Necessary Conditions for Control Effort Minimization of Euler-Lagrange Systems

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The $\mathcal{L}_1$ norm of the control vector is a suitable measure for the effort needed to control a vehicle, since, in several cases of practical interest, it can provide accurate estimate of the fuel consumption. In this paper, we address the problem of minimizing the weighted sum of the $\mathcal{L}_1$ norms of the control vectors of N vehicles moving in formation. Specifically, modeling each agent as a six degrees-of-freedom rigid body subject to external forces and moments, and holonomic and nonholonomic constraints, we give necessary conditions for minimizing the formation's control effort. In addition, we provide necessary conditions for the existence of singular controls for the abnormal and the normal optimal control problems. Two of our main results show that singular controls have order of singularity equal to one and are analytical in the junction between singular and non-singular arcs. In order to highlight the framework presented in this paper, we provide a numerical example concerning a formation of F-16 performing an Immelmann turn.

I. Introduction

Multi-agent formations are particularly valuable not only because they combine the strength, the speed, the precision, and the repeatability of each formation member, but also because formations guarantee redundancy, increased coverage, ease of reconfigurability, and the possibility of implementing distributed sensors and actuators [1, 2]. The relevance of autonomous vehicles formations is evidenced by the increase of space [3], aerial [4, 5], marine [6], terrestrial [7, 8], and industrial [7] applications they are employed in. Path planning is one of the main problems to address when designing missions involving multiple vehicles, since trajectories must meet the mission objectives and should be optimized with respect to some performance measure capturing minimum time or minimum control effort. The $\mathcal{L}_1$ norm of the control vector is a suitable measure of the effort needed to control aerospace vehicles. In fact, this performance measure allows effectively estimating the fuel consumption of unmanned aerial vehicles equipped with conventional fuel-based propulsion systems [9] and spacecraft equipped with rigidly mounted conventional thrusters [10].

Trajectories that guarantee minimum control effort for multi-agent formations are made of three segments, namely, *maximum thrust arcs*, *minimum* (or *null*) *thrust arcs*, and *singular arcs*. Specifically, optimal control vectors have maximum magnitude along maximum thrust arcs and minimum magnitude along null thrust arcs. Necessary conditions for local minima such as the Pontryagin's principle and the Legendre necessary condition do not provide any information about optimal control vectors along singular arcs [11]. In this paper, we address the problem of minimizing the weighted sum of the $\mathcal{L}_1$ norms of the control vectors of N vehicles moving in formation. Specifically, we prove necessary conditions for the existence of maximum thrust, null thrust, and singular control arcs for a formation of vehicles modeled as rigid bodies subject to external forces and moments, which are functions of the agents' position and velocity, and holonomic and nonholonomic constraints. Two of the most relevant results of this paper are that singular controls have order of singularity equal to one and are analytical in the junction between singular and non-singular arcs.

The study of singular control problems stemmed from navigation applications in aerospace engineering. Specifically, the existence of singular arcs was first hinted by studying a candidate fuel-optimal trajectory for space navigation, known as Lawden's spiral [12], whose optimality was disproven in [13]. Concerning the study of singular controls in space applications, it is worth to mention references [14] and [15], where the optimal rotation problem for symmetric spacecraft is completely addressed. Among the many works on singular controls for aircraft trajectories, we recall references [16] and [17], where time optimal trajectories are analyzed. Singular controls are also commonly found in optimal trajectories for rockets [18, 19] and robotic manipulators [20, 21]. Finally, an analysis of singular arcs in the time optimal problem for autonomous underwater vehicles can be found in reference [22]. None of the works currently available in the literature, however, addresses the singular control problem for the control effort minimization problem of systems of N rigid bodies subject to external forces and moments, and holonomic and nonholonomic constraints.

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A study of the control effort minimization problem for formations of N vehicles modeled as rigid bodies has already been addressed in [11], where the authors do not consider the singular control problem and prove necessary conditions for the existence of maximum and null thrust arcs under the simplifying assumptions that the system constraints are holonomic. Moreover, results proven in [11] for the normal optimal control problem are achieved applying the Pontryagin minimum principle to an upper bound of the Hamiltonian function. In this paper, exploiting some properties of singular controls, we prove necessary conditions for the existence of solutions of the normal optimal control problem that are less restrictive than those presented in [11].

In order to illustrate the framework developed in this paper, we provide a numerical example concerning a formation of two F-16 aircraft [23] performing an Immelmann turn. Numerical approaches to the optimal control problem are based on first order necessary conditions and therefore are not designed to find candidate optimal singular controls [24–26]. In order to partly address this problem, the authors in [27] recommend to isolate, possibly analytically, singular controls and attempt a numerical solution of the optimal control problem along singular arcs. However, the accuracy of results is not guaranteed [27]. The theoretical framework developed in this paper can be applied as follows. Firstly, a numerical solution of the control effort minimization problem is attempted without prior knowledge of the existence of singular controls. Next, the necessary conditions proven in this paper are applied to verify the validity of numerical results and identify singular controls. Successively, numerical simulations are iterated along singular arcs. The application of the necessary conditions proven in this paper is straightforward since, in the case of the abnormal optimal control problem, one needs to verify three scalar nonlinear differential equations and the positive definiteness of a three-by-three matrix function, whereas, in the case of the normal optimal control problem, one needs to verify a scalar nonlinear differential equation and the positiveness of a scalar function.

The contents of this paper are as follows. In Section II, we establish notation and define the physical variables needed to formulate the control effort minimization problem. In Section III, we provide sufficient conditions to write a minimal set of equations of motion of a system of N rigid bodies and we formally state the path planning optimization problem addressed in this paper. Section IV provides the necessary mathematical background and Section V discusses necessary conditions for the solution of the control effort minimization problem. In Section VI, we apply theoretical results proven in Section V to solve the control effort minimization problem for a formation of F-16 performing an Immelmann turn. Finally, in Section VII, we draw conclusions and highlight future research directions. Due to space limitations, we omit all the proofs in this paper. Detailed proofs of our results are provided in [28].

II. Notations and Definitions

The mathematical notation used in this paper is fairly standard. The symbol $\mathbb{N}$ denotes the set of positive integers, $\mathbb{R}$ the set of real numbers, $\overline{\mathbb{R}}_+$ the set of non-negative real numbers, $\mathbb{R}^n$ the set of $n \times 1$ column vectors on the field of real numbers, $\mathbb{R}^{n\times m}$ the set of real $n \times m$ matrices, and $\mathrm{int}(\mathcal{A})$ is the interior of $\mathcal{A} \subset \mathbb{R}^{n\times m}$. Given $t \in \mathbb{R}$, $\mathcal{I}(t)$ denotes a neighborhood of t. If $f : (t_1, t_2) \to \mathbb{R}^n$ is continuous with its first p derivatives, then $f(\cdot) \in C^p(t_1, t_2)$. The zero vector in $\mathbb{R}^n$ is denoted by 0_n or 0, the zero matrix in $\mathbb{R}^{n\times m}$ is denoted by $0_{n\times m}$ or 0, and the identity matrix in $\mathbb{R}^{n\times n}$ is denoted by I_n or I. Given $x \in \mathbb{R}^3$ such that $x \triangleq [x_1,\, x_2,\, x_3]^{\mathrm{T}}$, we define

$$x^{\times} \triangleq \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}.$$

The nullspace of $A \in \mathbb{R}^{n\times m}$ is denoted by $\mathcal{N}(A)$, the transpose of A is denoted by A^{T}, and the transposed inverse of $B \in \mathbb{R}^{n\times n}$ is denoted by $B^{-\mathrm{T}}$. The matrix B is non-negative (respectively, positive) definite, that is $B \geq 0$ (respectively, $B > 0$), if $B = B^{\mathrm{T}}$ and the eigenvalues of B are non-negative (respectively, positive.) Given $N \triangleq \begin{bmatrix} 0_{m\times m} & 0_{m\times(n-m)} \\ 0_{(n-m)\times m} & I_{n-m} \end{bmatrix}$ and $C \in \mathbb{R}^{l\times n}$, we define the operator $\mathcal{O} : \mathbb{R}^{l\times n} \mapsto \mathbb{R}^{l\times(n-m)}$ such that $D = \mathcal{O}N(C)$, where $CN = \begin{bmatrix} 0_{l\times m} & D \end{bmatrix}$. For conciseness, if $l = n$, then we denote $(\mathcal{O}N(C^{\mathrm{T}}))^{\mathrm{T}}$ by $\mathcal{O}^{\mathrm{T}}N(C)$ and $\mathcal{O}N((\mathcal{O}N(C))^{\mathrm{T}})$ by $\mathcal{O}N(C)\mathcal{O}$.

