Available online at www.sciencedirect.com

ScienceDirect

Journal of the Franklin Institute 354 (2017) 7385–7409

www.elsevier.com/locate/jfranklin

Robust observer-based control of nonlinear dynamical systems with state constraints

Andrea L'Afflitto*, Keyvan Mohammadi

School of Aerospace and Mechanical Engineering, The University of Oklahoma, Norman, OK, 73019-0390, USA

Received 8 March 2017; received in revised form 4 August 2017; accepted 12 September 2017
Available online 20 September 2017

Abstract

In this paper, we design a variable structure observer-based control system that guarantees asymptotic convergence of the plant's trajectory to the equilibrium point despite matched and unmatched uncertainties in the plant dynamics. Our control laws are functions of the estimated plant state and the proposed framework allows employing any estimator or observer, such as the Walcott and Żak observer, as long as the estimated state converges asymptotically to the plant state. *Barrier Lyapunov functions* guarantee that the closed-loop system's trajectory verifies the state constraints. This study is the first of its kind, since recently variable structure control architectures have been adapted to account for constraints on the state space or allow output-feedback, but observer-based variable structure control in the presence of state constraints has not been attempted before. A numerical simulation involving the roll dynamics of an unstable aircraft, whose aerodynamic coefficients are unknown, illustrates our theoretical framework.

1. Introduction

In most cases of practical interest, it is difficult to accurately model dynamical systems and estimate the parameters characterizing such models. Additional complexity in the problem of controlling nonlinear systems is given by the fact that it may not be possible to directly measure all the components of the system's state. Lastly, it is often needed to guarantee that the system's trajectory is confined to a constraint set at all times. In this paper, we present

* Corresponding author.
E-mail addresses: a.lafflitto@ou.edu (A. L'Afflitto), Keyvan.Mohammadi@ou.edu (K. Mohammadi).

https://doi.org/10.1016/j.jfranklin.2017.09.007
0016-0032/

a novel nonlinear observer-based control architecture, which guarantees that the system's trajectory converges asymptotically to an equilibrium point and is constrained to a simply connected open set.

Notorious robust control techniques for nonlinear dynamical systems are adaptive control [1,2] and sliding mode control [3,4,5, Ch. 7,6,7, Ch. 14]. The sliding mode control architecture, which was first devised in 1960s by Emel'yanov and Barbashin [4], consists in steering in finite-time the system's trajectory to a *sliding manifold*, which is designed so that if the system's trajectory reaches this manifold, then the system state asymptotically converges to zero. The control law that drives the system trajectory to the sliding manifold involves the signum function, and hence is discontinuous. It is well known that solutions of ordinary differential equations with discontinuous right-hand sides may not exist or may not be unique [8,9, Ch. 2]. Furthermore, in most cases of practical interest discontinuous control inputs induce an undesired effect known as *chattering*, which consists in high-frequency oscillations of the system's state about the sliding manifold [6,10, Ch. 14]. Despite the theoretical and practical challenges concerning sliding mode control, this technique has drawn considerable interest in aerospace [11], chemical [12], electrical [13], marine [14], and mechanical engineering [15] for its ease of implementation [16] and ability to compensate for disturbances and uncertainties [17]; for further details, see [18,19] and the numerous references therein.

The problem of designing sliding mode control laws, which account for constraints in the state space, received relatively less attention [20] and has been addressed for first-order sliding mode [21], second-order sliding mode [22], third-order sliding mode with box constraints [23], and within the context of model predictive control [24]. The classical sliding mode architecture has also been modified to allow observer-based and output-feedback control [25–28]. It is worthwhile to recall also both [29], where an output-feedback sliding mode control architecture is proposed for uncertain stochastic systems, and [30], where sensor faults are accounted for. However, to the authors' best knowledge, an observer-based sliding mode control architecture in the presence of state constraints has not been proposed before. Alternative approaches to the problem of designing control laws that guarantee some bounds on the tracking error involve the adaptive control framework [31], where single-input-single-output dynamical systems are considered, and the backstepping framework [32], where triangular systems are considered.

In this paper, we design robust, observer-based feedback control laws for nonlinear time-varying dynamical systems affected by matched and unmatched uncertainties and subject to state constraints on the closed-loop system's trajectory. A unique feature of this work is that the feedback control laws presented are functions of the estimated plant state, which is reconstructed by a dynamic observer using the information provided by the measured output. To meet our design goal, firstly we prove sufficient conditions for the closed-loop system's trajectory to converge to the sliding manifold in finite-time and asymptotically to the equilibrium point along the sliding manifold in the presence of constraints on the plant state. Successively, the effectiveness of our observer-based feedback controls is shown by proving that if a state-feedback control guarantees convergence of the plant trajectory and the estimated state verifies the constraints on the plant state and asymptotically converges to the actual plant state, then the feedback control law obtained by accounting for the *estimated* state guarantees convergence of the closed-loop system to the equilibrium point. *Barrier Lyapunov functions* [33,34] are employed to certify that the constraints on the plant state are verified and it is worthwhile to recall the recent publications [35], where time-varying barrier Lyapunov functions are used within the context of backstepping control, and [36,37], where barrier Lyapunov functions are used within the context of adaptive control.

Although the framework provided in this paper does not depend on a *specific* plant observer, we show how a popular nonlinear robust observer, that is, the Walcott and Żak observer [38–44], can be used within our observer-based feedback control framework. Alternative observers that can be employed within our framework are presented, for instance, in [45–49].

This paper is organized in two parts. Firstly, we prove sufficient conditions for strong uniform finite-time stability and uniform asymptotic stability of time-varying dynamical systems in the presence of state constraints. Successively, we prove sufficient conditions for asymptotic and finite-time convergence of dynamical systems, whose feedback controls account for the system's estimated state. A numerical example illustrates the applicability of our robust nonlinear observer-based feedback control architecture. Specifically, we consider the problem of stabilizing the roll dynamics of a delta-wing aircraft, whose aerodynamic coefficients are unknown. The proofs of our main results are provided in the Appendix.

2. Notation, definitions, and mathematical preliminaries

In this section, we establish notation, definitions, and review some preliminary results. Let $\mathbb{R}_+$ denote the set of positive real numbers, $\overline{\mathbb{R}}_+$ denote the set of nonnegative real numbers, $\mathbb{R}^n$ denote the set of $n\times 1$ real column vectors, $\mathbb{R}^{n\times m}$ denote the set of $n\times m$ real matrices, and $\mathcal{B}_\varepsilon(x)$ denote the *open ball centered* at x with *radius* ε. We write $\|\cdot\|$ both for the Euclidean vector norm and the corresponding equi-induced matrix norm, $\|\cdot\|_\infty$ both for the infinity vector norm and the corresponding equi-induced matrix norm, I_n or I for the $n\times n$ identity matrix, $0_{n\times m}$ or 0 for the zero $n\times m$ matrix, and A^{T} for the transpose of the matrix A.

Consider the nonlinear dynamical system given by

$$\dot{x}(t)=f(t,x(t)),\qquad x(t_0)=x_0,\qquad t\in\mathcal{I}_{t_0,x_0},\tag{1}$$

where, for every $t\in\mathcal{I}_{t_0,x_0}$, $x(t)\in\mathcal{D}\subseteq\mathbb{R}^n$, $\mathcal{I}_{t_0,x_0}\subseteq[t_0,\infty)$ is the maximal interval of existence of a solution $x(t)$ of (1), $\mathcal{D}$ is an open set with $0\in\mathcal{D}$, and $f:\mathcal{I}_{t_0,x_0}\times\mathcal{D}\to\mathbb{R}^n$ is such that, for every $(t,x)\in\mathcal{I}_{t_0,x_0}\times\mathcal{D}$, $f(t,0)=0$ and $f(\cdot,\ \cdot)$ is continuous in t and x. A continuously differentiable function $x:\mathcal{I}_{t_0,x_0}\to\mathcal{D}$ is said to be the *solution* of (1) on the interval $\mathcal{I}_{t_0,x_0}\subset\mathbb{R}$ if $x(\cdot)$ satisfies Eq. (1) for all $t\in\mathcal{I}_{t_0,x_0}$. As shown in [50], it follows from Peano's theorem [8, Th. 2.24] that the joint continuity of $f(\cdot,\ \cdot)$ implies that, for every $x\in\mathcal{D}$, there exists $\tau_0<t_0<\tau_1$ and a solution $x(\cdot)$ of Eq. (1) defined on the open interval $(\tau_0,\ \tau_1)$ such that $x(t_0)=x$. A solution $t\mapsto x(t)$ is said to be *right maximally* defined if x cannot be extended (either uniquely or nonuniquely) forward in time. We assume that all right maximal solutions to Eq. (1) exist on $[t_0,\ \infty)$, and hence, we assume that Eq. (1) is *forward complete*.

We assume that Eq. (1) possesses unique solutions in forward time for all initial conditions except possibly the origin in the following sense. For every $x\in\mathcal{D}\backslash\{0\}$ there exists $\tau_x>t_0$ such that, if $y_1:[t_0,\tau_1)\to\mathcal{D}$ and $y_2:[t_0,\tau_2)\to\mathcal{D}$ are two solutions of Eq. (1) with $y_1(t_0)=y_2(t_0)=x$, then $\tau_x\le\min\{\tau_1,\ \tau_2\}$ and $y_1(t)=y_2(t)$ for all $t\in[t_0,\ \tau_x)$. Without loss of generality, we assume that for each x, τ_x is chosen to be the largest such number in $[t_0,\ \infty)$. In this case, we denote by the continuously differentiable map $s^{t_0,x_0}(\cdot)\triangleq s(\cdot,t_0,x_0)$ the *trajectory* or the unique *solution curve* of Eq. (1) on $\mathcal{I}_{t_0,x_0}$ satisfying $s(0,t_0,x_0)=x_0$. Sufficient conditions for forward uniqueness in the absence of Lipschitz continuity can be found in [51] [9, 52, Sec. 10], and [53, Sec. 1].

The following definition introduces the notion of finite-time stability for time-varying nonlinear dynamical systems, which plays a key role in this paper. This statement is a generalization of the definition of finite-time stability for nonautonomous systems provided in [50], as the notion of Lyapunov stability is provided in terms of open sets, and not open balls; the importance of defining finite-time stability in terms of open sets will become clear while analyzing the stability properties of constrained dynamical systems. For the statement of this definition, recall that if $s^{t_0,x_0}(t) \to 0$ as $t \to T$, where $s^{t_0,x_0}(\cdot)$ denotes the solution of Eq. (1), then for every $\hat{\varepsilon} > 0$, there exists $\hat{T}(\hat{\varepsilon}, t_0, x_0) \in (t_0, T)$, such that if $t \in (\hat{T}(\hat{\varepsilon}, t_0, x_0), T)$, then $\|s^{t_0,x_0}(t)\| < \hat{\varepsilon}$. Alternatively, if $s^{t_0,x_0}(t) \to 0$ as $t \to T$ *uniformly* in t_0, then for every $\hat{\varepsilon} > 0$, there exists $\hat{T}(\hat{\varepsilon}, x_0) \in (t_0, T)$, such that if $t \in (\hat{T}(\hat{\varepsilon}, x_0), T)$, then $\|s^{t_0,x_0}(t)\| < \hat{\varepsilon}$.

Definition 2.1. The nonlinear dynamical system (1) is *finite-time stable* if there exist an open neighborhood $\mathcal{D}_0 \subseteq \mathcal{D}$ of the origin and a function $T : [0, \infty) \times \mathcal{D}_0 \backslash \{0\} \to (t_0, \infty)$, called the *settling-time function*, such that the following statements hold:

(i) *Finite-time convergence.* For every $(t_0, x_0) \in [0, \infty) \times \mathcal{D}_0 \backslash \{0\}$, $s^{t_0,x_0}(t)$ is defined on $[t_0, T(t_0, x_0))$, $s^{t_0,x_0}(t) \in \mathcal{D}_0 \backslash \{0\}$ for all $t \in [t_0, T(t_0, x_0))$, and $s^{t_0,x_0}(t) \to 0$ as $t \to T(t_0, x_0)$.

(ii) *Lyapunov stability.* For every $t_0 \in [0, \infty)$ and every open set $\mathcal{N}_\varepsilon \subseteq \mathcal{D}_0$ containing $x = 0$, there exists an open set $\mathcal{N}_\delta \subseteq \mathcal{D}_0$ containing $x = 0$, such that for every $x_0 \in \mathcal{N}_\delta \backslash \{0\}$, $s^{t_0,x_0}(t) \in \mathcal{N}_\varepsilon$ for all $t \in [t_0, T(t_0, x_0))$.

The nonlinear dynamical system (1) is *uniformly finite-time stable* if Eq. (1) is finite-time stable and the following statement holds:

(iii) *Uniform Lyapunov stability.* For every open set $\mathcal{N}_\varepsilon \subseteq \mathcal{D}_0$ containing $x = 0$, there exists an open set $\mathcal{N}_\delta \subseteq \mathcal{D}_0$ containing $x = 0$, such that for every $x_0 \in \mathcal{N}_\delta \backslash \{0\}$, $s^{t_0,x_0}(t) \in \mathcal{N}_\varepsilon$ for all $t \in [t_0, T(t_0, x_0))$ and for all $t_0 \in [0, \infty)$.

The nonlinear dynamical system (1) is *strongly uniformly finite-time stable* if Eq. (1) is uniformly finite-time stable and the following statement holds:

(iv) *Uniform finite-time convergence.* For every $(t_0, x_0) \in [0, \infty) \times \mathcal{D}_0 \backslash \{0\}$, $s^{t_0,x_0}(t)$ is defined on $[t_0, T(t_0, x_0))$, $s^{t_0,x_0}(t) \in \mathcal{D}_0 \backslash \{0\}$ for all $t \in [t_0, T(t_0, x_0))$, and $s^{t_0,x_0}(t) \to 0$ as $t \to T(t_0, x_0)$ uniformly in t_0 for all $t_0 \in [0, \infty)$.

The nonlinear dynamical system (1) is *globally finite-time stable* (respectively, *globally uniformly finite-time stable* or *globally strongly uniformly finite-time stable*) if it is finite-time stable (respectively, uniformly finite-time stable or strongly uniformly finite-time stable) with $\mathcal{D}_0 = \mathbb{R}^n$.

The following result proves that if (1) is finite-time stable, then its solution exists, is unique, and is defined for all $t \in [t_0, \infty)$.

