned} 0 &\geq V'(x)f(x) - \frac{1}{4}V'(x)G(x)R_2^{-1}(x)G^T(x)V'^T(x) + V'(x)G(x)u_e \\ &\geq V'(x)f(x) - \frac{1}{2}V'(x)G(x)R_2^{-1}(x)G^T(x)V'^T(x) + V'(x)G(x)u_e, \quad (x, u_e) \in \mathbb{R}^n \times \mathbb{R}^m. \end{aligned} \tag{90}$$

Let \mathcal{Q} be an open neighborhood of $\{x \in \mathbb{R}^n : f(x) - \frac{1}{2}G(x)R_2^{-1}(x)G^T(x)V'^T(x) + V'(x)G(x)u_e = 0\}$ and note that Eq. (90) implies

$$V'(x)f(x) - \frac{1}{2}V'(x)G(x)R_2^{-1}(x)G^T(x)V'^T(x) + V'(x)G(x)u_e \leq 0, \quad (x, u_e) \in \mathcal{Q} \times \mathbb{R}^m. \tag{91}$$

Now, Eq. (91) is equivalent to Eq. (69) with $\phi(x)$ given by Eq. (85), and hence, Eq. (84) implies Eq. (69) with $\phi(x)$ given by Eq. (85).

Next, Eqs. (83) and (85) imply Eq. (68) and, since

$$\begin{aligned} L(x, u) + V'(x)[f(x) + G(x)u] \\ &= L(x, u) + V'(x)[f(x) + G(x)u] - L(x, \phi(x)) - V'(x)[f(x) + G(x)\phi(x)] \\ &= [u - \phi(x)]^T R_2(x)[u - \phi(x)] \\ &\geq 0, \quad (x, u) \in \mathbb{R}^n \times \mathbb{R}^m, \end{aligned} \tag{92}$$

condition (71) is satisfied.

Finally, it follows from Eqs. (83) to (85) that

$$\begin{aligned} \{x \in \mathcal{Q} : V'(x)(f(x) + G(x)\phi(x)) = 0\} &= \{x \in \mathcal{N} : L(x, \phi(x)) = 0\} \\ &= \{x \in \mathcal{Q} : h(x) = y_e \text{ and } V'(x)G(x) = 0\} \\ &\subseteq F^{-1}(0), \end{aligned} \tag{93}$$

and, by assumption, every equilibrium point $x_e \in F^{-1}(0)$ of the closed-loop system (67) is Lyapunov stable. Since all of the conditions of Theorem 4.1 are satisfied, the result follows. \square

Remark 4.2. Theorem 4.1 requires the construction of a continuously differentiable $V(\cdot)$ and a state-feedback control law $\phi(\cdot)$ such that Eqs. (68)–(71) are satisfied. In contrast, Corollary 4.1 requires the construction of a continuously differentiable $V(\cdot)$ such that Eqs. (83) and (84) are satisfied and with the semistabilizing state-feedback control law $\phi(\cdot)$ explicitly given by Eq. (85).

Next, we consider the linear-quadratic regulator problem for semistabilization, that is, we seek controllers $u(\cdot)$ that minimize Eq. (4) and guarantee semistability of the linear system given by Eqs. (5) and (6). The feedback gain K that minimizes Eq. (4) and guarantees semistability of Eq. (5) can be characterized via a solution to a linear matrix inequality [10]. The following result provides a useful alternative in finding the optimal gain K via an algebraic Riccati equation.

Theorem 4.2. Consider the linear controlled dynamical system \mathcal{G} given by Eqs. (5) and (6) with quadratic performance measure (4), assume that the pair (A, B) is semicontrollable and the pair (A, C) is semiobservable, and let $P_{LS} = P_{LS}^T \geq 0$ be the least squares solution to the algebraic Riccati equation

$$0 = A^T P + PA + C^T C - PBR_2^{-1}B^T P. \tag{94}$$

Then, with $u = Kx = -R_2^{-1}B^T P_{LS}x$, the solution $x(t) = x_e$, $t \geq 0$, to Eq. (5) is globally

semistable,

$$J(x_0, K) = x_0^T \left[\int_0^\infty (\tilde{A}\tilde{A}^\#)^T e^{\tilde{A}t} (C^T C + K^T R_2 K) e^{\tilde{A}t} \tilde{A}\tilde{A}^\# dt \right] x_0, \tag{95}$$