Time is the only independent variable used in this paper and is denoted by t, and we let $t \in [t_1, t_2] \subset \mathbb{R}$, where t_1 is fixed and assigned a priori, and t_2 is found by solving the optimization problem. Given a system of N rigid bodies, the components of $q : [t_1, t_2] \to \mathbb{R}^{\gamma}$ are the *independent generalized coordinates*, which uniquely identify the system configuration at every $t \in [t_1, t_2]$. Specifically, the *position vector* of the center of mass of the αth rigid body, $\alpha = 1, \ldots, N$, in a given inertial reference frame is denoted by $r_\alpha : \mathbb{R}^{\gamma} \to \mathbb{R}^3$, the *attitude vector* of the αth rigid body in modified Rodrigues parameters [29] is denoted by $\sigma_\alpha : \mathbb{R}^{\gamma} \to \mathbb{R}^3$, the *state vector* of the αth rigid body is denoted by $x_\alpha \triangleq [r_\alpha^{\mathrm{T}},\, \sigma_\alpha^{\mathrm{T}}]^{\mathrm{T}}$, and the *system's configuration* at time t is given by $[x_1^{\mathrm{T}}(q(t)),\, \ldots,\, x_N^{\mathrm{T}}(q(t))]^{\mathrm{T}}$.

The mapping $x_\alpha(q(t))$, $t \in [t_1, t_2]$, is defined as the *trajectory* of the αth rigid body and the mapping $x_\alpha(q(t))$, $t \in [\hat{t}_1, \hat{t}_2] \subseteq [t_1, t_2]$, is defined as the *arc* of the αth body between $\hat{t}_1$ and $\hat{t}_2$.

The components of

$$q_{\rm dot}(t) \triangleq D(q)\dot{q}(t) + d(q) \tag{1}$$

are the *quasi-velocities*, where $D : \mathbb{R}^\gamma \to \mathbb{R}^{\gamma\times\gamma}$ is invertible and continuously differentiable and $d : \mathbb{R}^\gamma \to \mathbb{R}^\gamma$ is continuously differentiable. Detailed discussions about quasi-velocities can be found in [30, 31]. The vector $v_\alpha(q, q_{\rm dot}) \triangleq \dot{r}_\alpha(q)$, $\alpha = 1, \ldots, N$, denotes the *velocity* of the center of mass of the αth rigid body and $\omega_\alpha(q, q_{\rm dot}) \triangleq R_{\rm rod}^{-1}(\sigma_\alpha)\dot{\sigma}_\alpha(q)$ denotes the *angular velocity* of the αth rigid body in a principal body reference frame, where $R_{\rm rod}(\sigma_\alpha) \triangleq \frac{1}{4}(1 - \sigma_\alpha^{\rm T}\sigma_\alpha)I_3 + \frac{1}{2}\sigma_\alpha^\times + \frac{1}{2}\sigma_\alpha\sigma_\alpha^{\rm T}$ [29]. Lastly, the *augmented state vector* of the αth rigid body is denoted by $\tilde{x}_\alpha \triangleq [r_\alpha^{\rm T}, v_\alpha^{\rm T}, \sigma_\alpha^{\rm T}, \omega_\alpha^{\rm T}]^{\rm T}$, $\alpha = 1, \ldots, N$.

For a given set of real constants $\rho_{\alpha,1}$, $\rho_{\alpha,2}$, $\rho_{\alpha,3}$, and $\rho_{\alpha,4}$, $\alpha = 1, \ldots, N$, such that $0 \le \rho_{\alpha,1} < \rho_{\alpha,2}$ and $0 \le \rho_{\alpha,3} < \rho_{\alpha,4}$, define

$$\mathcal{G}_{\alpha,\rm tran} \triangleq \left\{z \in \mathbb{R}^3 : \rho_{\alpha,1} \le \|z\|_2 \le \rho_{\alpha,2}\right\} \cup \{0_3\},$$
$$\mathcal{G}_{\alpha,\rm rot} \triangleq \left\{z \in \mathbb{R}^3 : \rho_{\alpha,3} \le \|z\|_2 \le \rho_{\alpha,4}\right\} \cup \{0_3\},$$

and let $u_{\alpha,\rm tran} : [t_1, t_2] \to \mathcal{G}_{\alpha,\rm tran}$ (respectively, $u_{\alpha,\rm rot} : [t_1, t_2] \to \mathcal{G}_{\alpha,\rm rot}$) be the *force* (respectively, the *moment*) provided by the control system of the αth rigid body. The vector $u_{\alpha,\rm tran}$ (respectively, $u_{\alpha,\rm rot}$) is also referred to as the *αth translational control vector* (respectively, the *αth rotational control vector*.) The following definition is needed.

Definition II.1 ([32]) *If $u_{\alpha,\rm tran} : [t_1, t_2] \to \mathcal{G}_{\alpha,\rm tran}$ (respectively, $u_{\alpha,\rm rot} : [t_1, t_2] \to \mathcal{G}_{\alpha,\rm rot}$), $\alpha = 1, \ldots, N$, is such that i) $u_{\alpha,\rm tran}(\cdot)$ (respectively, $u_{\alpha,\rm rot}(\cdot)$) is continuous at the endpoints of $[t_1, t_2]$, ii) $u_{\alpha,\rm tran}(\cdot)$ (respectively, $u_{\alpha,\rm rot}(\cdot)$) is continuous for all $t \in (t_1, t_2)$ with the exception of a finite number of times t at which $u_{\alpha,\rm tran}(t)$ may have discontinuities of the first kind, and iii) $u_{\alpha,\rm tran}(\tau) = \lim_{t\to\tau^-} u_{\alpha,\rm tran}(t)$ (respectively, $u_{\alpha,\rm rot}(\tau) = \lim_{t\to\tau^-} u_{\alpha,\rm rot}(t)$), where $\tau \in [t_1, t_2]$ is a point of discontinuity of first kind for $u_{\alpha,\rm tran}(t)$ (respectively, $u_{\alpha,\rm rot}(\cdot)$), then $u_{\alpha,\rm tran}(\cdot)$ (respectively, $u_{\alpha,\rm rot}(\cdot)$) is an admissible control in $\mathcal{G}_{\alpha,\rm tran}$ (respectively, $\mathcal{G}_{\alpha,\rm rot}$.)*

Lastly, we define $\tilde{u} \triangleq [u_1^{\rm T}, \ldots, u_N^{\rm T}]^{\rm T}$, where $u_\alpha \triangleq [{\rm c}u_{\alpha,\rm tran}^{\rm T}, u_{\alpha,\rm rot}^{\rm T}]^{\rm T}$, $\alpha = 1, \ldots, N$, and ${\rm c} = 1$ is a real constant with units of distance.

III. Problem Statement

In this section, we provide a formal statement of the control effort minimization problem addressed in this paper. To this goal, in the following we state sufficient conditions that allow writing a minimal set of equations of motion for a system for N rigid bodies.