Proposition 2.1 ([50,54]). *Consider the nonlinear dynamical system $\mathcal{G}$ given by Eq.(1). Assume $\mathcal{G}$ is finite-time stable and let $\mathcal{D}_0 \subseteq \mathcal{D}$ and $T : [0, \infty) \times \mathcal{D}_0 \backslash \{0\} \to (t_0, \infty)$ be defined as in Definition 2.1. Then, for every $(t_0, x_0) \in [0, \infty) \times \mathcal{D}_0$, there exists a unique solution s(t,*

$t_0, x_0)$, $t \geq t_0$, *to* (1) *such that* $s(t, t_0, x_0) \in \mathcal{D}_0$, $t \in [t_0, T(t_0, x_0))$, *and such that* $s(t, t_0, x_0) = 0$, $t \geq T(t_0, x_0)$, *where* $T(t_0, 0) \triangleq t_0$.

It follows from Proposition 2.1 that if the zero solution $x(t) \equiv 0$ to Eq. (1) is finite-time stable, then the solutions of Eq. (1) define a continuous *global semiflow* on $\mathcal{D}_0$; that is, $s : [t_0, \infty) \times [0, \infty) \times \mathcal{D}_0 \to \mathcal{D}_0$ is jointly continuous and satisfies the consistency property $s(t_0, t_0, x) = x$ and the semigroup property $s(t, \tau, s(\tau, t_0, x)) = s(t, t_0, x)$ for every $x \in \mathcal{D}_0$ and $t \geq \tau \geq t_0$. In addition, it follows from Proposition 2.1 that we can extend $T(t_0, \cdot)$ to all of $\mathcal{D}_0$ by defining $T(t_0, 0) \triangleq t_0$, for all $t_0 \in [0, \infty)$. Now, by uniqueness of solutions it follows that $s(T(t_0, x) + t, t_0, x) = 0$, $t \in [0, \infty)$, and hence, it is easy to see from Definition 2.1 that

$$T(t_0, x) = \inf\{t \in [t_0, \infty) : s(t, t_0, x) = 0\}, \qquad (t_0, x) \in [0, \infty) \times \mathcal{D}_0. \tag{2}$$

Lastly, it follows from Definition 2.1 and Proposition 2.1 that if the zero solution $x(t) \equiv 0$ to Eq. (1) is finite-time stable, then it is asymptotically stable, and hence, finite-time stability is a stronger condition than asymptotic stability.

In this paper, we consider controlled, nonlinear, time-varying dynamical systems of the form

$$\dot{x}(t) = F(t, x(t), u(t)), \qquad x(t_0) = x_0, \qquad t \geq t_0, \tag{3}$$

$$y(t) = H(x(t), u(t)), \tag{4}$$

where $x(t) \in \mathcal{D}$, $t \geq t_0$, denotes the *plant state*, $u(t) \in U \subseteq \mathbb{R}^m$ denotes the *control input*, $y(t) \in \mathbb{R}^l$ denotes the *measured output*, $0 \in U$, $F : [t_0, \infty) \times \mathcal{D} \times U \to \mathbb{R}^n$ is such that $F(t, 0, 0) = 0$, $t \geq t_0$, $F(\cdot, x, u)$ is continuous in t, for all $(x, u) \in \mathcal{D} \times U$, and $F(t, \cdot, \cdot)$ is jointly continuous in x and u uniformly in t, for all $t \in [t_0, \infty)$, and $H : \mathcal{D} \times U \to \mathbb{R}^l$ is continuous on $\mathcal{D} \times U$ and such that $H(0, 0) = 0$. A piecewise continuous function $\phi : \mathcal{D} \to U$ such that $\|\phi(\cdot)\|$ is continuous and $\phi(0) = 0$ is called a *control law*, and if $u(t) = \phi(x(t))$, $t \geq t_0$, where $\phi(\cdot)$ is a control law and $x(\cdot)$ denotes the unique solution of the *closed-loop system*

$$\dot{x}(t) = F(t, x(t), \phi(x(t))), \qquad x(t_0) = x_0, \qquad t \geq t_0, \tag{5}$$

then $u(\cdot)$ is a *state-feedback control law*.

To address the problem of designing feedback control laws that do not rely on the prefect knowledge of the state vector, we consider dynamical systems in the form

$$\dot{x}(t) = F(t, x(t), \phi(x(t) + e(t))), \qquad x(t_0) = x_0, \qquad t \geq t_0, \tag{6}$$

where $\phi(\cdot)$ is a control law and $e : [t_0, \infty) \to \mathbb{R}^n$ is continuous and such that $e(t) \to 0$ as $t \to \infty$. The vector function $e(\cdot)$ denotes the *estimation error*, that is, the difference between the solution $x(\cdot)$ of Eq. (6) and the state of a nonlinear observer, which is designed to estimate $x(\cdot)$, based on the measured output $y(\cdot)$. If $u(t) = \phi(x(t) + e(t))$, $t \geq t_0$, where $\phi(\cdot)$ is a control law, $e(t) \to 0$ as $t \to \infty$ uniformly in t_0, and $x(\cdot)$ denotes the unique solution of Eq. (6), then $u(\cdot)$ is an *observer-based feedback control law*.

3. Sufficient conditions for finite-time and uniform asymptotic stability in the presence of state constraints

In this section, we provide sufficient conditions for strong uniform finite-time stability and uniform asymptotic stability [8, Def. 4.2] of the time-varying nonlinear dynamical system (1), which guarantee that the solution $x(t)$, $t \geq t_0$, of Eq. (1) is contained in the simply connected *constraint set*

$$\mathcal{C} \triangleq \{x \in \mathcal{D} : h(x) \geq 0\}, \tag{7}$$

where $h : \mathcal{D} \to \mathbb{R}$ is continuously differentiable and $h(0) > 0$. For the statement of these results, we denote the *interior* of $\mathcal{C}$ by $\mathring{\mathcal{C}}$, that is, $\mathring{\mathcal{C}} \triangleq \{x \in \mathcal{D} : h(x) > 0\}$; in this paper, we assume that the constraint set is a proper subset of the plant state space, that is, $\mathring{\mathcal{C}} \subset \mathbb{R}^n$.

Theorem 3.1. *Consider the nonlinear dynamical system (1) with $x_0 \in \mathring{\mathcal{C}}$. If there exist a continuously differentiable function $V : [t_0, \infty) \times \mathring{\mathcal{C}} \to \mathbb{R}$, class $\mathcal{K}$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, and real numbers $\theta \in (0, 1)$ and $k > 0$, such that*

$$\alpha(\|x\|) \leq \frac{V(t,x)}{h(x)} \leq \beta(\|x\|), \qquad (t,x) \in [t_0, \infty) \times \mathring{\mathcal{C}}, \tag{8}$$

$$\frac{1}{h(x)}\frac{\partial V(t,x)}{\partial t} + \left[\frac{1}{h(x)}\frac{\partial V(t,x)}{\partial x} - \frac{V(t,x)}{h^2(x)}\frac{\partial h(x)}{\partial x}\right] f(t,x) \leq -k\left(\frac{V(t,x)}{h(x)}\right)^{\theta}, \tag{9}$$

then (1) is strongly uniformly finite-time stable, and $x(t) \in \mathring{\mathcal{C}}$ for all $t \geq t_0$. Moreover, there exist a neighborhood $\mathcal{D}_0 \subseteq \mathring{\mathcal{C}}$ of the origin and a settling-time function $T : [0, \infty) \times \mathcal{D}_0 \to [t_0, \infty)$, such that

$$T(t_0, x_0) \leq \frac{1}{k(1-\theta)}\left[\frac{V(t_0, x_0)}{h(x_0)}\right]^{1-\theta}, \qquad (t_0, x_0) \in [0, \infty) \times \mathcal{D}_0, \tag{10}$$

and $T(\cdot, \cdot)$ is jointly continuous on $[0, \infty) \times \mathcal{D}_0$.

It follows from the proof of Theorem 3.1, which is reported in the Appendix, that if the continuously differentiable function $\frac{V(t,x)}{h(x)}$ satisfies Eqs. (8) and (9), then it is a *barrier Lyapunov function*, that is, $\frac{V(t,x)}{h(x)}$ is positive-definite, $\frac{V(t,x(t))}{h(x(t))}$ is finite for every $t \geq t_0$ along the trajectory of (1), and $\frac{V(t,x)}{h(x)} \to \infty$ as $\mathrm{dist}(x, \partial\mathcal{C}) \to 0$, where $\partial\mathcal{C} \triangleq \{x \in \mathcal{D} : h(x) = 0\}$ denotes the *boundary* of $\mathcal{C}$ and $\mathrm{dist}(\cdot, \cdot)$ denotes the *distance function* between subsets of $\mathbb{R}^n$ [55, p. 16]; the notion of barrier Lyapunov function is usually defined for time-invariant dynamical systems [33,34], whereas the dynamical system (1) is time-varying.

A weaker form of Theorem 3.1 could have been proven as a direct consequence of Theorem 3.2 of [50]. Specifically, if Eqs. (8) and (9) are satisfied, then it follows from Theorem 3.2 of [50] that for every $\varepsilon > 0$ and $t_0 \in [0, \infty)$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\mathcal{B}_\delta(0) \subseteq \mathring{\mathcal{C}}$, and if $x_0 \in \mathcal{B}_\delta(0)$, then $x(\cdot)$ converges to the equilibrium point $x = 0$ in finite-time and $x(t) \in \mathcal{B}_\varepsilon(0) \subseteq \mathring{\mathcal{C}}$, $t \geq t_0$. This is a considerably weaker result, since strong uniform finite-time stability of Eq. (1) is guaranteed only if the initial condition x_0 lays in an open ball, which is *entirely* contained in $\mathring{\mathcal{C}}$. Theorem 3.1, instead, proves that if Eq. (8) and (9) are satisfied, then for every $\varepsilon > 0$ and $t_0 \in [0, \infty)$ there exists $\delta = \delta(\varepsilon) > 0$ such that if $x_0 \in \mathcal{B}_\delta(0) \cap \mathring{\mathcal{C}}$, then $x(\cdot)$ converges to $x = 0$ in finite-time, and $x(t) \in \mathcal{B}_\varepsilon(0) \cap \mathring{\mathcal{C}}$, $t \geq t_0$.

It is worth to remind that convex sets are diffeomorphic to $\mathbb{R}^n$ [56, p. 60]. Therefore, if $\mathring{\mathcal{C}}$ is convex, then there exists a continuously differentiable function $\mathcal{T} : \mathring{\mathcal{C}} \to \mathbb{R}^n$, whose inverse exists and is continuously differentiable, and one can apply the results on global strong uniform stability of time-varying dynamical systems [50] to prove sufficient conditions for the solutions of Eq. (1) to be finite-time stable and contained in $\mathring{\mathcal{C}}$ for any initial condition. However, there does not exist any systematic approach to devise such $\mathcal{T}(\cdot)$.

The next result provides sufficient conditions for the time-varying dynamical system (1) to be uniformly asymptotically stable and such that $x(t) \in \mathring{\mathcal{C}}$, for all $t \geq t_0$. The proof of this theorem is virtually identically to the proof of Theorem 3.1, and therefore is omitted for brevity.

Theorem 3.2. *Consider the nonlinear time-varying dynamical system (1) with $x_0 \in \mathring{\mathcal{C}}$. If there exist a continuously differentiable function $V : [t_0, \infty) \times \mathring{\mathcal{C}} \to \mathbb{R}$ and class $\mathcal{K}$ functions $\alpha(\cdot)$, $\beta(\cdot)$, and $\lambda(\cdot)$ such that*

$$\alpha(\|x\|) \leq \frac{V(t,x)}{h(x)} \leq \beta(\|x\|), \qquad (t,x) \in [t_0, \infty) \times \mathring{\mathcal{C}}, \tag{11}$$

$$\frac{1}{h(x)}\frac{\partial V(t,x)}{\partial t} + \left[\frac{1}{h(x)}\frac{\partial V(t,x)}{\partial x} - \frac{V(t,x)}{h^2(x)}\frac{\partial h(x)}{\partial x}\right] f(t,x) \leq -\lambda(\|x\|), \tag{12}$$

then Eq. (1) is uniformly asymptotically stable, and $x(t) \in \mathring{\mathcal{C}}$ for all $t \geq t_0$.

It is worth to recall that the authors in [33] and [57] provide systematic approaches to find a barrier Lyapunov function and a feedback control law, such that a closed-loop system is asymptotically stable and the system's trajectory is contained in some constraint set. These results are achieved in [33] by mean of an iterative algorithm and in [57] by extending Artstein necessary and sufficient conditions for the existence of control Lyapunov functions [58].

4. Observer-Based Feedback Control and Trajectory Convergence

In this section, we provide sufficient conditions for the solution of the nonlinear dynamical system (6) to converge to the equilibrium point either asymptotically or in finite-time. Specifically, the next result proves that if $e : [t_0, \infty) \to \mathbb{R}^n$ is such that $\lim_{t\to\infty} e(t) = 0$ and the state-feedback control law $u(t) = \phi(x(t))$, $t \geq t_0$, guarantees strong uniform finite-time stability of Eq. (5), then the observer-based feedback control law $u(t) = \phi(x(t) + e(t))$ guarantees finite-time convergence of the solution of (6) to $x = 0$.

Theorem 4.1. *Consider the nonlinear dynamical system (6), assume that $e(\cdot)$ is continuous on $[t_0, \infty)$ and $e(t) \to 0$ as $t \to \infty$ uniformly in t_0, and suppose there exist a continuously differentiable function $V : [t_0, \infty) \times \mathcal{D} \to \mathbb{R}$, class $\mathcal{K}$ function $\alpha(\cdot)$ and $\beta(\cdot)$, and real numbers $\theta \in (0, 1)$ and $k > 1$, such that*

$$\alpha(\|x\|) \leq V(t,x) \leq \beta(\|x\|), \qquad (t,x) \in [t_0, \infty) \times \mathcal{D} \tag{13}$$

$$\frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x} F(t,x,\phi(x)) \leq -k(V(t,x))^{\theta}. \tag{14}$$

Then, there exists a compact set $\mathcal{M} \subset \mathcal{D}$, such that $0 \in \mathcal{M}$ and if $x_0 \in \mathcal{M}$, then $x(t) \to 0$ as $t \to t_1$ uniformly in t_0, for some finite-time $t_1 \geq t_0$.

The proof of Theorem 4.1 is reported in the Appendix. In this paper, we address the problem of designing nonlinear robust observers and observer-based feedback control laws, which guarantee uniform finite-time convergence to the origin of nonlinear dynamical systems.