where $\tilde{A} = A + BK$, and

$$J(x_0, K) = \min_{u(\cdot) \in \mathcal{S}(x_0)} J(x_0, u(\cdot)). \tag{96}$$

Proof. Let $F(x, u) = A(x - x_e) + B(u - u_e)$, $L(x, u) = (x - x_e)^T C^T C(x - x_e) + (u - u_e)^T R_2(u - u_e)$, $V(x) = (x - x_e)^T \hat{P}(x - x_e)$, $\hat{P} = \hat{P}^T \geq 0$, $\mathcal{Q} = \mathcal{D} = \mathbb{R}^n$, and $U = \mathbb{R}^m$, and note that Eq. (70) specializes to

$$(x - x_e)^T C^T C(x - x_e) + (u - u_e)^T R_2(u - u_e) + 2(x - x_e)^T \hat{P}[A(x - x_e) + B(u - u_e)] = 0. \tag{97}$$

Hence, $V'(x)F(x, \phi(x)) \leq 0$ for all $x \in \mathbb{R}^n$. Next, note that

$$\begin{aligned} L(x, u) + V'(x)F(x, u) &= L(x, u) + V'(x)F(x, u) - [L(x, \phi(x)) + V'(x)F(x, \phi(x))] \\ &= [u - \phi(x)]^T R_2 [u - \phi(x)] \\ &\geq 0, \quad x \in \mathbb{R}^n, \end{aligned} \tag{98}$$

so that conditions (69)–(71) of Theorem 4.1 hold.

Next, it follows from Eqs. (78) and (97) that $u = -R_2^{-1} B^T \hat{P}x = Kx$, and hence,

$$\begin{aligned} V'(x)F(x, \phi(x)) &= 2(x - x_e)^T \hat{P}(A + BK)(x - x_e) \\ &= (x - x_e)^T [(A + BK)^T \hat{P} + \hat{P}(A + BK)](x - x_e) \\ &= -(x - x_e)^T (C^T C + K^T R_2 K)(x - x_e). \end{aligned} \tag{99}$$

Now, note that Eq. (46) is equivalent to Eq. (94) with $K = -R_2^{-1} B^T P$ and, since semiobservability is preserved under full state-feedback [10], it follows that if (A, C) is semiobservable, then (\tilde{A}, \tilde{R}) is semiobservable, where $\tilde{R} \triangleq C^T C + K^T R_2 K$. Since (\tilde{A}, \tilde{R}) is semiobservable, it follows from *ii*) of Lemma 2.7 that $\mathcal{R}(\tilde{A}) \cap \mathcal{N}(\tilde{A}) = \{0\}$, which implies that \tilde{A} is group invertible [16, p. 119]. Thus, defining $L = I - \tilde{A}\tilde{A}^\#$ and considering the Lyapunov function candidate $\hat{V}(\hat{x}) = \hat{x}^T (\hat{P} + L^T L)\hat{x}$, where $\hat{x} \triangleq x - x_e$, global semistability follows as in the proof of Theorem 3.2. Now, it follows from Theorem 2.6 that the least squares solution P_{LS} of Eq. (94) is given by Eq. (47), and hence, taking $\hat{P} = P_{LS}$, Eq. (95) directly follows from Eq. (72). Finally, Eq. (96) is a restatement of Eq. (73). \square

Remark 4.3. It is important to note that unlike Theorem 4.1 and Corollary 4.1, in Theorem 4.2 we do not require the assumption that $u(\cdot) \in \mathcal{S}(x_0)$. Rather Lyapunov stability, and hence, boundedness of solutions of the closed-loop system follow from the hypothesis of the theorem.

Proposition 4.1. Consider the controlled linear dynamical system \mathcal{G} given by Eqs. (5) and (6) with quadratic performance measure (4), assume that the pair (A, B) is semicontrollable and the pair (A, C) is semiobservable, and let $P_{LS} = P_{LS}^T \geq 0$ be the least squares solution to Eq. (94). Then, with $u = Kx = -R_2^{-1} B^T P_{LS}x$, the equilibrium solution $x(t) \equiv x_e$ to Eq. (5) is globally semistable and Eq. (101) holds. Furthermore, Eq. (96) holds.