A. The Governing Equations of Motion

The kinetic energy of a system of N rigid bodies is given by *König's theorem* [30] and for our problem takes the form

$$T(q, q_{\rm dot}) = \frac{1}{2}\sum_{\alpha=1}^{N} m_\alpha v_\alpha^{\rm T}(q, q_{\rm dot})v_\alpha(q, q_{\rm dot}) + \frac{1}{2}\sum_{\alpha=1}^{N}\omega_\alpha^{\rm T}(q, q_{\rm dot})I_{{\rm in},\alpha}\omega_\alpha(q, q_{\rm dot}), \tag{2}$$

where $m_\alpha \in \mathbb{R}$ and $I_{{\rm in},\alpha} \in \mathbb{R}^3$ are the mass and the inertia matrix of the αth rigid body, respectively, which are assumed constant. The system's dynamic equations can be written as [30]

$$\begin{aligned}\frac{\rm d}{{\rm d}t}\frac{\partial T(q, q_{\rm dot})}{\partial q_{\rm dot}} &= \sum_{\alpha=1}^{N} m_\alpha v_\alpha^{\rm T}(q, q_{\rm dot})\frac{\rm d}{{\rm d}t}\frac{\partial v_\alpha(q, q_{\rm dot})}{\partial q_{\rm dot}} + \sum_{\alpha=1}^{N}\omega_\alpha^{\rm T}(q, q_{\rm dot})I_{{\rm in},\alpha}\frac{\rm d}{{\rm d}t}\frac{\partial\omega_\alpha(q, q_{\rm dot})}{\partial q_{\rm dot}}\\ &\quad+ \sum_{\alpha=1}^{N}\left(a(\tilde{x}_\alpha) + u_{\alpha,\rm tran}\right)^{\rm T}\frac{\partial v_\alpha(q, q_{\rm dot})}{\partial q_{\rm dot}} + \sum_{\alpha=1}^{N}\left(m(\tilde{x}_\alpha) + u_{\alpha,rot}\right)^{\rm T}\frac{\partial\omega_\alpha(q, q_{\rm dot})}{\partial q_{\rm dot}},\end{aligned} \tag{3}$$

where $a : \mathbb{R}^{12} \to \mathbb{R}^3$ and $m : \mathbb{R}^{12} \to \mathbb{R}^3$ are continuously differentiable and denote the *external forces* and *moments* acting on a rigid body, respectively. Since $a(\cdot)$ and $m(\cdot)$ are functions of the augmented state vector, external forces and moments acting on the αth vehicle, $\alpha = 1, \ldots, N$, are functions of the agent's translational and angular positions and velocities. The boundary conditions for (3) are given by (10) and (11) below.

In most cases of practical application, the vector of quasi-velocities (1) is constrained by the linear *nonholonomic constraint*

$$\tilde{N}_{\rm nh} q_{\rm dot}(t) = 0_\gamma, \tag{4}$$

where $\tilde{N}_{\rm nh} \triangleq \begin{bmatrix} I_\zeta & 0_{\zeta\times(\gamma-\zeta)} \\ 0_{(\gamma-\zeta)\times\zeta} & 0_{(\gamma-\zeta)\times(\gamma-\zeta)} \end{bmatrix}$. In this case, the first ζ scalar differential equations of (3) are identically equal to zero [30]. Remarkably, nonlinear nonholonomic constraints are not needed to address mechanical problems [33] and linear nonholonomic constraints can be always reduced to the form (4) by properly choosing $D(\cdot)$ in (1) [30].

The following result allows writing the the system's equations of motion (3) in explicit form.

Theorem III.1 *Consider a system of N rigid bodies, which equations of motion are given by (3). If, for all $q \in \mathbb{R}^\gamma$, $Z(q) > 0_{(\gamma-\zeta)\times(\gamma-\zeta)}$, where*

$$Z^{-1}(q) \triangleq \mathcal{O} N_{\rm nh} \Bigg(D^{-\rm T}(q) \Bigg[\sum_{\alpha=1}^{N} \left(\frac{\partial \sigma_\alpha(q)}{\partial q} \right)^{\rm T} R_{\rm rod}^{-\rm T}(\sigma_\alpha) {\rm I}_{{\rm in},\alpha} R_{\rm rod}^{-1}(\sigma_\alpha) \frac{\partial \sigma_\alpha(q)}{\partial q} + \sum_{\alpha=1}^{N} m_\alpha \left(\frac{\partial r_\alpha(q)}{\partial q} \right)^{\rm T} \frac{\partial r_\alpha(q)}{\partial q} \Bigg] D^{-1}(q) \Bigg) \mathcal{O},$$

then (3) is equivalent to the system of $2\gamma - \zeta$ scalar differential equations given by

$$\begin{bmatrix} \dot{q}(t) \\ \mathcal{O}^{\rm T} N_{\rm nh} \big(\dot{q}_{\rm dot}(t)\big) \end{bmatrix} = \begin{bmatrix} D^{-1}(q(t)) N_{\rm nh} q_{\rm dot}(t) - d(q(t)) \\ f_{\rm dyn}(q(t), q_{\rm dot}(t)) \end{bmatrix} + \begin{bmatrix} 0_{\gamma\times 6N} \\ M(q(t)) \end{bmatrix} \tilde{u}(t), \qquad t \in [t_1, t_2], \tag{5}$$

where $f_{\rm dyn} : \mathbb{R}^\gamma \times \mathbb{R}^\gamma \to \mathbb{R}^{\gamma-\zeta}$,

$$\begin{aligned} M(q) &\triangleq Z(q) \mathcal{O}^{\rm T} N_{\rm nh} \left[\left(\frac{\partial v_1(q, q_{\rm dot})}{\partial q_{\rm dot}} \right)^{\rm T}, \left(\frac{\partial \omega_1(q, q_{\rm dot})}{\partial q_{\rm dot}} \right)^{\rm T}, \ldots, \left(\frac{\partial v_N(q, q_{\rm dot})}{\partial q_{\rm dot}} \right)^{\rm T}, \left(\frac{\partial \omega_N(q, q_{\rm dot})}{\partial q_{\rm dot}} \right)^{\rm T} \right] \\ &= Z(q) \mathcal{O}^{\rm T} N_{\rm nh} \Bigg(D^{-\rm T}(q) \Bigg[\left(\frac{\partial r_1(q)}{\partial q} \right)^{\rm T}, \left(R_{\rm rod}^{-1}(\sigma_1) \frac{\partial \sigma_1(q)}{\partial q} \right)^{\rm T}, \ldots, \\ &\qquad \left(\frac{\partial r_N(q)}{\partial q} \right)^{\rm T}, \left(R_{\rm rod}^{-1}(\sigma_N) \frac{\partial \sigma_N(q)}{\partial q} \right)^{\rm T} \Bigg] \Bigg), \end{aligned} \tag{6}$$

and

$$N_{\rm nh} \triangleq I_\gamma - \tilde{N}_{\rm nh}. \tag{7}$$

In Theorem III.1, the explicit expression of $f_{\rm dyn}(\cdot,\cdot)$ is omitted for conciseness and the boundary conditions of (5) are given by (10) and (11) below.

Remark III.1 *Theorem III.1 provides a sufficient condition to write the equations of motion for a system of N rigid bodies as an explicit first order nonlinear differential equation that is affine in the control $\tilde{u}(\cdot)$. In addition, Theorem III.1 allows computing the kinetic energy $T(\cdot,\cdot)$ for the constrained dynamical system, that is, accounting for the nonholonomic constraints (4). Lastly, (5) represents a systems of $2\gamma - \zeta$ first order differential equations and hence is a minimal set of equations of motion for a system of N rigid bodies [30].*

The following example illustrates the result proven in Theorem III.1.