It follows from Theorem 3.2 of [50] that if (13) and (14) are satisfied, then the closed-loop system (5) is strongly uniformly finite-time stable. Hence, Theorem 4.1 proves that if the state-feedback control law $u=\phi(x)$ guarantees strong uniform finite-time stability of the closed-loop system and the estimation error $e(\cdot)$ is such that $\lim_{t\to\infty} e(t)=0$, then the observer-based feedback control law $u=\phi(x+e)$ guarantees uniform finite-time convergence of the system trajectory to the equilibrium point. Remarkably, it follows from the proof of Theorem 4.1 that larger values of k and smaller values of θ allow for larger sets $\mathcal{M}$.

The next result proves that if $\lim_{t\to\infty} e(t)=0$ and the state-feedback control law $u(t)=\phi(x(t))$ guarantees uniform asymptotic stability of Eq. (5), then the observer-based feedback control law $u(t)=\phi(x(t)+e(t))$ guarantees uniform asymptotic convergence of the solution of Eq. (6) to $x=0$. The proof of this theorem is omitted, since it is substantially identical to the proof of Theorem 4.1.

Theorem 4.2. *Consider the nonlinear dynamical system (6), assume that $e(t)\to 0$ as $t\to\infty$ uniformly in t_0, and suppose there exist a continuously differentiable function $V:[t_0,\infty)\times\mathcal{D}\to\mathbb{R}$ and class $\mathcal{K}$ functions $\alpha(\cdot)$, $\beta(\cdot)$, $\lambda(\cdot)$ such that Eq. (13) is satisfied and*

$$\frac{\partial V(t,x)}{\partial t}+\frac{\partial V(t,x)}{\partial x}F(t,x,\phi(x))\leq -\lambda(\|x\|),\qquad (t,x)\in[t_0,\infty)\times\mathcal{D}. \tag{15}$$

Then, there exists a compact set $\mathcal{M}\subset\mathcal{D}$, such that $0\in\mathcal{M}$ and if $x_0\in\mathcal{M}$, then $x(t)\to 0$ as $t\to\infty$ uniformly in t_0.

It is worth to remark that neither Theorem 4.1 nor Theorem 4.2 infer Lyapunov stability of the equilibrium point $x=0$. These results merely state finite-time or asymptotic convergence of the closed-loop system's trajectory to the equilibrium point.

5. A robust observer-based feedback controller for nonlinear dynamical systems

In this section, we apply the results of Sections 3 and 4 to design a robust observer-based feedback control law, which guarantees that the closed-loop trajectory is constrained to a simply connected set and asymptotically converges to the equilibrium point. This result is inspired by the sliding mode control architecture, as our controller steers the system trajectory to a sliding manifold in finite-time. However, despite the terminal sliding mode framework, our control architecture is observer-based and guarantees that the closed-loop system trajectory is constrained to a simply connected set.

We consider time-varying nonlinear dynamical systems of the form

$$\dot{x}(t)=f(x(t))+B(x(t))[G(x(t))E(x(t))u(t)+\delta_1(t,x(t),u(t))]+\delta_2(x(t)),$$
$$x(t_0)=x_0,\qquad t\geq t_0, \tag{16}$$

where $x(t)\in\mathcal{D}\subseteq\mathbb{R}^n$, $t\geq t_0$, $u(t)\in U\subseteq\mathbb{R}^m$, $f:\mathcal{D}\to\mathbb{R}^n$, $B:\mathcal{D}\to\mathbb{R}^{n\times m}$, $G:\mathcal{D}\to\mathbb{R}^{m\times m}$, and $E:\mathcal{D}\to\mathbb{R}^{m\times m}$ are continuous in x, $f(0)=0$, $E(x)$ is invertible for all $x\in\mathcal{D}$, $G(\cdot)$ is an unknown positive-definite diagonal matrix, that is, $G(x)\geq g_0 I_m$ for some $g_0>0$, $\delta_1:[t_0,\infty)\times\mathcal{D}\times U\to\mathbb{R}^m$ is such that $\delta_1(t,0,0)=0$, $t\geq t_0$, $\delta_1(\cdot,x,u)$ is continuous in t, for all $(x,u)\in\mathcal{D}\times U$, and $\delta_1(t,\cdot,\cdot)$ is jointly continuous in x and u uniformly in t, for all $t\in[t_0,\infty)$, and $\delta_2:\mathcal{D}\to\mathbb{R}^n$ is continuous in x and such that $\delta_2(0)=0$. The terms $\delta_1(\cdot,\cdot,\cdot)$ and $\delta_2(\cdot)$ are *unknown* and capture the matched and unmatched uncertainties, respectively. Assuming that the elements of $G(\cdot)$ are unknown allows accounting for failures in the control mechanism. The dynamical system (16) is in the same form as the dynamical model used in classic sliding mode design; for details, see [6, pp. 569-570].

For the statement of the following assumption, it is worth to recall that if $\mathcal{T}:\mathcal{D}\to\mathbb{R}^n$ is continuously differentiable, invertible, and $\mathcal{T}^{-1}(\cdot)$ is continuously differentiable, then $\mathcal{T}(\cdot)$ is a *diffeomorphism* [6, p. 508].

Assumption 5.1. Consider the nonlinear dynamical system (16) and the constraint set (7). There exists a diffeomorphism $\mathcal{T}:\mathring{\mathcal{C}}\to\mathbb{R}^n$, such that $\mathcal{T}(0)=0$ and

$$\frac{\partial\mathcal{T}(x)}{\partial x}B(x)=\begin{bmatrix}0_{(n-m)\times m}\\ I_m\end{bmatrix},\qquad x\in\mathring{\mathcal{C}}.\tag{17}$$

Let $\mathcal{T}(\cdot)$ be a diffeomorphism, such that Assumption 5.1 is satisfied and $\mathcal{T}(x)=[\eta^{\mathrm{T}},\xi^{\mathrm{T}}]^{\mathrm{T}}$, where $x\in\mathring{\mathcal{C}}$, $\eta\in\mathbb{R}^{n-m}$, and $\xi\in\mathbb{R}^m$. Then Eq. (16) is equivalent to

$$\dot\eta(t)=f_\eta(\eta(t),\xi(t))+\delta_\eta(\eta(t),\xi(t)),\qquad \eta(t_0)=[I_{n-m},0_{(n-m)\times m}]\mathcal{T}(x_0),\qquad t\geq t_0,\tag{18}$$

$$\dot\xi(t)=f_\xi(\eta(t),\xi(t))+G(\eta(t),\xi(t))E(\eta(t),\xi(t))u(t)+\delta_\xi(t,\eta(t),\xi(t),u(t)),\\ \xi(t_0)=[0_{m\times(n-m)},I_m]\mathcal{T}(x_0).\tag{19}$$

Assumption 5.1, which is fundamental to design also classical sliding mode controls [6, p. 564], postulates the separation of the matched uncertainties from the unmatched uncertainties. This assumption is usually verified by mechanical systems, where $f_\eta(\cdot,\cdot)$ captures the undisturbed kinematic equations and $f_\xi(\cdot,\cdot)$ captures the undisturbed, uncontrolled dynamic equations.

Define the simply connected sets

$$\mathcal{C}_\eta\triangleq\{\eta\in\mathbb{R}^{n-m}:h_\eta(\eta)\geq 0\},\tag{20}$$

$$\mathcal{C}_\xi\triangleq\{\xi\in\mathbb{R}^{m}:h_\xi(\xi)\geq 0\},\tag{21}$$

where $h_\eta:\mathbb{R}^{n-m}\to\mathbb{R}$ and $h_\xi:\mathbb{R}^m\to\mathbb{R}$ are continuously differentiable, $h_\eta(0)>0$, $h_\xi(0)>0$, and $\mathring{\mathcal{C}}_\eta\times\mathring{\mathcal{C}}_\xi\subseteq\mathcal{T}(\mathring{\mathcal{C}})$; note that if $(\eta,\xi)\in\mathring{\mathcal{C}}_\eta\times\mathring{\mathcal{C}}_\xi$, then $\mathcal{T}^{-1}(\eta,\xi)\in\mathring{\mathcal{C}}$, since $\mathcal{T}(\cdot)$ is continuous and surjective. The next lemma provides sufficient conditions for the existence of a state-feedback control law $\phi:\mathbb{R}^{n-m}\to\mathbb{R}^m$, such that $\phi(\mathring{\mathcal{C}}_\eta)\subseteq\mathring{\mathcal{C}}_\xi$ and the closed-loop system

$$\dot\eta(t)=f_\eta(\eta(t),\phi(\eta(t)))+\delta_\eta(\eta(t),\phi(\eta(t))),\quad \eta(t_0)=[I_{n-m},0_{(n-m)\times m}]\mathcal{T}(x_0),\quad t\geq t_0,\tag{22}$$

is asymptotically stable and $\eta(t)\in\mathring{\mathcal{C}}_\eta$, $t\geq t_0$. The existence of the state-feedback control law $\phi(\cdot)$ is fundamental to define a sliding manifold; for details, see [6, p. 564].

Lemma 5.1. *Consider the nonlinear time-invariant dynamical system (18) with $\eta(t_0)\in\mathring{\mathcal{C}}_\eta$. If there exist a continuously differentiable function $V:\mathring{\mathcal{C}}_\eta\to\mathbb{R}$, class $\mathcal{K}$ functions $\alpha(\cdot)$, $\beta(\cdot)$, and $\lambda(\cdot)$, and a state-feedback control law $\phi:\mathbb{R}^{n-m}\to\mathring{\mathcal{C}}_\xi$, such that $\phi(\mathring{\mathcal{C}}_\eta)\subseteq\mathring{\mathcal{C}}_\xi$ and*

$$\alpha(\|\eta\|)\leq\frac{V(\eta)}{h_\eta(\eta)}\leq\beta(\|\eta\|),\qquad \eta\in\mathring{\mathcal{C}}_\eta,\tag{23}$$

$$\left[\frac{1}{h_\eta(\eta)}\frac{\partial V(\eta)}{\partial \eta} - \frac{V(\eta)}{h_\eta^2(\eta)}\frac{\partial h_\eta(\eta)}{\partial \eta}\right]\left[f_\eta(\eta,\phi(\eta)) + \delta_\eta(\eta,\phi(\eta))\right] < -\lambda(\|\eta\|), \tag{24}$$

then the closed-loop system (22) is asymptotically stable, and $\eta(t) \in \mathring{\mathcal{C}}_\eta$ for all $t \geq t_0$.

Proof. The result is a direct consequence of Theorem 3.2 applied to the nonlinear time-invariant dynamical system (18). □

The next theorem, which is the main result of this section, provides a robust observer-based feedback control law, such that $x(t)$, $t \geq t_0$, lays in the interior of the constraint set $\mathcal{C}$ given by (7) and $x(t) \to 0$ as $t \to \infty$, despite the uncertainties captured by $G(\eta, \xi, u)$, $\delta_\xi(\eta, \xi, u)$, and $\delta_\eta(\eta, \xi)$. For the statement of this result, let

$$\mathcal{C}_z \triangleq \{z \in \mathbb{R}^m : h_z(z) \geq 0\} \tag{25}$$

be simply connected, $h_z : \mathbb{R}^m \to \mathbb{R}$ be continuously differentiable, $h_z(0) > 0$, and $\mathring{\mathcal{C}}_z \subseteq \{z \in \mathbb{R}^m : z = \xi - \phi(\eta), (\eta, \xi) \in \mathring{\mathcal{C}}_\eta \times \mathring{\mathcal{C}}_\xi\}$. Furthermore, let x_i denote the ith component of $x \in \mathbb{R}^n$, let the invertible matrix function $\hat{G} : \mathbb{R}^{n-m} \times \mathbb{R}^m \to \mathbb{R}^{m\times m}$ denote an *estimate* of $G(\cdot)$, and define

$$\psi(\eta, \xi, w) \triangleq -\lambda(\eta, \xi) + E^{-1}(\eta, \xi)w, \qquad (t, \eta, \xi, w) \in [t_0, \infty) \times \mathring{\mathcal{C}}_\eta \times \mathring{\mathcal{C}}_\xi \times \mathbb{R}^m, \tag{26}$$

$$\lambda(\eta, \xi) \triangleq E^{-1}(\eta, \xi)\hat{G}^{-1}(\eta, \xi)\left[f_\xi(\eta, \xi) - \frac{\partial \phi(\eta)}{\partial \eta}f_\eta(\eta, \xi)\right], \tag{27}$$

$$\Delta(t, \eta, \xi, w) \triangleq \delta(t, \eta, \xi, \psi(\eta, \xi, w)) + \left[I - G(\eta, \xi)\hat{G}^{-1}(\eta, \xi)\right]\left[f_\xi(\eta, \xi) - \frac{\partial \phi(\eta)}{\partial \eta}f_\eta(\eta, \xi)\right], \tag{28}$$

$$\delta(t, \eta, \xi, w) \triangleq \delta_\xi(t, \eta, \xi, w) - \frac{\partial \phi(\eta)}{\partial \eta}\delta_\eta(\eta, \xi), \tag{29}$$

where $\phi(\cdot)$ satisfies Lemma 5.1. Furthermore, let $e : [t_0, \infty) \to \mathcal{B}_{\delta_e}(0)$ be continuous and such that

$$[(\eta(t) + e_\eta(t))^{\mathrm{T}}, (\xi(t) + e_\xi(t))^{\mathrm{T}}]^{\mathrm{T}} \triangleq \mathcal{T}(x(t) + e(t)), \qquad t \geq t_0, \tag{30}$$

and $e(t) \to 0$ as $t \to \infty$ uniformly in t_0, where δ_e is sufficiently small, $x(\cdot)$ denotes the unique solution of Eq. (16) with $u = \psi(\eta + e_\eta, \xi + e_\xi, \gamma(\eta + e_\eta, \xi + e_\xi))$, $\gamma(\cdot, \cdot)$ is such that

$$\gamma_i(\eta, \xi) = -\mathrm{sign}(z_i(\eta, \xi))\mathrm{sign}\left(\frac{2}{h_z(z(\eta, \xi))} - \frac{z_i(\eta, \xi)}{h_z^2(z(\eta, \xi))}\frac{\partial h_z(z)}{\partial z_i}\right)\beta_i(\eta, \xi),$$
$$(\eta, \xi) \in \mathring{\mathcal{C}}_\eta \times \mathring{\mathcal{C}}_\xi, \qquad i = 1, \ldots, m, \tag{31}$$