Proof. Since (A, B) is semicontrollable and (A, C) is semiobservable, the conditions of Theorem 3.7 of [20] are satisfied, and hence, there exists a $n \times n$ matrix $P = P^T \geq 0$ such that Eq. (94) holds. Let $P_{LS} = \arg \min_{P \in \mathcal{P}} \|P\|_F$ be the least squares solution of Eq. (94), where \mathcal{P} denotes the set of all P satisfying (94). Now, noting that, with $K = -R_2^{-1} B^T P$, Eq. (46) is equivalent to Eq. (94), it follows from Theorems 2.6 and 4.2 that Eq. (5), with $u = -R_2^{-1} B^T P_{LS}x$ and P_{LS}

given by Eq. (47), is globally semistable and

$$J(x_0, K) = x_0^T P_{LS} x_0 \leq J(x_0, u(\cdot)), \tag{100}$$

where $K = -R_2^{-1} B^T P_{LS}$. \square

Proposition 4.2. Consider the controlled linear dynamical system \mathcal{G} given by Eqs. (5) and (6) with quadratic performance measure (4), assume the pair (A, B) is semistabilizable and the pair (A, C) is semidetactable, and assume that there exists $P = P^T \geq 0$ such that Eq. (94) holds. Then, with $u = Kx = -R_2^{-1} B^T P x$, the equilibrium solution $x(t) \equiv x_e$ to Eq. (5) is globally semistable and

$$J(x_0, K) = x_0^T \left[\int_0^\infty e^{\tilde{A}^T t} (C^T C + K^T R_2 K) e^{\tilde{A} t} dt \right] x_0, \tag{101}$$

where $\tilde{A} = A + BK$, and

$$\min_{u(\cdot) \in \mathcal{S}(x_0)} J(x_0, u(\cdot)) = J(x_0, K_*) \leq J(x_0, K) = 2x_0^T \tilde{A} \tilde{A}^\# x_0, \tag{102}$$

where $K_* = -R_2^{-1} B^T P_{LS}$ and $P_{LS} = P_{LS}^T \geq 0$ is the least squares solution to Eq. (94).

Proof. Global semistability of Eq. (5), with $u = -R_2^{-1} B^T P x$, and Eq. (101) follow directly from Theorem 2.7. To show Eq. (102), note that it follows from Eqs. (101) and (94) that

$$\begin{aligned} J(x_0, K) &= -x_0^T \left[\int_0^\infty e^{\tilde{A}^T t} (\tilde{A}^T P + P \tilde{A}) e^{\tilde{A} t} dt \right] x_0 \\ &= -x_0^T \left[e^{\tilde{A}^T t} P \Big|_{t=0}^\infty + P e^{\tilde{A} t} \Big|_{t=0}^\infty \right] x_0 \\ &= x_0^T \left[(\tilde{A} \tilde{A}^\#)^T P + P \tilde{A} \tilde{A}^\# \right] x_0. \end{aligned} \tag{103}$$

Since, by Theorem 2.5, $P = P_{LS} + \alpha z z^T$, where $\alpha \geq 0$ and $z \in \mathcal{N}(\tilde{A}^T)$ satisfies Eq. (44), it follows from Eq. (103) that

$$J(x_0, K) = x_0^T \left[(\tilde{A} \tilde{A}^\#)^T P_{LS} + P_{LS} \tilde{A} \tilde{A}^\# \right] x_0 = 2x_0^T P_{LS} \tilde{A} \tilde{A}^\# x_0.$$

Finally, with $F(x, u) = A(x - x_e) + B(u - u_e)$, $L(x, u) = (x - x_e)^T C^T C (x - x_e) + (u - u_e)^T R_2 (u - u_e)$, $V(x) = (x - x_e)^T P_{LS} (x - x_e)$, $Q = \mathcal{D} = \mathbb{R}^n$, and $U = \mathbb{R}^m$, it follows using similar arguments as in the proof of Theorem 4.2 that $J(x_0, K_*) = \min_{u(\cdot) \in \mathcal{S}(x_0)} J(x_0, u(\cdot))$. Hence, (102) holds. \square

Definition 4.1. A nonnegative-definite matrix $P \in \mathbb{R}^{n \times n}$ is a *semistabilizing solution* of Eq. (94) if $A - BR_2^{-1} B^T P$ is semistable. Furthermore, a semistabilizing solution P_{\min} of Eq. (94) is the *minimally semistabilizing solution* to Eq. (94) if $P \geq P_{\min}$ for every semistabilizing solution P to Eq. (94).