Example III.1 *Consider the rotational dynamics of a rigid body that is not subject to external forces and moments. In this case, $N = 1$, $q_{\rm dot} = \omega_1$, $D(q) = R_{\rm rod}^{-1}(\sigma_1)$, $r = 0$, $a(\tilde{x}_1) = 0$, and $m(\tilde{x}_1) = 0$. Thus, $Z(q) = R_{\rm rod}^{-1}(\sigma_1) \left[R_{\rm rod}^{-\rm T}(\sigma_1) {\rm I}_{{\rm in},\alpha} R_{\rm rod}^{-1}(\sigma_1) \right]^{-1} R_{\rm rod}^{-\rm T}(\sigma_1) = I_{\rm in,1}^{-1}$ and the dynamic equation (5) specializes to the Euler equations [30]*

$$\begin{bmatrix} \dot{\sigma}_1(t) \\ \dot{\omega}_1(t) \end{bmatrix} = \begin{bmatrix} R_{\rm rod}(\sigma_1)\omega_1(t) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ I_{\rm in,1}^{-1} u_{1,\rm rot}(t) \end{bmatrix}, \tag{8}$$

which can be equivalently deduced from (3). One can surmise from this example that angular velocities are an example of quasi-velocities. △

B. Path Planning Optimization Problem

Given $\tilde{x}_{\alpha,1} \in \mathbb{R}^{12}$, $\alpha = 1,\ldots,N$, the continuously differentiable manifold $s_2 : \mathbb{R}^{12N} \to \mathbb{R}^{\zeta_2}$, and $\mu_\alpha \in [0,1]$, $\alpha = 1,\ldots,N$, such that $\sum_{\alpha=1}^{N} \mu_\alpha = 1$, the trajectory optimization problem discussed in this paper can be formulated as follows. For all $\alpha = 1,\ldots,N$, find $u_{\alpha,\mathrm{tran}}(\cdot)$ and $u_{\alpha,\mathrm{rot}}(\cdot)$ among all admissible controls in $\mathcal{G}_{\alpha,\mathrm{tran}}$ and $\mathcal{G}_{\alpha,\mathrm{rot}}$, respectively, such that the performance measure

$$J[\tilde{u}(\cdot)] \triangleq \sum_{\alpha=1}^{N} \left(\mu_\alpha \int_{t_1}^{t_2} \|u_\alpha(t)\|_2 \,\mathrm{d}t \right) \tag{9}$$

is minimized and the system's equations of motion (5) hold with boundary conditions

$$[\tilde{x}_1^{\mathrm{T}}(q(t_1), q_{\mathrm{dot}}(t_1)), \ \ldots, \ \tilde{x}_N^{\mathrm{T}}(q(t_1), q_{\mathrm{dot}}(t_1))]^{\mathrm{T}} = [\tilde{x}_{1,1}^{\mathrm{T}}, \ \ldots, \ \tilde{x}_{N,1}^{\mathrm{T}}]^{\mathrm{T}}, \tag{10}$$

$$s_2([\tilde{x}_1^{\mathrm{T}}(q(t_2), q_{\mathrm{dot}}(t_2)), \ \ldots, \ \tilde{x}_N^{\mathrm{T}}(q(t_2), q_{\mathrm{dot}}(t_2))]^{\mathrm{T}}) = 0_{\zeta_2}. \tag{11}$$

In this paper, we assume that there exists at least one set of $2N$ admissible controls $\{u_{1,\mathrm{tran}}(\cdot),\ldots,u_{N,\mathrm{tran}}(\cdot), u_{1,\mathrm{rot}}(\cdot),\ldots,u_{N,\mathrm{rot}}(\cdot)\}$ such that (5), (10), and (11) are satisfied. The performance measure (9) is the weighted sum of the $\mathcal{L}_1$ norms of the control vectors $u_\alpha(\cdot)$, $\alpha = 1,\ldots,N$, and the constant μ_α in (9) measures the relative importance of minimizing the control effort of the αth body with respect to the others. In the following, we assume without loss of generality that $\mu_\alpha \neq 0$, for all $\alpha = 1,\ldots,N$. Equation (10) implies that the system's configuration is known at time t_1, which is usually the case for most applications, and (11) partly imposes the system's configuration at t_2. For instance, (11) can be used to impose that the N rigid bodies reach a surface described by s_2 at time t_2 with a prescribed velocity.

IV. Mathematical Background

In this section, we review some of the mathematical background needed for the scopes of this paper.

A. Pontryagin's Minimum Principle

In the following, we state Pontryagin's minimum principle, which is a first order necessary condition for the existence of admissible controllers $u_{\alpha,\mathrm{tran}} : [t_1,t_2] \to \mathcal{G}_{\alpha,\mathrm{tran}}$ and $u_{\alpha,\mathrm{rot}} : [t_1,t_2] \to \mathcal{G}_{\alpha,\mathrm{rot}}$, $\alpha = 1,\ldots,N$, that solve the trajectory optimization problem stated in Section III.B. Let $s : \mathbb{R}^n \to \mathbb{R}^m$ be a continuously differentiable manifold and let the manifold tangent to s at y_0 be given by

$$\mathcal{T}(s(\cdot), y_0) \triangleq \left\{ y \in \mathbb{R}^n : \left.\frac{\partial s(y)}{\partial y}\right|_{y=y_0} (y - y_0) = 0_m \right\}. \tag{12}$$

Every vector $v \in \mathbb{R}^n$ that is normal to the manifold tangent to $s(\cdot)$ at y_0, that is, such that $v^{\mathrm{T}} y = 0$, $y \in \mathcal{T}(s(\cdot), y_0)$, is said to verify the *transversality condition* for $s(\cdot)$ at y_0.

Given the dynamical system (5) with performance measure (9), the *Hamiltonian function* is defined as

$$\mathfrak{h}(q, q_{\mathrm{dot}}, \tilde{u}, p) \triangleq p_0 \sum_{\alpha=1}^{N} \mu_\alpha \|u_\alpha\|_2 + p_{\mathrm{dot}}^{\mathrm{T}}\big(D^{-1}(q) N_{\mathrm{nh}} q_{\mathrm{dot}} - d(q)\big) + p_{\mathrm{dyn}}^{\mathrm{T}}\big(f_{\mathrm{dyn}}(q, q_{\mathrm{dot}}) + M(q)\tilde{u}\big), \tag{13}$$

where $p_0 \in \overline{\mathbb{R}}_+$, $p(t) \triangleq [p_0, p_{\mathrm{dot}}^{\mathrm{T}}(t), p_{\mathrm{dyn}}^{\mathrm{T}}(t)]^{\mathrm{T}}$, the *costate vectors* $p_{\mathrm{dot}} : [t_1,t_2] \to \mathbb{R}^{\gamma}$ and $p_{\mathrm{dyn}} : [t_1,t_2] \to \mathbb{R}^{\gamma-\zeta}$ are the solutions of the *costate equation*

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} p_{\mathrm{dot}}(t) \\ p_{\mathrm{dyn}}(t) \end{bmatrix} = - \begin{bmatrix} \frac{\partial}{\partial q}\big(D^{-1}(q(t)) N_{\mathrm{nh}} q_{\mathrm{dot}}(t) - d(q)\big) & \mathcal{O}N_{\mathrm{nh}}\big(D^{-1}(q(t))\big) \\ \frac{\partial}{\partial q}\big(f_{\mathrm{dyn}}(q, q_{\mathrm{dot}}(t)) + M(q)\tilde{u}(t)\big) & \mathcal{O}N_{\mathrm{nh}}\big(\frac{\partial f_{\mathrm{dyn}}(q(t), q_{\mathrm{dot}})}{\partial q_{\mathrm{dot}}}\big) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} p_{\mathrm{dot}}(t) \\ p_{\mathrm{dyn}}(t) \end{bmatrix}, \tag{14}$$

$t \in [t_1, t_2]$, and the boundary conditions for (14) are given in Theorem IV.1 below. Finally, let

$$\mathfrak{m}(q(t), q_{\mathrm{dot}}(t), p(t)) \triangleq \min_{\tilde{u} \in \prod_{\alpha=1}^{N} (\mathcal{G}_{\alpha,\mathrm{tran}} \times \mathcal{G}_{\alpha,\mathrm{rot}})} \mathfrak{h}(q(t), q_{\mathrm{dot}}(t), \tilde{u}, p(t)). \tag{15}$$

In the this paper, we refer to the following theorem as *Pontryagin's minimum principle* [32].