$z_i(\cdot, \cdot)$ denotes the ith component of $z(\eta, \xi) \triangleq \xi - \phi(\eta)$, $i = 1, \ldots, m$, $k_i \in [0, 1)$, $\mathrm{sign}(\cdot)$ denotes the signum function, $\beta_i : \mathbb{R}^{n-m} \times \mathbb{R}^m \to \overline{\mathbb{R}}_+$ is continuous in its arguments and such that

$$\beta_i(\eta, \xi) \geq c + \frac{\rho_i(\eta, \xi)}{1 - k_i}, \tag{32}$$

$c > 0$, $\rho_i : \mathring{\mathcal{C}}_\eta \times \mathring{\mathcal{C}}_\xi \to \overline{\mathbb{R}}_+$ is continuous in its arguments, and

$$\left|\left[G^{-1}(\eta, \xi)\Delta(t, \eta, \xi, \psi(\eta, \xi, w))\right]_i\right| \le \rho_i(\eta, \xi) + k_i|w_i|,$$
$$(t, \eta, \xi, w) \in [t_0, \infty) \times \mathring{\mathcal{C}}_\eta \times \mathring{\mathcal{C}}_\xi \times \mathbb{R}^m. \quad (33)$$

Theorem 5.1. *Consider the nonlinear dynamical system (16) and the constraint sets (7), (20), and (25). Suppose Assumption 5.1 is verified, the conditions of Lemma 5.1 are satisfied, and there exist $k_i \in [0, 1)$, $i = 1, \ldots, m$, and continuous functions $\rho_i : \mathring{\mathcal{C}}_\eta \times \mathring{\mathcal{C}}_\xi \to \overline{\mathbb{R}}_+$, such that (33) is verified. Then, there exist compact sets $\mathcal{M}_\eta \subset \mathring{\mathcal{C}}_\eta$ and $\mathcal{M}_z \subset \mathring{\mathcal{C}}_z$, such that $0 \in \mathcal{M}_\eta$ and $0 \in \mathcal{M}_z$. Furthermore, if $(\eta(t_0), \xi(t_0) - \phi(\eta(t_0)) \in \mathcal{M}_\eta \times \mathcal{M}_z$, then the solution $x(\cdot)$ of (16), with*

$$u = \psi(\eta + e_\eta, \xi + e_\xi, \gamma(\eta + e_\eta, \xi + e_\xi)), \qquad (\eta, \xi, e_\eta, e_\xi) \in \mathring{\mathcal{C}}_\eta \times \mathring{\mathcal{C}}_\xi \times \mathcal{B}_{\delta_{e_\eta}}(0) \times \mathcal{B}_{\delta_{e_\xi}}(0), \quad (34)$$

where $\mathcal{B}_{\delta_{e_\eta}}(0) \times \mathcal{B}_{\delta_{e_\xi}}(0) \subseteq \mathcal{T}\left(\mathcal{B}_{\delta_e}(0)\right)$, is such that $x(t) \in \mathring{\mathcal{C}}$, $t \ge t_0$, and $x(t) \to 0$ as $t \to \infty$ uniformly in t_0.

Theorem 5.1, whose proof is reported in the Appendix, provides sufficient conditions to guarantee that the trajectory $x(t)$, $t \ge t_0$, of (16) with feedback control law (34) is contained in the interior of the constraint set (7), and asymptotically convergences to the equilibrium point $x = 0$. The classical sliding mode architecture guarantees asymptotic convergence for uncertain nonlinear plants in the same form as Eq. (16), and accounts for uncertainties, such that

$$\left\|G^{-1}(\eta, \xi)\Delta(t, \eta, \xi, \psi(\eta, \xi, w))\right\|_\infty \le \rho(\eta, \xi) + k\|w\|_\infty,$$
$$(t, \eta, \xi, w) \in [t_0, \infty) \times \mathbb{R}^{n-m} \times \mathbb{R}^m \times \mathbb{R}^m; \quad (35)$$

note that Eq. (35) implies Eq. (33). The classical sliding mode control does not account for constraints on the system's trajectory, which is assumed to be perfectly known.

The control law (34) is a function of the design parameters $c > 0$ and $k_i \in [0, 1)$, $i = 1, \ldots, m$. The parameter c is arbitrary and should be chosen as small as possible to minimize the amplitude of chattering in proximity of the sliding manifold. The parameters k_i, $i = 1, \ldots, m$, depend on the modeling assumptions on the plant and should be chosen as small as possible to reduce chattering. However, smaller values of c and k_i imply slower convergence to the sliding manifold.

Remark 5.1. If

$$\operatorname{sign}\left(\frac{2}{h_z(z(\eta, \xi))} - \frac{z_i(\eta, \xi)}{h_z^2(z(\eta, \xi))}\frac{\partial h_z(z)}{\partial z_i}\right) > 0, \qquad i = 1, \ldots, m, \quad (36)$$

for all $(\eta, \xi) \in \mathring{\mathcal{C}}_\eta \times \mathring{\mathcal{C}}_\xi$, then (31) reduces to

$$\gamma_i(\eta, \xi) = -\operatorname{sign}(z_i(\eta, \xi))\beta_i(\eta, \xi). \quad (37)$$

This condition is verified, for instance, if $h_z(z) < e^{2\omega} z_i^2$, for all $z \in \mathbb{R}^m$ and $i = 1, \ldots, m$, and

$$\mathring{\mathcal{C}}_z \subset \left\{z \in \mathbb{R}^m : \|z\| \le e^{-\omega}\right\}, \quad (38)$$

where $\omega \in \mathbb{R}$.

In order to guarantee that the solution $x(\cdot)$ of Eq. (16) with observer-based feedback control (34) is such that $x(t) \in \mathring{\mathcal{C}}$, $t \ge t_0$, Theorem 5.1 requires the estimation error $e(\cdot)$ to be such that $e(t) \in \mathcal{B}_{\delta_e}(0)$, $t \geq t_0$, where $\delta_e > 0$ is sufficiently small. Specifically, one needs to design a robust nonlinear observer, such that $\big(\eta(t) + e_\eta(t)\big) \in \mathring{\mathcal{C}}_\eta$, $t \geq t_0$, and $\big(\xi(t) + e_\xi(t)\big) \in \mathring{\mathcal{C}}_\xi$. This observer design problem is addressed in the next section.

6. The Walcott and Żak observer

In this section, we show how a popular nonlinear robust observer, that is, the Walcott and Żak observer [38–44], can be employed within the proposed observer-based feedback control framework. Specifically, consider the uncertain nonlinear dynamical system

$$\dot{x}(t) = Ax(t) + B\big[u(t) + \delta_1(t, x(t), u(t))\big], \qquad x(t_0) = x_0, \qquad t \geq t_0, \tag{39}$$

$$y(t) = Cx(t), \tag{40}$$

where $x(t) \in \mathbb{R}^n$, $t \geq 0$, $u(t) \in \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $\delta_1 : [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is *unknown*, $\delta_1(\cdot, \cdot, \cdot)$ is jointly continuous in t, x, and u, $\delta_1(t, \cdot, \cdot)$ is locally jointly Lipschitz continuous in x and u uniformly in t, for all t in compact subsets of $[t_0, \infty)$, and $C \in \mathbb{R}^{l\times n}$ has full row rank. The plant model (39) is sufficiently broad to model numerous problems of practical interest, since several system identification techniques can be applied to compute linear models that provide satisfactory first-order approximations of nonlinear dynamical systems [59,60]. To simplify the sensors' calibration process, it is common practice to consider some linear function of the measured state vector as the system output; the rank condition on the matrix C implies that there is no redundancy amongst the measurements captured by the output $y(\cdot)$.

Consider the *observer*

$$\dot{\hat{x}}(t) = A\hat{x}(t) + B[u(t) - v(t)] - K\big[C\hat{x}(t) - y(t)\big], \qquad \hat{x}(t_0) = \hat{x}_0, \qquad t \geq t_0, \tag{41}$$

where the *observer gain* $K \in \mathbb{R}^{n\times l}$ is such that $A_e \triangleq A + KC$ is Hurwitz, P is the symmetric, positive-definite solution of the Lyapunov equation

$$0 = A_e^{\mathrm{T}} P + P A_e + Q, \tag{42}$$

$Q \in \mathbb{R}^{n\times n}$ is symmetric and positive-definite, and $F \in \mathbb{R}^{m\times l}$ is such that

$$PB = C^{\mathrm{T}} F^{\mathrm{T}}. \tag{43}$$

The next result provides a feedback control law for the virtual control $v : [t_0, \infty) \to \mathbb{R}^m$ so that the state $\hat{x}(\cdot)$ of (41) converges to the plant state $x(t)$ of Eq. (39) exponentially in time. For the statement of this result, let $e(t) = x(t) - \hat{x}(t)$, $t \geq t_0$, denote the *estimation error*, and note that Eqs. (39)–(41) imply that

$$\dot{e}(t) = A_e e(t) + B[v(t) + \delta_1(t, x(t), u(t))], \qquad e(t_0) = x_0 - \hat{x}_0, \qquad t \geq t_0, \tag{44}$$

where $x(t)$ verifies Eq. (39).

Theorem 6.1 ([38]). *Consider the uncertain nonlinear dynamical systems (39) and (40), the observer (41), and the estimation error dynamics (44). If there exist a continuous function* $\rho : [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m \to \overline{\mathbb{R}}_+$ *such that*

$$\|\delta_1(t, x, u)\| \leq \rho(t, \hat{x}, y, u), \qquad (t, x, \hat{x}, y, u) \in [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m, \tag{45}$$

then the solution $e(t)$, $t \geq 0$, *of* Eq. (44) *with feedback control law*

$$v = \gamma(t, \hat{x}, y, u), \qquad (t, \hat{x}, y, u) \in [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m, \tag{46}$$

is such that $e(t) \in \mathcal{B}_{\delta_e}(0)$, $t \geq t_0$, *and* $e(t) \to 0$ *as* $t \to \infty$ *uniformly in* t_0 *and exponentially in time, where* $\gamma_i(t, \hat{x}, y, u) = -\mathrm{sign}(z_i(\hat{x}, y))\beta(t, \hat{x}, y, u)$, $z_i(\hat{x}, y)$ *denotes the ith component of* $z(\hat{x}, y) \triangleq F[y - C\hat{x}]$, $i = 1, \ldots, m$, $\beta(t, \hat{x}, y, u) \geq c_e + \rho(t, \hat{x}, y, u)$, $c_e > 0$, *and* $\|x_0 - \hat{x}_0\| < \delta_e$.

Remark 6.1. Theorem 6.1 guarantees that the norm of the estimation error is a decreasing function of time and exponentially converges to zero [38]. Hence, if the estimation error $e(t_0) = x_0 - \hat{x}_0$ is sufficiently small, that is, $(\eta(t_0) + e_\eta(t_0)) \in \mathring{\mathcal{C}}_\eta$ and $(\xi(t_0) + e_\xi(t_0)) \in \mathring{\mathcal{C}}_\xi$, where $\mathring{\mathcal{C}}_\eta \times \mathring{\mathcal{C}}_\xi \subseteq \mathcal{T}(\mathcal{B}_{\delta_e}(0))$, then the Walcott and Żak observer given by Eqs. (41) and (46) is suitable estimator for the observer-based feedback control framework developed herein. Similarly, the observers presented in [39–44], which are variations of the Walcott and Żak observer, can be employed within the proposed framework.

Although the estimation error $e(\cdot)$ can be neither measured directly nor computed integrating Eq. (44), $Ce(t) = y(t) - C\hat{x}(t)$, $t \geq t_0$, is well-defined, since the measured output $y(t)$ is known and $\hat{x}(t)$ is computed integrating Eq. (41) with the same control input $u(\cdot)$ as in Eq. (39) and the virtual control input $v(\cdot)$ given by Eq. (46). Thus, Eq. (41) with feedback control law Eq. (46) is well-defined. The gain matrix $K \in \mathbb{R}^{l \times n}$ in Eq. (41) plays an important role in the Walcott and Żak observer. Specifically, since $A_e = A + KC$, it follows from Eqs. (44) and (46) that K directly affects the rate of exponential convergence of the estimation error along the sliding manifold. Moreover, the matrices Q and K provide $n(n+1)/2 + ml$ design variables so that P and F verify both Eqs. (42) and (43).

7. Roll stabilization of a delta-wing aircraft

In this section, we apply the results developed in Sections 5 and 6 and provide an observer-based feedback robust nonlinear control that stabilizes the roll dynamics of a delta-wing aircraft; this aircraft configuration is unstable [61, pp. 285]. Specifically, our controller guarantees that the aircraft roll angle and roll rate are constrained to given intervals at all times and the system's state asymptotically converges to the equilibrium condition.

The roll dynamics of a delta-wing aircraft is captured by [61, pp. 285-290]

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\theta_1 & -\theta_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \theta_6 \end{bmatrix} \left[u(t) + \tilde{\delta}_1(x_1, x_2)\right], \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}, \quad t \geq 0, \tag{47}$$

$$y(t) = \hat{\theta}_6^{-1} x_1(t) + \hat{\theta}_6^{-1} x_2(t), \tag{48}$$

where $x_1(t)$ and $x_2(t) \in \mathbb{R}$ denote the aircraft roll angle and roll rate, respectively, $u(t) \in \mathbb{R}$ denotes the voltage applied to the aileron motor to induce a deflection of the control surface [62, Ex. 3.4], $\Phi \in \mathbb{R}$, $\tilde{\delta}_1(x_1, x_2) = \theta_6^{-1}\left[-\theta_3|x_1|x_2 + \theta_4|x_2|x_2 + \theta_5 x_1^3\right]$, the parameters $\theta_i \in \left[\underline{\theta}_i, \overline{\theta}_i\right]$, $i = 1, \ldots, 6$, are unknown, $\overline{\theta}_j, \underline{\theta}_j > 0$, $j = 1, 3, \ldots 6$, is known, $\overline{\theta}_2, \underline{\theta}_2 > 1$ is known, and $\hat{\theta}_6 \in \left[\frac{1}{2}\overline{\theta}_6, \overline{\theta}_6\right]$ is known. Our goal is to apply Theorems 5.1 and 6.1 and design an observer-based feedback control law, such that $x(t) \to 0$ as $t \to \infty$, where $x = [x_1, x_2]^{\mathrm{T}}$, $x(t) \in \mathring{\mathcal{C}}$, $t \geq 0$, and

$\mathcal{C} = \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R} : |x_1| \leq \overline{x}_1, |x_2| \leq \overline{x}_2\}$

and $\overline{x}_1, \overline{x}_2 > 0$.