It follows from Definition 4.1 that the least squares solution P_{LS} to Eq. (94) is the minimally semistabilizing solution to Eq. (94). Given the linear dynamical system given by Eqs. (5) and (6), if the pair (A, B) is semicontrollable and the pair (A, C) is semiobservable, then it follows from Lemma 2.3 that, for every $K \in \mathbb{R}^{m \times n}$, the pair $(A + BK, B)$ is semicontrollable, and by Proposition 2.1 in [10], it follows that, for every $R_2 \in \mathbb{R}^{n \times n}$ such that $R_2 = R_2^T > 0$, the pair $(A + BK, C^T C + K^T R_2 K)$ is semiobservable. Furthermore, if the pair (A, C) is semiobservable, then (A, C) is semidetactable and it follows from Theorems 2.3 and 2.5 that every solution $P = P^T \geq 0$ of Eq. (46)

is given by

$$P = \int_0^\infty e^{\tilde{A}^T t} (C^T C + K^T R_2 K) e^{\tilde{A} t} dt + z z^T, \tag{104}$$

where $\tilde{A} = A + BK$ and $z \in \mathcal{N}(\tilde{A}^T)$. Now, if $K = -R_2^{-1} B^T P$, then Eq. (46) is equivalent to Eq. (94), where P can be computed using the Schur decomposition of the Hamiltonian matrix [13, pp. 853–859], and the least squares solution $P_{LS} = P_{LS}^T \geq 0$ of Eq. (94) is given by $P_{LS} = P - z z^T$, where z is the solution of the optimization problem

$$\min_{z \in \mathbb{R}^n} \|P - z z^T\|_F \tag{105}$$

subject to

$$0 \leq P - z z^T, \tag{106}$$

$$0 = (A^T - P B R_2^{-1} B^T) z. \tag{107}$$

One might surmise that Theorem 4.2 and Proposition 4.1 give different values for $J(x_0, K)$. However, note that

$$\begin{aligned} J(x_0, u(\cdot)) &= \int_0^\infty [(x(t) - x_e)^T C^T C (x(t) - x_e) + (u(t) - u_e)^T R_2 (u(t) - u_e)] dt \\ &= \int_0^\infty [(x_0 - x_e)^T e^{\tilde{A}^T t} (C^T C + K^T R_2 K) e^{\tilde{A} t} (x_0 - x_e)] dt \\ &= x_0^T \left[\int_0^\infty (\tilde{A} \tilde{A}^\#)^T e^{\tilde{A}^T t} (C^T C + K^T R_2 K) e^{\tilde{A} t} \tilde{A} \tilde{A}^\# dt \right] x_0 \end{aligned} \tag{108}$$

and, since

$$\begin{aligned} J(x_0, u(\cdot)) &= \int_0^\infty [(x(t) - x_e)^T C^T C (x(t) - x_e) + (u(t) - u_e)^T R_2 (u(t) - u_e)] dt \\ &= \int_0^\infty x^T(t) (C^T C + K^T R_2 K) x(t) dt, \end{aligned} \tag{109}$$

it follows that

$$\begin{aligned} J(x_0, u(\cdot)) &= \int_0^\infty [(x(t) - x_e)^T C^T C (x(t) - x_e) + (u(t) - u_e)^T R_2 (u(t) - u_e)] dt \\ &= \int_0^\infty x^T(t) (C^T C + K^T R_2 K) x(t) dt \\ &= x_0^T \int_0^\infty e^{\tilde{A}^T t} (C^T C + K^T R_2 K) e^{\tilde{A} t} dt x_0. \end{aligned} \tag{110}$$

Hence, Eqs. (95) and (101) are equivalent.

Finally, in light of Theorem 4.2 and Lemma 4.3 of [11] the following result is immediate.

Proposition 4.3. Consider the linear controlled dynamical system \mathcal{G} given by Eqs. (5) and (6). If the pair (A, B) is semicontrollable, the pair (A, C) is semiobservable, and \mathcal{G} , with $u = Kx$, is semistable, then

$$P = \int_0^\infty (\tilde{A} \tilde{A}^\#)^T e^{\tilde{A}^T t} (C^T C + K^T R_2 K) e^{\tilde{A} t} \tilde{A} \tilde{A}^\# dt \tag{111}$$

satisfies

$$0 = \tilde{A}^T (\tilde{A}^T P + P\tilde{A} + C^T C + K^T R_2 K) \tilde{A}, \tag{112}$$

or, equivalently, Eq. (46).