Theorem IV.1 *For all $\alpha = 1, \ldots, N$, let $u^*_{\alpha,\mathrm{tran}}(t)$ and $u^*_{\alpha,\mathrm{rot}}(t)$, $t \in [t_1, t_2]$, be admissible controls in $\mathcal{G}_{\alpha,\mathrm{tran}}$ and $\mathcal{G}_{\alpha,\mathrm{rot}}$, respectively, that minimize the performance measure (9) subject to the dynamic equation (5) and the constraints (10) and (11). Then there exist $p^*_0 \in \overline{\mathbb{R}}_+$, $p^*_{\mathrm{dyn}}(t)$, and $p^*_{\mathrm{dot}}(t)$ such that i) $|p^*_0| + \|p^*_{\mathrm{dyn}}(t)\|_2 + \|p^*_{\mathrm{dot}}(t)\|_2 \neq 0$, $t \in [t_1, t_2]$, ii) (14) is satisfied, iii) $p^*_{\mathrm{dyn}}(t_1)$ and $p^*_{\mathrm{dot}}(t_1)$ are arbitrary, iv) $p^*_{\mathrm{dyn}}(t_2)$ and $p^*_{\mathrm{dot}}(t_2))$ satisfy the transversality condition for s_2 at $q^*(t_2)$, and v) $\mathfrak{h}(q^*(t), q^*_{\mathrm{dot}}(t), \tilde{u}^*(t), p^*(t))$ attains its minimum (15) almost everywhere on $[t_1, t_2]$, which is equal to zero, except on a finite number of points.*

Pontryagin's minimum principle is a necessary condition for strong local minima, and hence, it provides *candidate* optimal control vectors. We say the optimization problem is *normal* if $p^*_0 \neq 0$, otherwise the optimization problem is *abnormal*. For normal problems, we assume without loss of generality that $p^*_0 = 1$.

B. Necessary Conditions for Singular Controls

In this paper, we define singular controls and singular arcs as follows.

Definition IV.1 *Consider the performance measure (9) subject to (5), (14), (10), (11), and conditions iii) and iv) of Theorem IV.1, and let $u_{\alpha,\mathrm{tran}}(\cdot)$ (respectively, $u_{\alpha,\mathrm{rot}}(\cdot)$) be an admissible control in $\mathcal{G}_{\alpha,\mathrm{tran}}$ (respectively, $\mathcal{G}_{\alpha,\mathrm{rot}}$), $\alpha = 1, \ldots, N$. If*

$$\frac{\partial}{\partial u_{\alpha,\mathrm{tran}}} \mathfrak{h}(q(t), q_{\mathrm{dot}}(t), \tilde{u}, p(t)) = 0_3^{\mathrm{T}}, \qquad t \in [\hat{t}_{1,\alpha,\mathrm{tran}}, \hat{t}_{2,\alpha,\mathrm{tran}}],$$

(respectively,

$$\frac{\partial}{\partial u_{\alpha,\mathrm{rot}}} \mathfrak{h}(q(t), q_{\mathrm{dot}}(t), \tilde{u}, p(t)) = 0_3^{\mathrm{T}} \qquad t \in [\hat{t}_{1,\alpha,\mathrm{rot}}, \hat{t}_{2,\alpha,\mathrm{rot}}],)$$

where $[\hat{t}_{1,\alpha,\mathrm{tran}}, \hat{t}_{2,\alpha,\mathrm{tran}}] \subset [t_1, t_2]$ (respectively, $[\hat{t}_{1,\alpha,\mathrm{rot}}, \hat{t}_{2,\alpha,\mathrm{rot}}] \subset [t_1, t_2]$), then $u_{\alpha,\mathrm{tran}}(t)$ (respectively, $u_{\alpha,\mathrm{rot}}(t)$), is a singular translational (respectively, rotational) control. Furthermore, if $u_{\alpha,\mathrm{tran}}(t)$ is a singular translational control and $u_{\alpha,\mathrm{rot}}(t)$ is a singular rotational control, $t \in [\hat{t}_{1,\alpha}, \hat{t}_{2,\alpha}] \subset [t_1, t_2]$, then $x_\alpha(t)$, $t \in [\hat{t}_{1,\alpha}, \hat{t}_{2,\alpha}]$, is a singular arc.

Pontryagin's minimum principle and the Legendre-Clebsch necessary condition hold along singular arcs. However, these theorem do not provide any useful information to identify singular translational and rotational controls. In these cases, one can apply the following theorem, known as the *generalized Legendre-Clebsch necessary condition* [34–36].

Theorem IV.2 *For all $\alpha = 1, \ldots, N$, let $u^*_{\alpha,\mathrm{tran}}(\cdot) \in \mathrm{int}(\mathcal{G}_{\alpha,\mathrm{tran}})$ (respectively, $u^*_{\alpha,\mathrm{rot}}(\cdot) \in \mathrm{int}(\mathcal{G}_{\alpha,\mathrm{rot}})$) be an admissible control in $\mathcal{G}_{\alpha,\mathrm{tran}}$ (respectively, $\mathcal{G}_{\alpha,\mathrm{rot}}$) that minimizes the performance measure (9) subject to (5), (14), (10), (11), and conditions iii) and iv) of Theorem IV.1. Then, there exists $[\hat{t}_{1,\alpha,\mathrm{tran}}, \hat{t}_{2,\alpha,\mathrm{tran}}] \subseteq [t_1, t_2]$ (respectively, $[\hat{t}_{1,\alpha,\mathrm{rot}}, \hat{t}_{2,\alpha,\mathrm{rot}}] \subseteq [t_1, t_2]$) and an integer $\nu_{\alpha,\mathrm{tran}}$ (respectively, $\nu_{\alpha,\mathrm{rot}}$) such that*

$$\frac{\mathrm{d}^\kappa}{\mathrm{d}t^\kappa} \frac{\partial}{\partial u^*_{\alpha,\mathrm{tran}}} \mathfrak{h}(q^*(t), q^*_{\mathrm{dot}}(t), \tilde{u}^*, p^*(t)) = 0_3^{\mathrm{T}}, \qquad t \in [\hat{t}_{1,\alpha,\mathrm{tran}}, \hat{t}_{2,\alpha,\mathrm{tran}}], \tag{16}$$

(respectively,

$$\frac{\mathrm{d}^\lambda}{\mathrm{d}t^\lambda} \frac{\partial}{\partial u^*_{\alpha,\mathrm{rot}}} \mathfrak{h}(q^*(t), q^*_{\mathrm{dot}}(t), \tilde{u}^*, p^*(t)) = 0_3^{\mathrm{T}}, \qquad t \in [\hat{t}_{1,\alpha,\mathrm{rot}}, \hat{t}_{2,\alpha,\mathrm{rot}}],) \tag{17}$$

for $\kappa = 0, \ldots, 2\nu_{\alpha,\mathrm{tran}} - 1$ (respectively, $\lambda = 0, \ldots, 2\nu_{\alpha,\mathrm{rot}} - 1$). Furthermore, $u^*_{\alpha,\mathrm{tran}}(t)$ (respectively, $u^*_{\alpha,\mathrm{rot}}(t)$) appears explicitly in the left-hand-side of (16) (respectively, (17)) for $\kappa = 2\nu_{\alpha,\mathrm{tran}}$ (respectively, $\lambda = 2\nu_{\alpha,\mathrm{rot}}$.) In conclusion,

$$(-1)^{\nu_{\alpha,\mathrm{tran}}} \frac{\partial}{\partial u^*_{\alpha,\mathrm{tran}}} \left(\frac{\mathrm{d}^{2\nu_{\alpha,\mathrm{tran}}}}{\mathrm{d}t^{2\nu_{\alpha,\mathrm{tran}}}} \frac{\partial}{\partial u^*_{\alpha,\mathrm{tran}}} \mathfrak{h}(q^*(t), q^*_{\mathrm{dot}}(t), \tilde{u}^*, p^*(t)) \right)^{\mathrm{T}} \geq 0_{3\times 3}, \qquad t \in [\hat{t}_{1,\alpha,\mathrm{tran}}, \hat{t}_{2,\alpha,\mathrm{tran}}], \tag{18}$$

(respectively,

$$(-1)^{\nu_{\alpha,\mathrm{rot}}} \frac{\partial}{\partial u^*_{\alpha,\mathrm{rot}}} \left(\frac{\mathrm{d}^{2\nu_{\alpha,\mathrm{rot}}}}{\mathrm{d}t^{2\nu_{\alpha,\mathrm{rot}}}} \frac{\partial}{\partial u^*_{\alpha,\mathrm{rot}}} \mathfrak{h}(q^*(t), q^*_{\mathrm{dot}}(t), \tilde{u}^*, p^*(t)) \right)^{\mathrm{T}} \geq 0_{3\times 3}, \qquad t \in [\hat{t}_{1,\alpha,\mathrm{rot}}, \hat{t}_{2,\alpha,\mathrm{rot}}].) \tag{19}$$

The integer $\nu_{\alpha,\mathrm{tran}}$ (respectively, $\nu_{\alpha,\mathrm{rot}}$) is the *order of the translational* (respectively, *rotational*) *singularity*. If $\nu_{\alpha,\mathrm{tran}} = 0$ (respectively, $\nu_{\alpha,\mathrm{rot}} = 0$), then (16) (respectively, (17)) reduces to the Euler-Lagrange necessary condition and (18) (respectively, (19)) reduces to the Legendre necessary condition. A discussion about the order of singularity is given in reference [37]. Pontryagin's principle applies to strong local minima while Theorem IV.2 applies to weak local minima. Thus, the generalized Legendre-Clebsch necessary condition yields for a smaller class of candidate optimal controls. The next result allows us finding candidate optimal translational and rotational singular controls.