Let $\hat{\theta}_i \in [\underline{\theta}_i, \overline{\theta}_i]$, $i = 1, 2, 6$, denote an estimate of θ_i. The nonlinear dynamical system (47) and (48) is equivalent to

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\hat{\theta}_1 & -\hat{\theta}_2\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\ \hat{\theta}_6\end{bmatrix}[u(t) + \delta_1(x_1(t), x_2(t), u(t))],$$
$$[x_1(0), x_2(0)]^{\mathrm{T}} = [x_{10}, x_{20}]^{\mathrm{T}}, \qquad t \geq 0, \tag{49}$$

$$y(t) = \hat{\theta}_6^{-1} x_1(t) + \hat{\theta}_6^{-1} x_2(t), \tag{50}$$

where

$$\delta_1(x_1, x_2, u) = \hat{\theta}_6^{-1}\Big[\theta_6 \tilde{\delta}_1(x_1, x_2) + (\hat{\theta}_1 - \theta_1)x_1 + (\hat{\theta}_2 - \theta_2)x_2 + (\theta_6 - \hat{\theta}_6)u\Big],$$

and Eqs. (49) and (50) is in the same form as Eq. (16) with $n = 2$, $m = 1$, $x = [x_1, x_2]^{\mathrm{T}}$, $t_0 = 0$, $f(x) = \begin{bmatrix}0 & 1\\ -\hat{\theta}_1 & -\hat{\theta}_2\end{bmatrix}x$, $B(x) = \Big[0, \hat{\theta}_6\Big]^{\mathrm{T}}$, $G(x) = E(x) = 1$, and $\delta_2(x) = 0$. In this case, Assumption 5.1 is verified by $\mathcal{T}(x) = \hat{\theta}_6^{-1} x$, $x \in \mathbb{R}^2$, $\eta = \hat{\theta}_6^{-1} x_1$, and $\xi = \hat{\theta}_6^{-1} x_2$. Moreover, $f_\eta(\eta, \xi) = \xi$, $f_\xi(\eta, \xi) = -\hat{\theta}_1 \eta - \hat{\theta}_2 \xi$, $\delta_\eta(\eta, \xi) = 0$, $\delta_\xi(t, \eta, \xi, u) = \delta_1(\hat{\theta}_6 \eta, \hat{\theta}_6 \xi, u)$,

$$\mathcal{C}_\eta = \Big\{\eta \in \mathbb{R} : |\eta| \leq \hat{\theta}_6^{-1} \overline{x}_1\Big\}, \tag{51}$$

$$\mathcal{C}_\xi = \Big\{\xi \in \mathbb{R} : |\xi| \leq \hat{\theta}_6^{-1} \overline{x}_2\Big\}, \tag{52}$$

$$\mathcal{C}_z = \Big\{z \in \mathbb{R} : |z| \leq \hat{\theta}_6^{-1}(\overline{x}_2 + q\overline{x}_1)\Big\}, \tag{53}$$

$h_\eta(\eta) = \left(\hat{\theta}_6^{-1} \overline{x}_1\right)^2 - \eta^2$, $h_\xi(\xi) = \left(\hat{\theta}_6^{-1} \overline{x}_2\right)^2 - \xi^2$, and $h_z(z) = \hat{\theta}_6^{-2}(\overline{x}_2 + q\overline{x}_1)^2 - z^2$.

Next, consider the continuously differentiable function

$$V(\eta) = \eta^2, \qquad \eta \in \mathring{\mathcal{C}}_\eta, \tag{54}$$

and the state-feedback control law $\phi(\eta) = -q\eta$, where $q \in \left(0, \overline{x}_2 \overline{x}_1^{-1}\right)$, and note that $\frac{V(\eta)}{h_\eta(\eta)}$ is positive-definite for all $\eta \in \mathring{\mathcal{C}}_\eta$, Eq. (23) is verified with $\alpha(\|\eta\|) = \beta(\|\eta\|) = \frac{V(\eta)}{\left|(\hat{\theta}_6^{-1} \overline{x}_1)^2 - \eta^2\right|}$, and $\phi(\mathring{\mathcal{C}}_\eta) \subset \mathring{\mathcal{C}}_\xi$. Furthermore, it holds that

$$\left[\frac{1}{h_\eta(\eta)} \frac{\partial V(\eta)}{\partial \eta} - \frac{V(\eta)}{h_\eta^2(\eta)} \frac{\partial h_\eta(\eta)}{\partial \eta}\right]\left[f_\eta(\eta, \phi(\eta)) + \delta_\eta(\eta, \phi(\eta))\right]$$
$$= -2q \frac{h_\eta(\eta) + \eta^2}{h_\eta^2(\eta)} \eta^2 = -2q \frac{\left(\hat{\theta}_6^{-1} \overline{x}_1\right)^2 \eta^2}{\left(\hat{\theta}_6^{-1} \overline{x}_1\right)^2 - \eta^2}, \quad \eta \in \mathring{\mathcal{C}}_\eta, \tag{55}$$

which implies that Eq. (24) is verified. Hence, the conditions of Lemma 5.1 are satisfied and the closed-loop system

$$\dot{\eta}(t) = -q\eta(t), \qquad \eta(0) = \hat{\theta}_6^{-1}x_{10}, \qquad t \geq 0, \tag{56}$$

is asymptotically stable and $\eta(t) \in \mathring{\mathcal{C}}_\eta$, $t \geq 0$. Finally, in this case it holds that

$$\left[G^{-1}(\eta, \xi)\Delta(t, \eta, \xi, \psi(\eta, \xi, w))\right] = \delta_\xi(t, \eta, \xi, \psi(\eta, \xi, w)), \qquad [t_0, \infty) \times \mathring{\mathcal{C}}_\eta \times \mathring{\mathcal{C}}_\xi \times \mathbb{R}^m,$$

where

$$\psi(\eta, \xi, w) = \hat{\theta}_1\eta + \left(\hat{\theta}_2 - q\hat{\theta}_1\right)\xi + w, \tag{57}$$

and Eq. (33) is satisfied with

$$\begin{aligned}\rho_1(\eta, \xi) = {} & \overline{\theta}_3\hat{\theta}_6|\eta||\xi| + \overline{\theta}_4\hat{\theta}_6\xi^2 + \overline{\theta}_5\hat{\theta}_6^2|\eta|^3 + \left(\hat{\theta}_1 - \underline{\theta}_1\right)|\eta| + \left(\hat{\theta}_2 - \underline{\theta}_2\right)|\xi| \\ & + \hat{\theta}_6^{-1}\left(\overline{\theta}_6 - \hat{\theta}_6\right)\left[\hat{\theta}_1|\eta| + \left(\hat{\theta}_2 - q\hat{\theta}_1\right)|\xi|\right], \qquad (\eta, \xi) \in \mathring{\mathcal{C}}_\eta \times \mathring{\mathcal{C}}_\xi,\end{aligned} \tag{58}$$

$$k_1 = \hat{\theta}_6^{-1}\left(\overline{\theta}_6 - \hat{\theta}_6\right). \tag{59}$$

In this case, the conditions of Theorem 5.1 are verified and it follows from Remark 5.1 and Eq. (31) that

$$\gamma_1(\eta, \xi) = -\text{sign}(\xi + q\eta)\beta_1(\eta, \xi), \qquad (\eta, \xi) \in \mathring{\mathcal{C}}_\eta \times \mathring{\mathcal{C}}_\xi, \tag{60}$$

where $\beta_1(\eta, \xi) \geq c + (1 - k_1)^{-1}\rho_1(\eta, \xi)$ and $c > 0$.

Let $\mathcal{M}_\eta$ be a compact set, such that $0 \in \mathcal{M}_\eta \subset (-\hat{\theta}_6^{-1}\overline{x}_1, \hat{\theta}_6^{-1}\overline{x}_1) = \mathring{\mathcal{C}}_\eta$ and let $\mathcal{M}_z$ be a compact set, such that $0 \in \mathcal{M}_z \subset \left(-\hat{\theta}_6^{-1}\overline{x}_2 - q\hat{\theta}_6^{-1}\overline{x}_1, \hat{\theta}_6^{-1}\overline{x}_2 + q\hat{\theta}_6^{-1}\overline{x}_1\right) = \mathring{\mathcal{C}}_z$. It follows from Theorem 5.1 that if $\left(\hat{\theta}_6^{-1}x_{10}, \hat{\theta}_6^{-1}(x_{20} + qx_{10})\right) \in \mathcal{M}_\eta \times \mathcal{M}_z$ and the estimated state $(\eta(t) + e_\eta(t), \xi(t) + e_\xi(t)) \in \mathring{\mathcal{C}}_\eta \times \mathring{\mathcal{C}}_\xi$, $t \geq 0$, then the solution $x(t) = [x_1(t), x_2(t)]^{\mathrm{T}}$, $t \geq 0$, of (47) with feedback control (34) is such that $x(t) \in \mathring{\mathcal{C}}$, $t \geq 0$, and $x(t) \to 0$ as $t \to \infty$.

Next, we apply the Walcott and Żak observer outlined in Theorem 6.1 to estimate the trajectory of Eq. (49). Note that Eqs. (49) and (50) is in the same form as Eqs. (39) and (40) with $A = \begin{bmatrix} 0 & 1 \\ -\hat{\theta}_1 & -\hat{\theta}_2 \end{bmatrix}$, $B = \left[0, \hat{\theta}_6\right]^{\mathrm{T}}$, $C = \hat{\theta}_6^{-1}[1, 1]$, and $\delta_1(t, x, u) = \delta_1(x, u)$, $(t, x, u) \in [0, \infty) \times \mathbb{R}^2 \times \mathbb{R}$, and consider the observer (41), where $x(t)$ satisfies Eq. (49). Defining the estimation error as $e(t) = x(t) - \hat{x}(t)$, $t \geq 0$, it follows from Eqs. (49), (41), and (57) that

$$\begin{aligned}\dot{e}(t) = {} & A_{\mathrm{e}}e(t) + B\left[v(t) + \delta_1(\hat{\theta}_6\eta(t), \hat{\theta}_6\xi(t), \psi(\eta(t), \xi(t), \gamma_1(\eta(t), \xi(t))))\right], \\ & e(0) = x_0 - \hat{x}_0, \qquad t \geq 0,\end{aligned} \tag{61}$$

where $A_{\mathrm{e}} = A + KC$.

In this case, it holds that

$$|\delta_1(\hat{\theta}_6e_1, \hat{\theta}_6e_2, \psi(e_1, e_2, \gamma_1(e_1, e_2)))| \leq \rho_2(e_1, e_2), \qquad (e_1, e_2) \in \mathbb{R} \times \mathbb{R}, \tag{62}$$

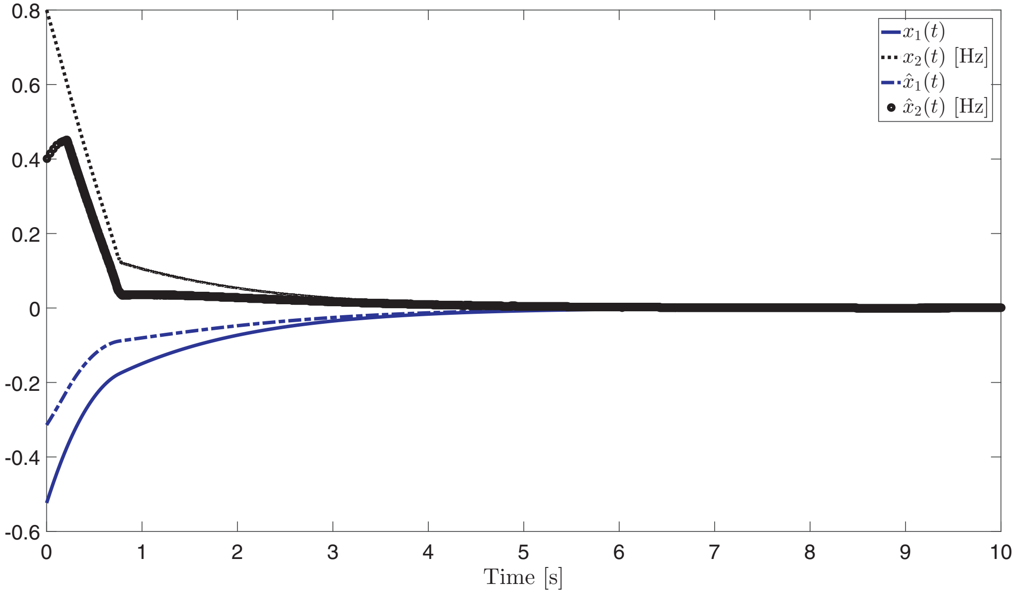

Fig. 7.1. Closed-loop aircraft roll dynamics and estimated roll dynamics versus time.

where $e = \hat{\theta}_6[e_1, e_2]^{\mathrm{T}}$ and $\rho_2(e_1, e_2) \triangleq \rho_1(e_1, e_2) + k_1|\psi(e_1, e_2, \gamma_1(e_1, e_2))|$, and $\rho_2(e_1, e_2) \leq \rho(\alpha)$, $(e_1, e_2, \alpha) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, where

$$\rho(\alpha) \triangleq \max_{\alpha \in \{e_1, e_2\}} \rho_2(\alpha, \alpha).$$

Since both $e_1(\cdot)$ and $e_2(\cdot)$ are continuous in their arguments, it follows from Eqs. (57), (58), and (60) that both $\alpha(t) = \hat{\theta}_6^{-1}\|x(t) - \hat{x}(t)\|_\infty$, $t \geq 0$, and $\rho(\alpha(t))$ are continuous, where $\|\cdot\|_\infty$ denotes the infinity norm of its argument. Now, if there exist $K \in \mathbb{R}^2$, $F \in \mathbb{R}$, and a symmetric positive-definite matrix $Q \in \mathbb{R}^{2\times 2}$ such that Eqs. (42) and (43) are verified, then it follows from Theorem 6.1 that the estimation error dynamics (61) with feedback control law (46) is such that $\|x(\cdot) - \hat{x}(\cdot)\|$ is a monotonically decreasing function and converges to zero exponentially fast. In this problem, the feedback control law $v = \gamma(\hat{x}, y)$ can be computed as follows. It follows from Eq. (58) that $\rho(\cdot)$ is a monotonically increasing function and note that

$$\begin{aligned}\hat{\theta}_6^{-1} \max\left\{|x_1 - \hat{x}_1|, |x_2 - \hat{x}_2|\right\} &\leq \hat{\theta}_6^{-1}|x_1 - \hat{x}_1 + x_2 - \hat{x}_2| + \hat{\theta}_6^{-1} \min\left\{|x_1 - \hat{x}_1|, |x_2 - \hat{x}_2|\right\} \\ &= |y - C\hat{x}| + \hat{\theta}_6^{-1} \min\left\{|x_1 - \hat{x}_1|, |x_2 - \hat{x}_2|\right\}.\end{aligned}$$