5. Illustrative numerical examples

In this section, we provide two numerical examples to highlight the optimal semistabilization framework developed in this paper.

5.1. Optimal consensus control for multiagent formations

For the first example, we use the optimal semistabilization framework to design consensus controllers for multiagent networks of single integrator systems. Specifically, the consensus problem involves the design of a dynamic protocol algorithm that guarantees semistability and system state equipartition [21], that is, $\lim_{t \rightarrow \infty} x_i(t) = \alpha \in \mathbb{R}$ for $i = 1, \dots, n$, where $x_i(t)$ denotes the i th component of the system state vector $x(t)$. To address the consensus problem of n agents exchanging information with collective dynamics given by Eqs. (5) and (6), we set the entries a_{ij} , $i, j = 1, \dots, n$, of the system matrix A such that, if agent j receives information from the agent i , $i \neq j$, then $a_{ij} = 1$, otherwise $a_{ij} = 0$, and $a_{ii} = -\sum_{j=1, j \neq i}^n a_{ij}$.

Here, we design a control law $u = Kx$ such that Eq. (5) with $u = Kx$ is semistable, the performance measure (4) is minimized in the sense of Eq. (73), and

$$x_e = \lim_{t \rightarrow \infty} x(t) = \alpha e, \tag{113}$$

where $e \triangleq [1, \dots, 1]^T$ and $\alpha \in \mathbb{R} \setminus \{0\}$ [22]. In order to account for the constraint (113), we introduce a terminal steady state constraint to the performance measure (4) so that

$$J(x_0, u(\cdot)) = \lim_{\tau \rightarrow \infty} \left\{ \mu^T(x(\tau) - \alpha e) + \int_0^\tau [(x(t) - x_e)^T C^T C(x(t) - x_e) + (u(t) - u_e)^T R_2 (u(t) - u_e)] dt \right\}, \tag{114}$$

where $\mu \in \mathbb{R}^n$, is minimized in the sense of Eq. (73). This optimization problem is in the form of a Bolza problem [23, Chapter 2], whereas the optimization problems discussed in Section 3 are in the form of Lagrange problems.

To account for the terminal consensus constraint, we introduce the additional scalar state $x_{n+1} : [0, \infty) \rightarrow \mathbb{R}$ and define $\hat{x} \triangleq [x^T, x_{n+1}]^T$ so that

$$\dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t), \quad \hat{x}(0) = \begin{bmatrix} x_0 \\ \lim_{\tau \rightarrow \infty} \frac{\mu^T(x(\tau) - \alpha e)}{\tau} \end{bmatrix}, \quad t \geq 0, \tag{115}$$

$$y(t) = \hat{C}\hat{x}(t), \tag{116}$$

where

$$\hat{A} \triangleq \begin{bmatrix} A & 0_n \\ 0_n^T & 0 \end{bmatrix}, \quad \hat{B} \triangleq \begin{bmatrix} B \\ 0_m^T \end{bmatrix}, \quad \hat{C} \triangleq [C \ 0_n],$$

and 0_n denotes the n -dimensional zero vector. In this case, the performance measure (114) can be rewritten as

$$J(x_0, u(\cdot)) = \int_0^\infty [(\hat{x}(t) - \hat{x}_e)^T \hat{C}^T \hat{C} (\hat{x}(t) - \hat{x}_e) + (u(t) - u_e)^T R_2 (u(t) - u_e)] dt, \tag{117}$$

where \hat{x}_e is an equilibrium point of Eq. (115) for some $u_e \in \mathbb{R}^m$. Note that if the pair (A, B) is semistabilizable and the pair (A, C) is semidetectable, then it follows from Definitions 2.2 and 2.3 that the pair (\hat{A}, \hat{B}) is semistabilizable and the pair (\hat{A}, \hat{C}) is semidetectable. Hence, it follows from Theorem 4.2 that the solution $\hat{x}(t) = \hat{x}_e, t \geq 0$, to Eq. (115) with $u = K\hat{x}$ and $K = -R_2^{-1} \hat{B}^T \hat{P}_{LS}$ is globally semistable, where \hat{P}_{LS} is the least squares solution of

$$0 = \hat{A}^T \hat{P} + \hat{P} \hat{A} + \hat{C}^T \hat{C} - \hat{P} \hat{B} R_2^{-1} \hat{B}^T \hat{P}, \tag{118}$$

and Eqs. (95) and (96) hold with $\tilde{A} = \hat{A} + \hat{B}K$.