Corollary IV.1 *Assume the conditions of Theorem IV.2 are verified. Then, a candidate optimal singular translational (respectively, rotational) control* $u^*_{\alpha,\mathrm{tran}}(t)$, $t \in [\hat{t}_{1,\alpha,\mathrm{tran}}, \hat{t}_{2,\alpha,\mathrm{tran}}]$ *(respectively,* $u^*_{\alpha,\mathrm{rot}}(t)$, $t \in [\hat{t}_{1,\alpha,\mathrm{rot}}, \hat{t}_{2,\alpha,\mathrm{rot}}]$*),* $\alpha = 1, \ldots, N$, *is such that*

$$\frac{\mathrm{d}^{2\nu_{\alpha,\mathrm{rot}}}}{\mathrm{d}t^{2\nu_{\alpha,\mathrm{tran}}}} \frac{\partial}{\partial u^*_{\alpha,\mathrm{tran}}} \mathfrak{h}(q^*(t), q^*_{\mathrm{dot}}(t), \tilde{u}^*, p^*(t)) = 0_3^{\mathrm{T}}, \qquad t \in [\hat{t}_{1,\alpha,\mathrm{tran}}, \hat{t}_{2,\alpha,\mathrm{tran}}], \tag{20}$$

(*respectively,*

$$\frac{\mathrm{d}^{2\nu_{\alpha,\mathrm{rot}}}}{\mathrm{d}t^{2\nu_{\alpha,\mathrm{rot}}}} \frac{\partial}{\partial u^*_{\alpha,\mathrm{rot}}} \mathfrak{h}(q^*(t), q^*_{\mathrm{dot}}(t), \tilde{u}^*, p^*(t)) = 0_3^{\mathrm{T}}, \qquad t \in [\hat{t}_{1,\alpha,\mathrm{rot}}, \hat{t}_{2,\alpha,\mathrm{rot}}].) \tag{21}$$

The next theorem characterizes the junction between singular and nonsingular controls. For the statement of this result, we recall that $f : [t_1, t_2] \to \mathbb{R}^n$ is piecewise analytic in a neighborhood $\mathcal{I}(\hat{t})$ of $\hat{t} \in (t_1, t_2)$ if $f(t)$ possesses derivatives of all orders and $f(t)$, $t \in \mathcal{I}(\hat{t})$, agrees with its Taylor series except a finite number of points.

Theorem IV.3 ([38]) *Consider the performance measure* (9) *subject to* (5), (14), (10), (11), *and conditions* $iii)$ *and* $iv)$ *of Theorem IV.1. Let* $u^*_{\alpha,\mathrm{tran}}(t) \in \mathrm{int}(\mathcal{G}_{\alpha,\mathrm{tran}})$, $t \in [\hat{t}_{1,\alpha,\mathrm{tran}}, \hat{t}_{2,\alpha,\mathrm{tran}}] \subset [t_1, t_2]$ *(respectively,* $u^*_{\alpha,\mathrm{rot}}(t) \in \mathrm{int}(\mathcal{G}_{\alpha,\mathrm{rot}})$, $t \in [\hat{t}_{1,\alpha,\mathrm{rot}}, \hat{t}_{2,\alpha,\mathrm{rot}}] \subset [t_1, t_2]$*),* $\alpha = 1, \ldots, N$, *be a candidate optimal singular translational (respectively, rotational) control with order of translational (respectively, rotational) singularity* $\nu_{\alpha,\mathrm{tran}}$ *(respectively,* $\nu_{\alpha,\mathrm{rot}}$*.) Assume that* $u^*_{\alpha,\mathrm{tran}}(t)$ *(respectively,* $u^*_{\alpha,\mathrm{rot}}(t)$*), is piecewise analytic in some neighborhood of* $\hat{t}_{1,\alpha,\mathrm{tran}}$ *and* $\hat{t}_{2,\alpha,\mathrm{tran}}$ *(respectively,* $\hat{t}_{1,\alpha,\mathrm{rot}}$ *and* $\hat{t}_{1,\alpha,\mathrm{rot}}$*), where it holds that*

$$(-1)^{\nu_{\alpha,\mathrm{tran}}} \frac{\partial}{\partial u^*_{\alpha,\mathrm{tran}}} \left(\frac{\mathrm{d}^{2\nu_{\alpha,\mathrm{tran}}}}{\mathrm{d}t^{2\nu_{\alpha,\mathrm{tran}}}} \frac{\partial}{\partial u^*_{\alpha,\mathrm{tran}}} \mathfrak{h}(q^*(t), q^*_{\mathrm{dot}}(t), \tilde{u}^*, p^*(t)) \right)^{\mathrm{T}} > 0_{3\times 3} \tag{22}$$

(*respectively,*

$$(-1)^{\nu_{\alpha,\mathrm{rot}}} \frac{\partial}{\partial u^*_{\alpha,\mathrm{rot}}} \left(\frac{\mathrm{d}^{2\nu_{\alpha,\mathrm{rot}}}}{\mathrm{d}t^{2\nu_{\alpha,\mathrm{rot}}}} \frac{\partial}{\partial u^*_{\alpha,\mathrm{rot}}} \mathfrak{h}(q^*(t), q^*_{\mathrm{dot}}(t), \tilde{u}^*, p^*(t)) \right)^{\mathrm{T}} > 0_{3\times 3}.) \tag{23}$$

In addition, let $\kappa_{1,\alpha,\mathrm{tran}} \in \mathbb{N} \cup \{0\}$ *and* $\kappa_{2,\alpha,\mathrm{tran}} \in \mathbb{N} \cup \{0\}$ *(respectively,* $\kappa_{1,\alpha,\mathrm{rot}} \in \mathbb{N} \cup \{0\}$ *and* $\kappa_{2,\alpha,\mathrm{rot}} \in \mathbb{N} \cup \{0\}$*),* $\alpha = 1, \ldots, N$, *be the smallest integers such that*

$$\frac{\mathrm{d}^{\kappa_{1,\alpha,\mathrm{tran}}} u^*_{\alpha,\mathrm{tran}}(t)}{\mathrm{d}t^{\kappa_{1,\alpha,\mathrm{tran}}}}$$

is discontinuous at $t = \hat{t}_{1,\alpha,\mathrm{tran}}$ *and*

$$\frac{\mathrm{d}^{\kappa_{2,\alpha,\mathrm{tran}}} u^*_{\alpha,\mathrm{tran}}(t)}{\mathrm{d}t^{\kappa_{2,\alpha,\mathrm{tran}}}}$$

is discontinuous at $t = \hat{t}_{2,\alpha,\mathrm{tran}}$ *(respectively,*

$$\frac{\mathrm{d}^{\kappa_{1,\alpha,\mathrm{rot}}} u^*_{\alpha,\mathrm{rot}}(t)}{\mathrm{d}t^{\kappa_{1,\alpha,\mathrm{rot}}}}$$

is discontinuous at $t = \hat{t}_{1,\alpha,\mathrm{rot}}$ *and*

$$\frac{\mathrm{d}^{\kappa_{2,\alpha,\mathrm{rot}}} u^*_{\alpha,\mathrm{rot}}(t)}{\mathrm{d}t^{\kappa_{2,\alpha,\mathrm{rot}}}}$$

is discontinuous at $t = \hat{t}_{2,\alpha,\mathrm{rot}}$*.) Then* $\nu_{\alpha,\mathrm{tran}} + \kappa_{1,\alpha,\mathrm{tran}}$ *and* $\nu_{\alpha,\mathrm{tran}} + \kappa_{2,\alpha,\mathrm{tran}}$ *(respectively,* $\nu_{\alpha,\mathrm{rot}} + \kappa_{1,\alpha,\mathrm{rot}}$ *and* $\nu_{\alpha,\mathrm{rot}} + \kappa_{2,\alpha,\mathrm{rot}}$*) are odd integers.*