Thus, it follows from [6, p. 188] that

$$\rho(\alpha(t)) \leq \rho\big(2|y(t) - C\hat{x}(t)|\big) + \rho\Big(2\hat{\theta}_6^{-1} \min\left\{|x_1(0) - \hat{x}_1(0)|, |x_2(0) - \hat{x}_2(0)|\right\}\Big), \tag{63}$$

for all $t \geq 0$, and Eq. (61) specializes to

$$\gamma(\hat{x}, y) = -\mathrm{sign}\big(y - C\hat{x}\big)\big[\rho\big(2|y - C\hat{x}|\big) + c_{\mathrm{e}}\big], \qquad (\hat{x}, y) \in \mathbb{R}^2 \times \mathbb{R}, \tag{64}$$

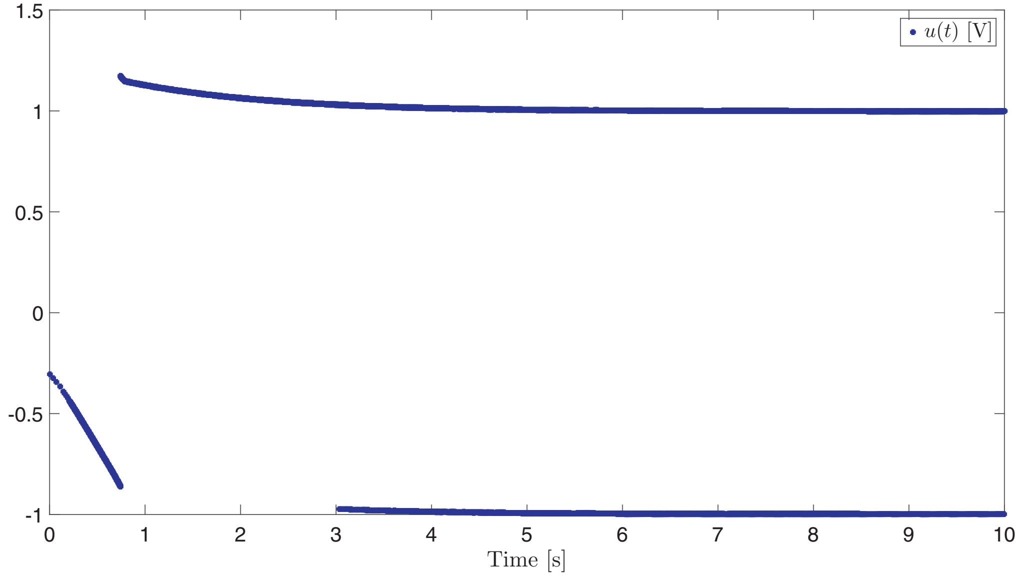

Fig. 7.2. Output-feedback control versus time.

where $c_e \geq \rho\left(2\hat{\theta}_6^{-1}\min\left\{\left|x_1(0)-\hat{x}_1(0)\right|,\left|x_2(0)-\hat{x}_2(0)\right|\right\}\right)$; although $x(0)$ may not be precisely known, it is reasonable to assume that an estimate of $x(0)$ is available. Alternative approaches to estimate both the aircraft roll angle and the roll rate consist in applying Kalman filters [49, Ch. 8] or extended Kalman filters [49, Ch. 13] to data measured by the gyroscopes installed on the vehicle [63].

Let $\theta_1 = 0.018$, $\theta_2 = 1.113$, $\theta_3 = 0.062$, $\theta_4 = 0.009$, $\theta_5 = 0.0021$, $\theta_6 = 0.790$, $\overline{\theta}_i = 0.1$, $i = 1, 3, 4, 5$, $\overline{\theta}_2 = 2.1$, $\overline{\theta}_6 = 1$, $\underline{\theta}_j = 10^{-3}$, $j = 1, 3, \ldots, 6$, $\underline{\theta}_2 = 1.001$, $\hat{\theta}_p = 0.03$, $p = 1, 3, 4, 5$, $\hat{\theta}_2 = 1.05$, $\hat{\theta}_6 = 0.9$, $\overline{x}_1 = 2\pi/9$, $\overline{x}_2 = 1\,\text{Hz}$, $q = \frac{1}{2}\overline{x}_2\overline{x}_1^{-1}$, $K = 0$, $F = 1$, $P = \begin{bmatrix} 2F\left(\hat{\theta}_1+\hat{\theta}_2\right) & F \\ F & F \end{bmatrix}$, $Q = \begin{bmatrix} 2F\hat{\theta}_1 & 0 \\ 0 & 2F\left(\hat{\theta}_2-1\right) \end{bmatrix}$, and $c = 1$. Fig. 7.1 shows the trajectory of the closed-loop system using the observer-based feedback control law $u = \psi(\eta + e_\eta, \xi + e_\xi, \gamma_1(\eta + e_\eta, \xi + e_\xi))$, which is shown in Fig. 7.2; clearly, $x(t) \in \mathring{\mathcal{C}}$, $t \geq 0$, $x(t) \to 0$ as $t \to \infty$, and $\|x(0) - \hat{x}(0)\| \geq \|x(t) - \hat{x}(t)\|$, $t \geq t_0$, and $\|x(t) - \hat{x}(t)\| \to 0$ as $t \to \infty$, that is, the closed-loop system converges to $x \equiv 0$ and the estimated state converges to the system's state. Fig. 7.2 clearly shows that the control input is affected by chattering for all $t \geq 3$.

8. Conclusion

This paper presents an observer-based feedback sliding mode architecture for time-varying nonlinear dynamical systems in the presence of uncertainties in the plant model and constraints on the system's trajectory. The proposed control law guarantees finite-time convergence of the closed-loop trajectory to the sliding manifold, while the state constraints are verified by *barrier Lyapunov functions*. Moreover, it is shown how the proposed framework can be integrated with existing nonlinear estimators, such as the Walcott and Żak observer. Our theoretical results are illustrated by a numerical example involving the roll dynamics of an unstable aircraft.

The numerical example clearly shows that the proposed control algorithm is affected by chattering. Future work directions involve the development of a chattering-free observer-based feedback control algorithm, based on the framework presented in this paper. To this goal, one possibility involves the introduction of a *boundary layer*, where the signum function is approximated by the saturation, the sigmoid, or the arctangent functions [64,65]. Alternatively, higher-order sliding mode methods [66] and observer-based sliding mode controllers [67] will be considered to mitigate chattering.

Acknowledgment

This work was supported in part by the NOAA/Office of Oceanic and Atmospheric Research under NOAA-University of Oklahoma Cooperative Agreement #NA16OAR4320115, U.S. Department of Commerce, and the National Science Foundation under Grant no. 1700640.

Appendix. Proofs of the main results

In this section, we present the proofs of the main theoretical results developed in this paper.

Proof of Theorem 3.1. Firstly, we use a contradiction argument to prove that if $x_0 \in \mathring{\mathcal{C}}$, then $x(t) \in \mathring{\mathcal{C}}$ for all $t \geq t_0$. Specifically, suppose *ad absurdum* there exists $T^* \in (t_0, \infty) \cap \mathcal{I}_{t_0,x_0}$, such that $\lim_{t\to T^*} h(x(t)) = 0$ along the trajectory of (1), where $\mathcal{I}_{t_0,x_0}$ denotes the maximal interval of existence of the solution $x(\cdot)$ of Eq. (1). It follows from the continuity of $h(\cdot)$ on $\mathcal{D}$ and $x(\cdot)$ on $\mathcal{I}_{t_0,x_0}$ that $\lim_{t\to T^*} h(x(t)) = h(\lim_{t\to T^*} x(t)) = h(x(T^*)) = 0$, which implies that $x(T^*) \neq 0$, since $0 \in \mathring{\mathcal{C}}$ and $h(x) > 0$, $x \in \mathring{\mathcal{C}}$. Moreover, it follows from Eq. (8) that $V(t, x) = 0$ if and only if $x = 0$ and hence it follows from the continuity of $V(\cdot, x(\cdot))$ that $\lim_{t\to T^*} V(t, x(t)) = V(T^*, x(T^*)) \neq 0$. Therefore, $\lim_{t\to T^*} \frac{V(t, x(t))}{h(x(t))} = \infty$. Now, since

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{V(t, x(t))}{h(x(t))} = \frac{1}{h(x(t))}\frac{\partial V(t, x(t))}{\partial t} + \left[\frac{1}{h(x(t))}\frac{\partial V(t, x(t))}{\partial x} - \frac{V(t, x(t))}{h^2(x(t))}\frac{\partial h(x(t))}{\partial x}\right] f(t, x(t)), \qquad t \geq t_0, \tag{65}$$

along the trajectory of Eq. (1), it follows from Eq. (9) that $\frac{V(t, x(t))}{h(x(t))}$, $t \geq t_0$, is a decreasing function of time, that is, if $x_0 \in \mathring{\mathcal{C}}$ and $x_0 \neq 0$, then

$$\frac{V(t, x(t))}{h(x(t))} < \frac{V(t_0, x_0)}{h(x_0)} \leq \beta(\|x_0\|) < \infty, \qquad t > t_0. \tag{66}$$

However, the fact that $\lim_{t\to T^*} \frac{V(t, x(t))}{h(x(t))} = \infty$ contradicts Eq. (66). Hence, if $x_0 \in \mathring{\mathcal{C}}$, then $x(t) \in \mathring{\mathcal{C}}$ for all $t \geq t_0$, that is, $\mathring{\mathcal{C}}$ is positively invariant with respect to Eq. (1).

Next, we prove uniform Lyapunov stability of (1). Let $\varepsilon > 0$, consider the open set $\mathcal{B}_\varepsilon(0) \cap \mathring{\mathcal{C}}$ containing $x = 0$, and let $\delta = \delta(\varepsilon) > 0$ be such that $\beta(\delta) = \alpha(\varepsilon)$. It follows from Eq. (8) that, for all $(t_0, x_0) \in [0, \infty) \times \mathcal{B}_\delta(0) \cap \mathring{\mathcal{C}}$,

$$\alpha(\|x(t)\|) \leq \frac{V(t, x(t))}{h(x(t))} \leq \frac{V(t_0, x_0)}{h(x_0)} < \beta(\delta) = \alpha(\varepsilon), \qquad t \geq t_0. \tag{67}$$

Thus, $x(t) \in \mathcal{B}_\varepsilon(0) \cap \mathring{\mathcal{C}}$, $t \geq t_0$, since $\mathring{\mathcal{C}}$ is positively invariant with respect to Eq. (1) for every $x_0 \in \mathcal{B}_\delta(0) \cap \mathring{\mathcal{C}}$, and uniform Lyapunov stability of Eq. (1) is proven.

Next, note that the solution to

$$\dot{v}(t) = -kv^\theta(t), \qquad v(t_0) = v_0 = \frac{V(t_0, x_0)}{h(x_0)}, \qquad t \geq t_0, \tag{68}$$

with $(t_0, x_0) \in [0, \infty) \times \mathring{\mathcal{C}}$, is given by

$$v(t) = \begin{cases} \left[v_0^{1-\theta} - k(1-\theta)t\right]^{\frac{1}{1-\theta}}, & t_0 \leq t < t_1, \quad v_0 \neq 0, \\ 0, & t \geq t_1, \quad v_0 \neq 0, \\ 0, & t \geq t_0, \quad v_0 = 0, \end{cases} \tag{69}$$

where

$$t_1 = \frac{1}{k(1-\theta)} \left[\frac{V(t_0, x_0)}{h(x_0)}\right]^{1-\theta}. \tag{70}$$

Now, let $w : [0, \infty) \to \mathbb{R}$ be a continuously differentiable function such that

$$\dot{w}(t) \leq -kv^\theta(t), \qquad w(t_0) = \frac{V(t_0, x_0)}{h(x_0)}, \qquad t \geq t_0, \tag{71}$$

where $v(t)$ is given by Eq. (69). Then, it follows from Eqs. (68), (71), and the comparison lemma [8, p. 126] that

$$w(t) \leq v(t), \qquad t \geq 0. \tag{72}$$

Thus, it follows from Eqs. (9), (68), (69), (71), and (72), with $w(t) = \left[\frac{V(t, x(t))}{h(x(t))}\right]^{1-\theta}$, $t \geq t_0$, that

$$\alpha(\|x(t)\|) \leq \frac{V(t, x(t))}{h(x(t))} \leq v(t), \qquad t \geq t_0, \tag{73}$$

and hence, using Eqs. (8), (69), and (73),

$$x(t) = 0, \qquad t \geq t_1, \tag{74}$$

where t_1 is given in Eqs. (70).

Next, we prove that there exist a neighborhood $\mathcal{D}_0 \subseteq \mathring{\mathcal{C}}$ of the origin and a settling-time function $T : [0, \infty) \times \mathcal{D}_0 \to (t_0, \infty)$ such that Eq. (10) is satisfied. Since $s(t_0, t_0, x_0) = x_0$ and $s(\cdot, \cdot, \cdot)$ is continuous, $\inf\{t \in (t_0, \infty) : s(t, t_0, x_0) = 0\} > t_0$, $x_0 \in \mathcal{B}_\delta(0) \cap \mathring{\mathcal{C}} \setminus \{0\}$. Furthermore, it follows from Eq. (74) that $\inf\{t \in (t_0, \infty) : s(t, t_0, x) = 0\} < \infty$, $x_0 \in \mathcal{B}_\delta(0) \cap \mathring{\mathcal{C}} \setminus \{0\}$. Now, defining $\mathcal{D}_0 \triangleq \mathcal{B}_\delta(0) \cap \mathring{\mathcal{C}}$ and $T : [0, \infty) \times \mathcal{D}_0 \to (t_0, \infty)$ by Eqs. (69) and (70), (10) is immediate.

Next, we prove strong uniform finite-time stability of (1). For $t \geq T(t_0, x_0)$, uniform finite-time convergence of $x(t)$ to zero is immediate for all $(t_0, x_0) \in [0, \infty) \times \mathcal{D}_0$. Alternatively, for every $t < T(t_0, x_0)$ and $\varepsilon > 0$, there exists $\delta = \alpha^{-1}\left(\frac{\varepsilon^{1-\theta}}{k(1-\theta)}\right)$ such that if $\|x(t)\| \leq \alpha^{-1}(v(t)) < \varepsilon$, then $T(t_0, x_0) - t \le t_1 - t < \delta$, which proves uniform finite-time convergence of (1). Consequently, the time-varying nonlinear dynamical system (1) is strongly uniformly finite-time stable.