Next, define $\hat{\mu} \triangleq [\mu^T, 0]^T$ and note that if $\hat{x}(t) = [x^T(t), x_{n+1}(t)]^T, t \geq 0$, is the solution of Eq. (115) with $u = K\hat{x}$, then

$$\lim_{\tau \rightarrow \infty} \frac{\mu^T(x(\tau) - \alpha e)}{\tau} = \mu^T \lim_{\tau \rightarrow \infty} \frac{x(\tau)}{\tau} = \hat{\mu}^T \lim_{\tau \rightarrow \infty} \frac{e^{\tilde{A}\tau} \hat{x}(0)}{\tau} = \hat{\mu}^T \tilde{A} \lim_{\tau \rightarrow \infty} e^{\tilde{A}\tau} \hat{x}(0). \tag{119}$$

Now, it follows from Proposition 11.8.1 of [13] that

$$\lim_{\tau \rightarrow \infty} \frac{\mu^T(x(\tau) - \alpha e)}{\tau} = \hat{\mu}^T \tilde{A} \lim_{\tau \rightarrow \infty} e^{\tilde{A}\tau} \hat{x}(0) = \hat{\mu}^T \tilde{A} (I_{n+1} - \tilde{A} \tilde{A}^\#) \hat{x}(0) = \hat{\mu}^T (\tilde{A} - \tilde{A} \tilde{A} \tilde{A}^\#) \hat{x}(0) = 0, \tag{120}$$

and hence, the system given by Eqs. (115) and (116) with $u = K\hat{x}$ is equivalent to

$$\dot{\hat{x}}(t) = \tilde{A} \hat{x}(t), \quad \hat{x}(0) = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}, \quad t \geq 0, \tag{121}$$

$$y(t) = \hat{C} \hat{x}(t). \tag{122}$$

For our simulation, we consider five agents so that

$$A = \begin{bmatrix} -2 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 1 & 1 & -4 & 1 & 1 \\ 0 & 1 & 1 & -2 & 0 \\ 1 & 1 & 0 & 0 & -2 \end{bmatrix}, \quad B = I_5, \tag{123}$$

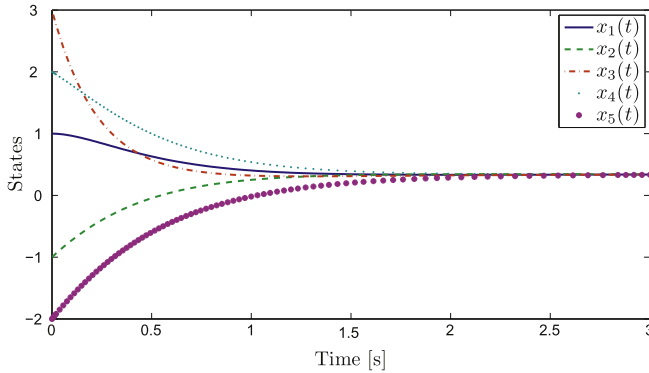


Fig. 1. State trajectories of the closed-loop system.

and set

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad R_2 = I_5. \tag{124}$$

Note that the pair (A, B) is controllable, and hence, semistabilizable, and the pair (A, C) is semidetactable but not observable. In this case, the least squares solution of Eq. (118) is given by

$$\hat{P}_{LS} = \begin{bmatrix} 0.1963 & -0.0513 & 0.0115 & -0.0646 & -0.0919 & 0 \\ -0.0513 & 0.2261 & -0.0082 & 0.0360 & -0.2024 & 0 \\ 0.0115 & -0.0082 & 0.1320 & 0.0417 & -0.1770 & 0 \\ -0.0646 & 0.0360 & 0.0417 & 0.2533 & -0.2663 & 0 \\ -0.0919 & -0.2024 & -0.1770 & -0.2663 & 0.7376 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{125}$$

For $x_0 = [1, -1, 3, 2, -2]^T$ the trajectories of the closed-loop system are shown in Fig. 1.

5.2. Rotational/translational proof-mass actuator

Consider the mechanical system adopted from [3] shown in Fig. 2 involving an eccentric rotational inertia on a translational oscillator giving rise to nonlinear coupling between the undamped oscillator and the rotational rigid body mode. The oscillator cart of mass M is connected to a fixed support via a linear spring of stiffness k . The cart is constrained to one-dimensional motion and the rotational proof-mass actuator consists of a mass m and mass moment of inertia I located at a distance e from the cart's center of mass.