Recall that if $\nu_{\alpha,\mathrm{tran}} > 1$ (respectively, $\nu_{\alpha,\mathrm{rot}} > 1$), then $u^*_{\alpha,\mathrm{tran}}(t)$ (respectively, $u^*_{\alpha,\mathrm{rot}}(t)$) is measurable but not analytic in the neighborhoods of $\hat{t}_{1,\alpha,\mathrm{tran}}$ and $\hat{t}_{2,\alpha,\mathrm{tran}}$ (respectively, $\hat{t}_{1,\alpha,\mathrm{rot}}$ and $\hat{t}_{2,\alpha,\mathrm{rot}}$) [39]. Thus, if $\nu_{\alpha,\mathrm{tran}} > 1$ and $\nu_{\alpha,\mathrm{rot}} > 1$, then Theorem IV.3 cannot be applied and the control vector $\tilde{u}^*(\cdot)$ is affected by chattering at the junction between singular and non-singular arcs [40].

V. Control Effort Minimization Problem for Systems of Rigid Bodies

In this section, we provide necessary conditions to solve the control effort minimization problem for a system of N rigid bodies. Specifically, in Sections V.A and V.B, we state necessary conditions to solve the abnormal and the normal control effort minimization problems, respectively. It follows from the definition of abnormal optimization problem that the results provided in Section V.A hold for *all* performance measures subject to (5), (10), and (11) that allow abnormal optimal control problems. For conciseness, given $p_{\rm dyn} : [t_1, t_2] \to \mathbb{R}^{\gamma-\zeta}$ that verifies (14) and conditions iii) and iv) of Theorem IV.1, we define the *auxiliary costate vector* as

$$\hat{p}_{\rm dyn}(t) \triangleq \mathcal{O}N_{\rm nh}\big(D^{-1}(q(t))\big)Z(q(t))p_{\rm dyn}(t), \qquad t \in [t_1, t_2]. \tag{24}$$

A. The Abnormal Optimization Problem

In this section, we provide necessary conditions to solve the abnormal control effort minimization problem, that is, we assume that $p_0^* = 0$ in Theorem IV.1. In addition, we assume without loss of generality that the translational control vectors of the first ν rigid bodies and the rotational control vectors of the first κ rigid bodies are singular.

The following lemma provides necessary conditions for $u^*_{\alpha,{\rm tran}}(\cdot)$, $\alpha = 1, \ldots, \nu$, and $u^*_{\beta,{\rm rot}}(\cdot)$, $\beta = 1, \ldots, \kappa$, to be candidate optimal singular controls and for $u^*_{\lambda,{\rm tran}}(\cdot)$, $\lambda = \nu+1, \ldots, N$, and $u^*_{\chi,{\rm rot}}(t)$, $\chi = \kappa+1, \ldots, N$, to be candidate optimal controls on maximum thrust arcs. Furthermore, the next result proves that there does not exist a null thrust arc in the abnormal control effort minimization problem.

Lemma V.1 *Consider the performance measure* (9) *subject to* (5), (14), (10), (11), *and conditions iii) and iv) of Theorem IV.1. If $p_0^* = 0$,*

$$\hat{p}^*_{\rm dyn}(t) \in \mathcal{N}\left(\frac{\partial r_\alpha(q^*)}{\partial q^*}\right), \qquad t \in [\hat{t}_{1,\alpha,{\rm tran}}, \hat{t}_{2,\alpha,{\rm tran}}] \subset [t_1, t_2], \qquad \alpha = 1, \ldots, \nu, \tag{25}$$

then $u^*_{\alpha,{\rm tran}}(t)$ is a candidate optimal singular translational control. In addition, if

$$\hat{p}^*_{\rm dyn}(t) \in \mathcal{N}\left(\frac{\partial \sigma_\beta(q^*)}{\partial q^*}\right), \qquad t \in [\hat{t}_{1,\beta,{\rm rot}}, \hat{t}_{2,\beta,{\rm rot}}] \subset [t_1, t_2], \qquad \beta = 1, \ldots, \kappa, \tag{26}$$

then $u^*_{\beta,{\rm rot}}(t)$ is a candidate optimal singular rotational control. Furthermore, the candidate optimal translational control $u^*_{\lambda,{\rm tran}}(t)$ is parallel to $-\frac{\partial r_\lambda(q^*)}{\partial q^*}\hat{p}^*_{\rm dyn}(t)$, $t \in [t_1, t_2]$, $\lambda = \nu+1, \ldots, N$, the candidate optimal rotational control $u^*_{\chi,{\rm rot}}(t)$ is parallel to $-R^{-1}_{\rm rod}(\sigma_\chi(q^*))\frac{\partial \sigma_\chi(q^*)}{\partial q^*}\hat{p}^*_{\rm dyn}(t)$, $t \in [t_1, t_2]$, $\chi = \kappa+1, \ldots, N$, $\|u^*_{\lambda,{\rm tran}}(t)\|_2 = \rho_{\lambda,2}$, and $\|u^*_{\chi,{\rm rot}}(t)\|_2 = \rho_{\chi,4}$.

Example V.1 *Consider the dynamical system presented in Example III.1. In this case, the Hamiltonian function* (13) *specializes to*

$$\mathfrak{h}(\sigma_1^*, \omega_1^*, u^*_{1,{\rm rot}}, p^*) = p^{*{\rm T}}_{\rm dot}\omega_1^* + u^{*{\rm T}}_{1,{\rm rot}}R^{-1}_{\rm rod}(\sigma_1^*)\hat{p}^*_{\rm dyn} = p^{*{\rm T}}_{\rm dot}\omega_1^* + u^{*{\rm T}}_{1,{\rm rot}}I^{-1}_{{\rm in},1}p^*_{\rm dyn} \tag{27}$$

and (26) *specializes to $I^{-1}_{{\rm in},1}p^*_{\rm dyn}(t) = 0_3$, $t \in [\hat{t}_{1,1,{\rm rot}}, \hat{t}_{2,1,{\rm rot}}]$, which implies that $p^*_{\rm dyn}(t) = 0_3$. Thus, if $p_0^* = 0$ and $p^*_{\rm dyn}(t) = 0_3$, $t \in [\hat{t}_{1,1,{\rm rot}}, \hat{t}_{2,1,{\rm rot}}]$, then it follows from Lemma V.1 that $u^*_{1,{\rm rot}}(t)$ is a singular rotational control. Alternatively, if $p_0^* = 0$ and $p^*_{\rm dyn}(t) \neq 0_3$, $t \in [\hat{t}_{1,1,{\rm rot}}, \hat{t}_{2,1,{\rm rot}}]$, then it follows from Lemma V.1 that $-u^*_{1,{\rm rot}}(t)$ and $I^{-1}_{{\rm in},1}p^*_{\rm dyn}(t)$ are parallel and $\|u^*_{1,{\rm rot}}(t)\| = \rho_{1,4}$. It follows from the dynamic equation* (8) *and the costate equation* (14) *that $p^*_{\rm dyn}(t)$ is constant, $t \in [t_1, t_2]$. Hence, $u^*_{1,{\rm rot}}(t)$ is constant and if $u^*_{1,{\rm rot}}(t)$ is a condidate optimal rotational singular control, then $[\hat{t}_{1,1,{\rm rot}}, \hat{t}_{2,1,{\rm rot}}] = [t_1, t_2]$.* $\triangle$

The next result shows that the order of singularity is one for abnormal optimal control problems involving mechanical systems.