Lastly, it follows from the finite-time stability of Eq. (1) and Proposition 3.4 of [50] that $T(\cdot, \cdot)$ can be extended to $\overline{\mathbb{R}}_+$ and $T(t_0, 0) = 0$. Moreover, the right-hand side of Eq. (10) is jointly continuous at $(t_0, 0)$, $t_0 \in [0, \infty)$, and hence, by Proposition 3.4 of [50], it is jointly continuous on $[0, \infty) \times \mathcal{D}_0$. □

Proof of Theorem 4.1. It follows from Eq. (13) that $x = 0$ is a local minimizer of $V(t, x)$, $(t, x) \in [t_0, \infty) \times \mathcal{D}$, and hence $\left[\frac{\partial V(t,x)}{\partial t}, \frac{\partial V(t,x)}{\partial x}\right]^{\mathrm{T}} = 0$, for all $t \in [t_0, \infty)$ and $x = 0$, since $V(\cdot, \cdot)$ is continuously differentiable on $[t_0, \infty) \times \mathcal{D}$. Furthermore, it follows from Eq. (14) and the continuity of $\left\|\frac{\partial V(t,x)}{\partial x}\right\|$ on $[t_0, \infty) \times \mathcal{D}$ that for every $\varepsilon > 0$, $k > 0$, and $\theta \in (0, 1)$ there exist a compact set $\mathcal{M} \subset \mathcal{D}$ and a continuous function $g : \mathcal{M} \to [0, k)$, such that $0 \in \mathcal{M}$ and

$$\varepsilon \left\| \frac{\partial V(t,x)}{\partial x} \right\| \le g(x)(V(t,x))^{\theta}, \qquad (t,x) \in [t_0, \infty) \times \mathcal{M}. \tag{75}$$

Next, it follows from Eq. (14) that the total derivative of $V(\cdot, \cdot)$ along the trajectory of Eq. (6) is given by

$$\begin{aligned} &\frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x} F(t, x, \phi(x+e)) \\ &= \frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x} F(t, x, \phi(x)) + \frac{\partial V(t,x)}{\partial x}[F(t, x, \phi(x+e)) - F(t, x, \phi(x))] \\ &\le -k(V(t,x))^{\theta} + \frac{\partial V(t,x)}{\partial x}[F(t, x, \phi(x+e)) - F(t, x, \phi(x))], \end{aligned} \tag{76}$$

for all $(t, x, e) \in [t_0, \infty) \times \mathcal{D} \times \mathbb{R}^n$. Since $F(t, \cdot, \cdot)$ is jointly continuous in x and u uniformly in t, for all $t \in [t_0, \infty)$, and $\|\phi(\cdot)\|$ is continuous on $\mathcal{D}$, for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that if $\|e\| < \delta$, then $\|F(t, x, \phi(x+e)) - F(t, x, \phi(x))\| < \varepsilon$, for all $(t, x, e) \in [t_0, \infty) \times \mathcal{M} \times \mathcal{B}_{\delta}(0)$. Therefore,

$$\frac{\partial V(x)}{\partial x} F(t, x, \phi(x+e)) \le -k(V(x))^{\theta} + \varepsilon \left\| \frac{\partial V(t,x)}{\partial x} \right\|, \quad (t, x, e) \in [t_0, \infty) \times \mathcal{M} \times \mathcal{B}_{\delta}(0), \tag{77}$$

and it follows from the continuity of $g(\cdot)$ on $\mathcal{M}$, Eq. (75), and Theorem 2.13 of [8] that

$$\frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x} F(t, x, \phi(x+e)) \le (\overline{g} - k)(V(t,x))^{\theta}, \quad (t, x, e) \in [t_0, \infty) \times \mathcal{M} \times \mathcal{B}_{\delta}(0), \tag{78}$$

where $\overline{g} \triangleq \max_{x \in \mathcal{M}} g(x)$ and $\overline{g} < k$.

Since $e(t) \to 0$ as $t \to \infty$ uniformly in t_0, for every $\varepsilon > 0$ there exists $T(\varepsilon) > t_0$, such that $\|e(t)\| < \delta(\varepsilon)$ for all $t > T(\varepsilon)$. Therefore, for every initial condition $x_0 \in \mathcal{M}$, it holds that

$$\frac{\partial V(t, x(t))}{\partial t} + \frac{\partial V(t, x(t))}{\partial x} F(t, x(t), \phi(x(t) + e(t))) \le (\overline{g} - k)(V(t, x(t)))^{\theta}, \qquad t > T(\varepsilon), \tag{79}$$

along the trajectory of Eq. (6). Uniform finite-time convergence of $x(\cdot)$ to zero now follows from Eqs. (13) and (79) by proceeding as in the proof of Theorem 3.1. □

Proof of Theorem 5.1. It follows from Assumption 5.1 that there exists a diffeomorphism $\mathcal{T}(x) = [\eta^{\mathrm{T}}, \xi^{\mathrm{T}}]^{\mathrm{T}}$ such that $\mathcal{T}(0) = 0$, Eq. (17) is satisfied, and Eq. (16) is equivalent to Eqs. (18) and (19). The nonlinear dynamical system Eq. (18) with $\xi = \phi(\eta)$ is equivalent to Eq. (22) and it follows from Lemma 5.1 that the equilibrium point $\eta(t) \equiv 0$, $t \geq t_0$, of Eq. (22) is asymptotically stable and $\eta(t) \in \mathring{\mathcal{C}}_\eta$, $t \geq t_0$.

Consider the nonlinear dynamical system

$$\dot{\eta}(t) = f_\eta(\eta(t), \phi(\eta(t) + e_\eta(t))) + \delta_\eta(\eta(t), \phi(\eta(t) + e_\eta(t))),$$
$$\eta(t_0) = [I_{n-m}, 0_{(n-m)\times m}]\mathcal{T}(x_0), \qquad t \geq t_0, \tag{80}$$

and recall that by assumption, $e_\eta(t) \in \mathcal{B}_{\delta_{e_\eta}}(0)$, $t \geq t_0$, and $\lim_{t\to\infty} e_\eta(t) = 0$ uniformly in t_0. Thus, it follows from Eqs. (23), (24), and Theorem 4.2 that there exists a compact set $\mathcal{M}_\eta \subset \mathring{\mathcal{C}}_\eta$, such that $0 \in \mathcal{M}_\eta$ and if $\eta(t_0) \in \mathcal{M}_\eta$, then $\eta(t) \to 0$ as $t \to \infty$ uniformly in t_0. Furthermore, proceeding as in the proofs of Theorems 4.1 and 3.1, one can show that $\eta(t) \in \mathcal{M}_\eta \subset \mathring{\mathcal{C}}_\eta$, $t \geq t_0$. Thus, the observer-based feedback control $\xi = \phi(\eta + e_\eta)$ guarantees that the solution of the dynamical system (18) asymptotically converges to $\eta(t) \equiv 0$ and lays in the constraint set $\mathring{\mathcal{C}}_\eta$ at all times.

Let $[\eta^{\mathrm{T}}(t), \xi^{\mathrm{T}}(t)]^{\mathrm{T}}$, $t \geq t_0$, denote the solution of Eqs. (18) and (19) with observer-based feedback control law (34). Next, we prove that $(\eta(t), \xi(t)) \in \mathring{\mathcal{C}}_\eta \times \mathring{\mathcal{C}}_\xi$, $t \geq t_0$, $\|\xi(t) - \phi(\eta(t))\| \to 0$ as $t \to t_1$, for some finite-time $t_1 \geq t_0$, and $\|\xi(t) - \phi(\eta(t))\|$ is sufficiently small, that is, $\|\xi(t) - \phi(\eta(t))\| < \|\phi(\eta(t) + e_\eta(t)) - \phi(\eta(t))\|$. Specifically, since $\|\phi(\cdot)\|$ is continuous on $\mathring{\mathcal{C}}_\eta$ and $e_\eta(t) \in \mathcal{B}_{\delta_{e_\eta}}(0)$, $t \geq t_0$, in the following we prove that the solution $[\eta^{\mathrm{T}}(t), \xi^{\mathrm{T}}(t)]^{\mathrm{T}}$, $t \geq t_0$, of Eqs. (18) and (19) is such that $z(t) = (\xi(t) - \phi(\eta(t))) \in \mathcal{N}_z$, $t \geq t_0$, where $\mathcal{N}_z \subseteq \mathring{\mathcal{C}}_z$ is an open connected set such that $0 \in \mathcal{N}_z$, and $z(t) \to 0$ as $t \to t_1$ uniformly in t_0.

It follows from Eqs. (18) and (19) that

$$\dot{z}(t) = f_\xi(\eta(t), \xi(t)) - \frac{\partial \phi(\eta(t))}{\partial \eta} f_\eta(\eta(t), \xi(t)) + G(\eta(t), \xi(t))E(\eta(t), \xi(t))u(t)$$
$$+\delta(t, \eta(t), \xi(t), u(t)), \qquad z(t_0) = \xi(t_0) - \phi(\eta(t_0)), \qquad t \geq t_0, \tag{81}$$

and Eqs. (81) with $u = \psi(\eta + e_\eta, \xi + e_\xi, w)$, where $\psi(\cdot, \cdot, \cdot)$ is given by Eq. (26), is equivalent to

$$\dot{z}(t) = G(\eta(t), \xi(t))w(t) + \Delta(t, \eta(t), \xi(t), w(t)) + \Gamma(t, \eta(t), \xi(t), w(t), e_\eta(t), e_\xi(t))$$
$$-\Gamma(t, \eta(t), \xi(t), w(t), 0, 0), \qquad z(t_0) = \xi(t_0) - \phi(\eta(t_0)), \qquad t \geq t_0, \tag{82}$$

where $\Delta(\cdot, \cdot, \cdot, \cdot)$ is given by Eq. (28) and

$$\Gamma(t, \eta, \xi, w, e_\eta, e_\xi) \triangleq \delta(t, \eta, \xi, \psi(\eta + e_\eta, \xi + e_\xi, w)) - G(\eta, \xi)E(\eta, \xi)\lambda(\eta + e_\eta, \xi + e_\xi)$$
$$+G(\eta, \xi)E(\eta, \xi)E^{-1}(\eta + e_\eta, \xi + e_\xi)w,$$
$$(t, \eta, \xi, w, e_\eta, e_\xi) \in [t_0, \infty) \times \mathbb{R}^{n-m} \times \mathbb{R}^m \times \mathbb{R}^m \times \mathcal{B}_{\delta_{e_\eta}}(0) \times \mathcal{B}_{\delta_{e_\xi}}(0). \tag{83}$$

Now, consider the nonlinear dynamical system (82) with $w = \gamma(\eta, \xi)$ and $e_\eta = e_\xi = 0$, let

$$V(z) = \frac{\|z\|^2}{h_z(z)}, \qquad z \in \mathcal{N}_z, \tag{84}$$

and define

$$\delta_z \triangleq \min_{i=1,\ldots,m} \min_{z\in\hat{\mathcal{M}}_z} (1-k_i)\left|\frac{2}{\sqrt{h_z(z)}} - \frac{z_i}{\sqrt{h_z^3(z)}}\frac{\partial h_z(z)}{\partial z_i}\right|, \tag{85}$$

where $\hat{\mathcal{M}}_z \subset \mathcal{N}_z$ is a compact set, whose smallest open cover is $\mathcal{N}_z$ [55, pp. 618-619]; the existence of $\delta_z > 0$ is guaranteed by Theorem 2.13 of [8], since $\left|\frac{2}{\sqrt{h_z(z)}} - \frac{z_i}{\sqrt{h_z^3(z)}}\frac{\partial h_z(z)}{\partial z_i}\right|$ is continuous on $\hat{\mathcal{M}}_z$.

Since $\sum_{i=1}^m z_i^{2\theta} \geq \left(\sum_{i=1}^m z_i^2\right)^\theta$, $\theta \in (0, 1)$, it follows from Eqs. (84), (82), (33), and (31) that

$$\begin{aligned}
\dot{V}(z) &= \sum_{i=1}^m \left[\frac{2}{h_z(z)} - \frac{z_i}{h_z^2(z)}\frac{\partial h_z(z)}{\partial z_i}\right] z_i\big[g_i(\eta,\xi)\gamma_i(\eta,\xi) + \Delta_i(t,\eta,\xi,\gamma(\eta,\xi))\big] \\
&\leq \sum_{i=1}^m g_i(\eta,\xi)\Bigg[\left(\frac{2}{h_z(z)} - \frac{z_i}{h_z^2(z)}\frac{\partial h_z(z)}{\partial z_i}\right) z_i\gamma_i(\eta,\xi) \\
&\quad + \left|\frac{2}{h_z(z)} - \frac{z_i}{h_z^2(z)}\frac{\partial h_z(z)}{\partial z_i}\right| |z_i|(\rho_i(\eta,\xi) + k_i|\gamma_i(\eta,\xi)|)\Bigg] \\
&\leq \sum_{i=1}^m \left|\frac{2}{h_z(z)} - \frac{z_i}{h_z^2(z)}\frac{\partial h_z(z)}{\partial z_i}\right| g_i(\eta,\xi)|z_i|[\rho_i(\eta,\xi) + (k_i-1)\beta_i(\eta,\xi)] \\
&\leq -cg_0\delta_z\sqrt{V(z)},
\end{aligned} \tag{86}$$

where $g_i(\cdot,\ \cdot)$ denotes the element on the ith row and ith column of $G(\cdot,\ \cdot)$ and g_0 is such that $G(\eta,\ \xi) \geq g_0 I_m > 0$, $(\eta, \xi) \in \mathbb{R}^{n-m} \times \mathbb{R}^m$. Hence, Eq. (13) is satisfied with $\alpha(\|z\|) = \beta(\|z\|) = \|z\|^2|h_z(z)|^{-1}$ and Eq. (14) is satisfied with $k = cg_0\delta_z$ and $\theta = \frac{1}{2}$. Since $e_\eta(t) \to 0$ and $e_\xi(t) \to 0$ as $t \to \infty$, and $V(\cdot)$ is a class $\mathcal{K}$ function, it follows from Eq. (84), (86), and Theorem 4.1 that there exists a compact set $\mathcal{M}_z \subseteq \hat{\mathcal{M}}_z \subset \mathcal{N}_z \subseteq \mathring{\mathcal{C}}_z$, such that $0 \in \mathcal{M}_z$ and if $z(t_0) \in \mathcal{M}_z$, then the solution $z(\cdot)$ of Eq. (82) is such that $z(t) \to 0$ as $t \to t_1$ uniformly in t_0, for some finite-time $t_1 > t_0$. Furthermore, proceeding as in the proofs of Theorems 4.1 and 3.1, one can prove that $z(t) \in \mathcal{M}_z \subset \mathcal{N}_z$, $t \geq t_0$. The result now follows recalling that the nonlinear dynamical system Eq. (16) is equivalent to Eqs. (18) and (19) or, alternatively, Eqs. (18) and (81), and if $(\eta(t), z(t)) \in \mathring{\mathcal{C}}_\eta \times \mathring{\mathcal{C}}_z$, then $(\eta(t), \xi(t)) \in \mathring{\mathcal{C}}_\eta \times \mathring{\mathcal{C}}_\xi$ and hence, $x(t) \in \mathring{\mathcal{C}}$. □

References

[1] N. Hovakimyan, C. Cao, $\mathcal{L}_1$ Adaptive Control Theory: Guaranteed Robustness with Fast Adaptation, Society for Industrial and Applied Mathematics, Boston, MA, 2010.