Letting $q, \dot{q}, \theta, \dot{\theta}, u_1,$ and u_2 denote the translational position and velocity of the cart, the angular position and velocity of the rotational proof mass, and the force acting on the cart and the moment acting on the rotating mass, respectively, the dynamic equations of motion are given by

$$(M + m)\ddot{q}(t) + me \left[\ddot{\theta}(t) \cos \theta(t) - \dot{\theta}^2(t) \sin \theta(t) \right] + kq(t) = u_1(t), \tag{126}$$

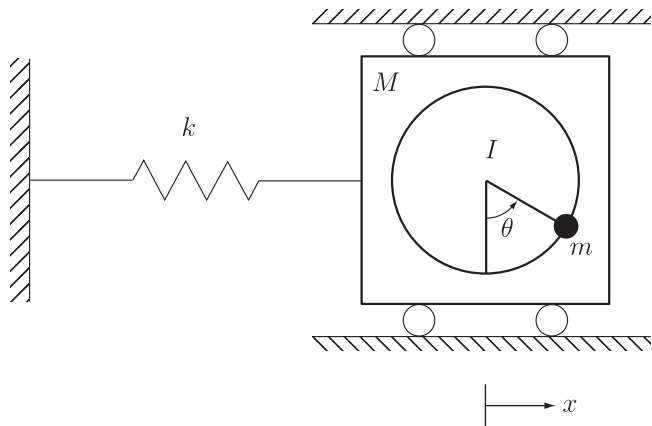


Fig. 2. Rotational/translational proof-mass actuator.

$$(I + me^2)\ddot{\theta}(t) + me\ddot{q}(t) \cos \theta(t) = u_2(t), \tag{127}$$

where $t \geq 0$, $q(0) = q_0$, $\dot{q}(0) = \dot{q}_0$, $\theta(0) = \theta_0$, and $\dot{\theta}(0) = \dot{\theta}_0$.

For this example, we seek a state feedback controller $u = [u_1, u_2]^T = \phi(x)$, where $x = [q, \dot{q}, \dot{\theta}, \theta]^T$, such that the performance measure

$$J(x(0), u(\cdot)) = \int_0^\infty [(x(t) - x_e)^T R_1 (x(t) - x_e) + (u(t) - u_e)^T (u(t) - u_e)] dt, \tag{128}$$

where

$$R_1 = \frac{1}{4} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

is minimized in the sense of Eq. (73), and Eqs. (126) and (127) are semistable.

Next, note that Eqs. (126) and (127) with performance measure (128) can be cast in the form of Eq. (79) with performance measure (82). In this case, Corollary 4.1 can be applied with $n=4$, $m=2$, $l=4$, $y = Cx$, $R_1 = C^T C$, and $R_2(x) = I_4$, $x \in \mathbb{R}^4$, to characterize the optimal semistabilizing controllers. The explicit expression of $f(x) + G(x)u$ is omitted for brevity. Specifically, Eq. (84) specializes to

$$0 = (x - x_e)^T R_1 (x - x_e) + V'(x)f(x) + V'(x)G(x)u_e - \frac{1}{4}V'(x)G(x)G^T(x)V'^T(x), \tag{129}$$

which is satisfied by

$$V(x) = \frac{1}{2}(x - x_e)^T P(x)(x - x_e), \quad x \in \mathbb{R}^4, \tag{130}$$

where

$$P(x) \triangleq \begin{bmatrix} k & 0 & 0 & 0 \\ 0 & M + m & me \cos \theta & 0 \\ 0 & me \cos \theta & I + me^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{131}$$

In this case, Eq. (85) specializes to

$$\phi(x) = -\frac{1}{2} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} (x - x_e), \quad x \in \mathbb{R}^4. \tag{132}$$

Note that the state feedback control law (132) is equivalent to a virtual damper applied to the translational mass M and the rotational mass m .