[2] P. Ioannou, B. Fidan, Adaptive Control Tutorial, Society for Industrial and Applied Mathematics, Boston, MA, 2006.

[3] U. Itkis, Control Systems of Variable Structure, Wiley, New York, NY, 1976.

[4] V. Utkin, Variable structure systems with sliding modes, IEEE Trans. Autom. Control 22 (2) (1977) 212–222, doi:10.1109/TAC.1977.1101446.

[5] J.-J. Slotine, W. Li, Applied Nonlinear Control, Prentice Hall, Princeton, NJ, 1991.

[6] H.K. Khalil, Nonlinear Systems, Pearson Education, Prentice Hall, Princeton, NJ, 2002.

[7] W. Perruquetti, J.P. Barbot, Sliding Mode Control In Engineering, CRC Press, New York, NY, 2002.

[8] W.M. Haddad, V. Chellaboina, Nonlinear Dynamical Systems and Control: A Lyapunov-Based Approach, Princeton University Press, Princeton, NJ, 2008.

[9] A. Filippov, Differential Equations with Discontinuous Right-Hand Sides, Mathematics and its Applications (Soviet Series), Kluwer Academic Publishers, Dordrecht, The Netherlands, 1988.
[10] L. Friedman, A. Levant, Higher order sliding modes, in: W. Perruquetti, J.P. Barbot (Eds.), Sliding Mode Control in Enginkeering, Marcel Dekker, Inc., Basel, Switzerland, 2002.
[11] Y. Shtessel, J. Buffington, S. Banda, Tailless aircraft flight control using multiple time scale reconfigurable sliding modes, IEEE Trans. Control Syst. Technol. 10 (2) (2002) 288–296, doi:10.1109/87.987075.
[12] G.A. de Andrade, D.J. Pagano, J.L. Guzmn, M. Berenguel, I. Fernndez, F.G. Acin, Distributed sliding mode control of pH in tubular photobioreactors, IEEE Trans. Control Syst. Technol. 24 (4) (2016) 1160–1173, doi:10.1109/TCST.2015.2480840.
[13] F. Betin, D. Pinchon, G.A. Capolino, A time-varying sliding surface for robust position control of a DC motor drive, IEEE Trans. Ind. Electron. 49 (2) (2002) 462–473, doi:10.1109/41.993280.
[14] S. Soylu, B.J. Buckham, R.P. Podhorodeski, A chattering-free sliding-mode controller for underwater vehicles with fault-tolerant infinity-norm thrust allocation, Ocean Eng. 35 (16) (2008) 1647–1659. doi: https://doi.org/10.1016/j.oceaneng.2008.07.013.
[15] Y. Jin, P.H. Chang, M. Jin, D.G. Gweon, Stability guaranteed time-delay control of manipulators using nonlinear damping and terminal sliding mode, IEEE Trans. Ind.Electron. 60 (8) (2013) 3304–3317, doi:10.1109/TIE.2012.2200216.
[16] L. Friedman, A. Levant, Introduction to high-order sliding modes, in: W. Perruquetti, J.P. Barbot (Eds.), Sliding Mode Control in Engineering, Marcel Dekker, Inc., Basel, Switzerland, 2002. ch. 1.
[17] A. Levant, Higher-order sliding modes, differentiation and output-feedback control, Int. J. Control 76 (9-10) (2003) 924–941, doi:10.1080/0020717031000099029.
[18] A. Pisano, E. Usai, Sliding mode control: a survey with applications in math, Math. Comput. Simul. 81 (5) (2011) 954–979. https://doi.org/10.1016/j.matcom.2010.10.003.
[19] J.Y. Hung, W. Gao, J.C. Hung, Variable structure control: a survey, IEEE Trans. Ind. Electron. 40 (1) (1993) 2–22, doi:10.1109/41.184817.
[20] G.P. Incremona, M. Rubagotti, A. Ferrara, Sliding mode control of constrained nonlinear systems, IEEE Trans. Autom. Control PP (99) (2016), doi:10.1109/TAC.2016.2605043. 1–1
[21] M. Innocenti, M. Falorni, State constrained sliding mode controllers, in: Proceedings of American Control Conference, vol. 1, 1998, pp. 104–108, doi:10.1109/ACC.1998.694638.
[22] F. Dinuzzo, A second order sliding mode controller with polygonal constraints, in: Proceedings of IEEE Conference on Decision and Control, 2009, pp. 6715–6719, doi:10.1109/CDC.2009.5400384.
[23] A. Ferrara, G.P. Incremona, M. Rubagotti, Third order sliding mode control with box state constraints, in: Proceedings of IEEE Conference on Decision and Control, 2014, pp. 4727–4732, doi:10.1109/CDC.2014.7040126.
[24] M. Rubagotti, D.M. Raimondo, A. Ferrara, L. Magni, Robust model predictive control with integral sliding mode in continuous-time sampled-data nonlinear systems, IEEE Trans. Autom. Control 56 (3) (2011) 556–570, doi:10.1109/TAC.2010.2074590.
[25] J.M. Daly, D.W.L. Wang, Output feedback sliding mode control in the presence of unknown disturbances, Syst. Control Lett. 58 (3) (2009) 188–193. https://doi.org/10.1016/j.sysconle.2008.10.007.
[26] A. Seuret, C. Edwards, S.K. Spurgeon, E. Fridman, Static output feedback sliding mode control design via an artificial stabilizing delay, IEEE Trans. Autom. Control 54 (2) (2009) 256–265, doi:10.1109/TAC.2008.2010889.
[27] S. Dadras, H.R. Momeni, Fractional-order dynamic output feedback sliding mode control design for robust stabilization of uncertain fractional-order nonlinear systems, Asian J. Control 16 (2) (2014) 489–497, doi:10.1002/asjc.711.
[28] J.-L. Chang, Sliding mode control design for mismatched uncertain systems using output feedback, Int. J. Control Autom. Syst. 14 (2) (2016) 579–586, doi:10.1007/s12555-014-0370-3.
[29] L. Wu, Y. Gao, J. Liu, H. Li, Event-triggered sliding mode control of stochastic systems via output feedback, Automatica 82 (2017) 79–92. https://doi.org/10.1016/j.automatica.2017.04.032.
[30] H. Li, Y. Gao, P. Shi, H.K. Lam, Observer-based fault detection for nonlinear systems with sensor fault and limited communication capacity, IEEE Trans. Autom. Control 61 (9) (2016) 2745–2751, doi:10.1109/TAC.2015.2503566.
[31] C.P. Bechlioulis, G.A. Rovithakis, Adaptive control with guaranteed transient and steady state tracking error bounds for strict feedback systems, Automatica 45 (2) (2009) 532–538. https://doi.org/10.1016/j.automatica.2008.08.012.
[32] C. Hua, L. Zhang, X. Guan, Reduced-order observer-based output feedback control of nonlinear time-delay systems with prescribed performance, Int. J. Syst. Sci. 47 (6) (2016) 1384–1393, doi:10.1080/00207721.2014.927937.

[33] K.P. Tee, S.S. Ge, E.H. Tay, Barrier Lyapunov functions for the control of output-constrained nonlinear systems, Automatica 45 (4) (2009) 918–927. https://doi.org/10.1016/j.automatica.2008.11.017.
[34] B. Ren, S.S. Ge, K.P. Tee, T.H. Lee, Adaptive neural control for output feedback nonlinear systems using a barrier Lyapunov function, IEEE Trans. Neural Netw. 21 (8) (2010) 1339–1345, doi:10.1109/TNN.2010.2047115.
[35] Y.J. Liu, S. Lu, D. Li, S. Tong, Adaptive controller design-based ABLF for a class of nonlinear time-varying state constraint systems, IEEE Trans. Syst. Man Cybern. Syst. 47 (7) (2017) 1546–1553, doi:10.1109/TSMC.2016.2633007.
[36] Y.-J. Liu, S. Tong, Barrier Lyapunov functions for Nussbaum gain adaptive control of full state constrained nonlinear systems, Automatica 76 (2017) 143–152. https://doi.org/10.1016/j.automatica.2016.10.011.
[37] Y.J. Liu, J. Li, S. Tong, C.L.P. Chen, Neural network control-based adaptive learning design for nonlinear systems with full-state constraints, IEEE Trans. Neural Netw. Learn. Syst. 27 (7) (2016) 1562–1571, doi:10.1109/TNNLS.2015.2508926.
[38] B. Walcott, S.H. Żak, State observation of nonlinear uncertain dynamical systems, IEEE Trans. Autom. Control 32 (2) (1987) 166–170, doi:10.1109/TAC.1987.1104530.
[39] S.H. Żak, S. Hui, Output feedback variable structure controllers and state estimators for uncertain/nonlinear dynamic systems, IEE Proc. D - Control Theory Appl. 140 (1) (1993) 41–50, doi:10.1049/ip-d.1993.0006.
[40] C. Edwards, S.K. Spurgeon, C.P. Tan, N. Patel, Sliding-Mode Observers, Springer, London, UK, pp. 221–242.
[41] S. Hui, S.H. Żak, Observer design for systems with unknown inputs, Int. J. Appl. Math. Comput. Sci. 15 (4) (2005) 431–446.
[42] S.K. Spurgeon, Sliding mode observers: a survey, Int. J. Syst. Sci. 39 (8) (2008) 751–764, doi:10.1080/00207720701847638.
[43] C. Edwards, S.K. Spurgeon, R.J. Patton, Sliding mode observers for fault detection and isolation, Automatica 36 (4) (2000) 541–553, doi:10.1016/S0005-1098(99)00177-6.
[44] A. Steinberg, M. Corless, Output feedback stabilization of uncertain dynamical systems, IEEE Trans. Autom. Control 30 (10) (1985) 1025–1027, doi:10.1109/TAC.1985.1103804.
[45] F.E. Thau, Observing the state of non-linear dynamic systems, Int. J. Control 17 (3) (1973) 471–479, doi:10.1080/00207177308932395.
[46] S.R. Kou, D.L. Elliott, T.J. Tarn, Exponential observers for nonlinear dynamic systems, Inf. Control 29 (3) (1975) 204–216.
[47] S. Bonnabel, P. Martin, P. Rouchon, Non-linear symmetry-preserving observers on Lie groups, IEEE Trans. Autom. Control 54 (7) (2009) 1709–1713, doi:10.1109/TAC.2009.2020646.
[48] A. Astolfi, D. Karagiannis, R. Ortega, Nonlinear and Adaptive Control with Applications, Springer, London, UK, 2007.
[49] D. Simon, Optimal State Estimation: Kalman, $\mathcal{H}_\infty$, and Nonlinear Approaches, Wiley, Toronto, ON, 2006.
[50] W.M. Haddad, A. L'Afflitto, Finite-time partial stability and stabilization, and optimal feedback control, J. Frankl. Inst. 352 (6) (2015) 2329–2357. https://doi.org/10.1016/j.jfranklin.2015.03.022.
[51] R. Agarwal, V. Lakshmikantham, Uniqueness and Nonuniqueness Criteria for Ordinary Differential Equations, World Scientific, Singapore, 1993.
[52] M. Kawski, Stabilization of nonlinear systems in the plane, Syst. Control Lett. 12 (1989) 169–175.
[53] T. Yoshizawa, Stability theory by Liapunov's second methods, Publications of the Mathematical society of Japan, Mathematical Society of Japan, Tokyo, Japan, 1966.
[54] W.M. Haddad, S.G. Nersesov, L. Du, Finite-time stability for time-varying nonlinear dynamical systems, in: S. Sivasundaram, J.V. Devi, Z. Drici, F. Mcrae (Eds.), Advances in Nonlinear Analysis: Theory Methods and Application, Cambridge Scientific Publishers, Cambridge, UK, 2009, pp. 139–150.
[55] E. Kreyszig, Introductory Functional Analysis with Applications, Wiley, New York, NY, 1989.
[56] S. Gonnord, N. Tosel, Calcul différentiel: Thèmes d'analyse pour l'agrégation, Ellipses, Paris, France, 1998.
[57] M.Z. Romdlony, B. Jayawardhana, Uniting control Lyapunov and control barrier functions, in: Proceedings of IEEE Conference on Decision and Control, 2014, pp. 2293–2298, doi:10.1109/CDC.2014.7039737.
[58] Z. Artstein, Stabilization with relaxed controls, Nonlinear Anal.: Theory Methods Appl. 7 (11) (1983) 1163–1173. https://doi.org/10.1016/0362-546X(83)90049-4.
[59] K. Keesman, System Identification: An Introduction, Springer, London, UK, 2011.
[60] R. Pintelon, J. Schoukens, System Identification: A Frequency Domain Approach, Wiley, New York, NY, 2012.
[61] E. Lavretsky, K. Wise, Robust and Adaptive Control: With Aerospace Applications, Springer, London, UK, 2012.
[62] A. L'Afflitto, A Mathematical Perspective on Flight Dynamics and Control, Springer, London, UK, 2017.

[63] V.K. Madyastha, V.C. Ravindra, S. Mallikarjunan, A. Goyal, Extended Kalman filter vs. error state Kalman filter for aircraft attitude estimation, in: Proceedings of AIAA Guidance, Navigation, and Control Conference, 2011, pp. 1–23, doi:10.1109/CDC.2014.7039737.

[64] J.-J. Slotine, S.S. Sastry, Tracking control of non-linear systems using sliding surfaces, with application to robot manipulators, Int. J. Control 38 (2) (1983) 465–492.

[65] J.A. Burton, A.S.I. Zinober, Continuous approximation of variable structure control, Int. J. Syst. Sci. 17 (6) (1986) 875–885, doi:10.1080/00207728608926853.

[66] G. Bartolini, A. Ferrara, E. Usani, Chattering avoidance by second-order sliding mode control, IEEE Trans. Autom. Control 43 (2) (1998) 241–246, doi:10.1109/9.661074.

[67] A. Levant, Sliding order and sliding accuracy in sliding mode control, Int. J.Control 58 (6) (1993) 1247–1263.