Finally, to show boundedness of solutions of the closed-loop system (126) and (127) with $u = \phi(x)$ given by Eq. (132), note that the largest invariant set of $\mathcal{M} = \{x \in \mathbb{R}^4 : V'(x)[f(x) + G(x)\phi(x)] = 0\}$ is $\mathcal{Z} \triangleq \{(0, 0, 0, \theta), \theta \in \mathbb{R}\}$. Now, Lyapunov stability of $x_e = [0, 0, 0, \theta_e]^T \in \mathcal{Z}$ for every $\theta_e \in \mathbb{R}$ follows from Theorem 2 of [24] by noting that $V(x_e) = 0$, $V(x) \geq 0, x \in \mathbb{R}^4$, $V'(x)[f(x) + G(x)\phi(x)] = -\dot{q}^2 - \dot{\theta}^2 \leq 0, x \in \mathbb{R}^4$, $[\dot{q}(t), \dot{\theta}(t)]^T = [0, 0]^T, t \in \mathbb{R}$, if and only if $[q(t), \theta(t)]^T = [q_e, \theta_e]^T, q_e \in \mathbb{R}$, and $x(t) \equiv \hat{x}_e \triangleq [q_e, 0, 0, \theta_e]^T \in \mathcal{M}, t < 0$, if and only if $q_e = 0$. Hence, it follows from Corollary 4.1 that the solution $x(t) \equiv x_e, t \geq 0$, is semistable.

Let $M = 2 \text{ kg}$, $m = 1 \text{ kg}$, $e = 0.2 \text{ m}$, $k = 10 \text{ N/m}$, $I = 4 \text{ kg} \cdot \text{m}^2$, $q_0 = 1 \text{ m}$, $\dot{q}_0 = 0 \text{ m/s}$, $\theta_0 = \pi/2$, and $\dot{\theta}_0 = 2 \text{ Hz}$. Fig. 3 shows the state trajectories of the controlled system versus

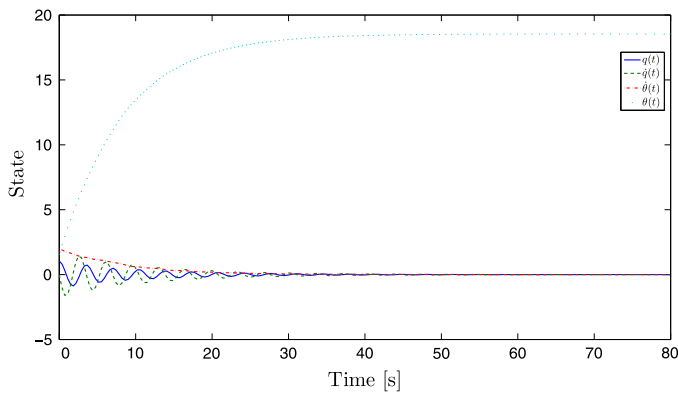


Fig. 3. Closed-loop system trajectories versus time.

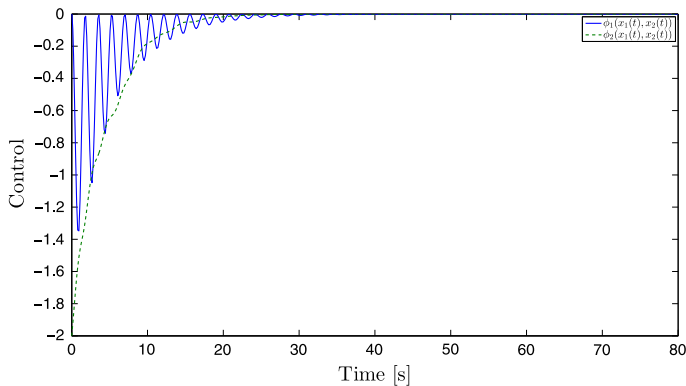


Fig. 4. Control signal versus time.

time. Fig. 4 shows the control signal versus time. Finally,

$$J(x(0), \phi(x(\cdot))) = \frac{1}{2}(x(0) - x_e)^T P(x(0) - x_e)(x(0) - x_e) = 26.16 \text{ N} \cdot \text{m} \quad (133)$$

and $\theta_e = 18.5407$.

6. Conclusion

In this paper, we presented an optimal control framework for addressing optimal linear and nonlinear semistabilizing controllers with quadratic and nonlinear-nonquadratic cost functionals. Specifically, we considered dynamical systems on the infinite interval and utilized a steady-state Hamilton–Jacobi–Bellman-type approach to characterize optimal nonlinear feedback controllers that guarantee Lyapunov stability and convergence for closed-loop systems having a continuum of equilibria. The proposed semistabilization framework was then used to design optimal controllers for consensus protocols for multiagent systems. Extensions of this work for addressing linear and nonlinear semistable singular control problems are currently under development.

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