Theorem V.1 *Consider the performance measure* (9) *subject to* (5), (14), (10), (11), *and conditions iii) and iv) of Theorem IV.1. If $p_0^* = 0$, $\hat{p}^*_{\rm dyn}(t) \in \mathcal{N}\big(\frac{\partial r_\alpha(q^*)}{\partial q^*}\big)$, $\alpha = 1, \ldots, \nu$, $t \in [\hat{t}_{1,\alpha,{\rm tran}}, \hat{t}_{2,\alpha,{\rm tran}}] \subset [t_1, t_2]$, and $\hat{p}^*_{\rm dyn}(t) \in \mathcal{N}\big(\frac{\partial \sigma_\beta(q^*)}{\partial q^*}\big)$, $\beta = 1, \ldots, \kappa$, $t \in [\hat{t}_{1,\beta,{\rm rot}}, \hat{t}_{2,\beta,{\rm rot}}] \subset [t_1, t_2]$, then the order of singularity of the candidate optimal singular translational control $u^*_{\alpha,{\rm tran}}(t)$, $t \in [\hat{t}_{1,\alpha,{\rm tran}}, \hat{t}_{2,\alpha,{\rm tran}}]$, is one, that is, $\nu_{\alpha,{\rm tran}} = 1$, and the order of the singularity of the candidate optimal singular rotational control $u^*_{\beta,{\rm rot}}(t)$, $t \in [\hat{t}_{1,\beta,{\rm rot}}, \hat{t}_{2,\beta,{\rm rot}}]$, is one, that is, $\nu_{\beta,{\rm rot}} = 1$.*

Example V.2 *Consider the dynamical system presented in Example III.1. If $p_0^* = 0$ and $p_{\rm dyn}^*(t) = 0_3$, $t \in [t_1, t_2]$, then it follows from Theorem V.1 that the order of singularity of the candidate optimal singular translational control $u_{\alpha,\rm tran}^*(t)$ is one, that is, $\nu_{1,\rm rot} = 1$. Alternatively, the order of singularity of candidate optimal singular controls for the performance measure (9) subject to (8), (14), (10), (11), and conditions $iii)$ and $iv)$ of Theorem IV.1 can be found by directly applying Theorem IV.2 with $N = 1$, $q_{\rm dot} = \omega_1$, $D(q) = R_{\rm rod}^{-1}(\sigma_1)$, $r = 0$, $a(\tilde{x}_1) = 0$, and $m(\tilde{x}_1) = 0$. Specifically, it follows from (27) that, for $t \in [t_1, t_2]$,*

$$\frac{\partial}{\partial u_{1,\rm rot}^*}\mathfrak{h}(\sigma_1^*(t), \omega_1^*(t), u_{1,\rm rot}^*, p^*(t)) = \hat{p}_{\rm dyn}^{*\rm T}(t) R_{\rm rod}^{-1}(\sigma_1^*(t)) \tag{28}$$

$$= p_{\rm dyn}^{*\rm T}(t) I_{\rm in,1}^{-1}, \tag{29}$$

$$\frac{\rm d}{{\rm d}t}\frac{\partial}{\partial u_{1,\rm rot}^*}\mathfrak{h}(\sigma_1^*(t), \omega_1^*(t), u_{1,\rm rot}^*, p^*(t)) = p_{\rm dyn}^{*\rm T}(t) I_{\rm in,1}^{-1}\Bigg[R_{\rm rod}(\sigma_1^*(t))\frac{\rm d}{{\rm d}t}R_{\rm rod}^{-1}(\sigma_1^*(t)) + \left(\frac{\rm d}{{\rm d}t}R_{\rm rod}(\sigma_1^*(t))\right)R_{\rm rod}^{-1}(\sigma_1^*(t))\Bigg], \tag{30}$$

and

$$\begin{aligned}\frac{{\rm d}^2}{{\rm d}t^2}\frac{\partial}{\partial u_{1,\rm rot}^*}\mathfrak{h}(\sigma_1^*(t), \omega_1^*(t), u_{1,\rm rot}^*, p^*(t)) = p_{\rm dyn}^{*\rm T}(t) I_{\rm in,1}^{-1}\Bigg[&\frac{\rm d}{{\rm d}t}R_{\rm rod}^{-1}(\sigma_1^*(t))\frac{\rm d}{{\rm d}t}R_{\rm rod}^{-1}(\sigma_1^*(t)) \\ &+ R_{\rm rod}(\sigma_1^*(t))\frac{{\rm d}^2}{{\rm d}t^2}R_{\rm rod}^{-1}(\sigma_1^*(t)) \\ &+ \left(\frac{{\rm d}^2}{{\rm d}t^2}R_{\rm rod}(\sigma_1^*(t))\right)R_{\rm rod}^{-1}(\sigma_1^*(t)) \\ &+ \left(\frac{\rm d}{{\rm d}t}R_{\rm rod}(\sigma_1^*(t))\right)\frac{\rm d}{{\rm d}t}R_{\rm rod}^{-1}(\sigma_1^*(t))\Bigg].\end{aligned} \tag{31}$$

Now, it follows from (8) that $u_{1,\rm rot}(\cdot)$ explicitly appears both in $\frac{{\rm d}^2}{{\rm d}t^2}R_{\rm rod}^{-1}(\sigma_1(t))$ and $\frac{{\rm d}^2}{{\rm d}t^2}R_{\rm rod}(\sigma_1(t))$ and hence the order of the singularity of $u_{1,\rm rot}^*(t)$ is one, that is $\nu_{1,\rm rot} = 1$. $\triangle$

For the statement of the next results, let $a = [a_1, \ldots, a_n]^{\rm T}$ and $a_i \in \mathbb{R}$, $i = 1, \ldots, n$, be the ith element of a, $r_\alpha = [r_{\alpha,1}, r_{\alpha,2}, r_{\alpha,3}]^{\rm T}$ and $r_{\alpha,i} \in \mathbb{R}$ be the ith element of r_α, $\sigma_\beta = [\sigma_{\beta,1}, \sigma_{\beta,2}, \sigma_{\beta,3}]^{\rm T}$ and $\sigma_{\beta,i} \in \mathbb{R}$ be the ith element of σ_β, $p_{\rm dyn} = [p_{{\rm dyn},1}, \ldots, p_{{\rm dyn},\gamma-\zeta}]^{\rm T}$ and $p_{{\rm dyn},i} \in \mathbb{R}$ be the ith element of $p_{\rm dyn}$, $\hat{p}_{\rm dyn} = [\hat{p}_{{\rm dyn},1}, \ldots, \hat{p}_{{\rm dyn},\gamma}]^{\rm T}$ and $\hat{p}_{{\rm dyn},i} \in \mathbb{R}$ be the ith element of $\hat{p}_{\rm dyn}$, and $A \in \mathbb{R}^{n\times m}$ and $A_{(i,j)}$ be the entry of A on the ith row and jth column. Theorems V.2 and V.3 below provide necessary conditions for optimality of singular controls.

Theorem V.2 *Consider the performance measure (9) subject to (5), (14), (10), (11), and conditions $iii)$ and $iv)$ of Theorem IV.1. If $p_0^* = 0$ and $\hat{p}_{\rm dyn}^*(t) \in \mathcal{N}\big(\frac{\partial r_\alpha(q^*)}{\partial q^*}\big)$, $\alpha = 1, \ldots, \nu$, $t \in [\hat{t}_{1,\alpha,\rm tran}, \hat{t}_{2,\alpha,\rm tran}] \subset [t_1, t_2]$, then the candidate optimal singular translational control $u_{\alpha,\rm tran}^*(\cdot)$ is such that*

$$\frac{{\rm d}^2}{{\rm d}t^2}\left(\frac{\partial r_\alpha(q^*)}{\partial q^*}\hat{p}_{\rm dyn}^*(t)\right) = 0_3, \qquad t \in [\hat{t}_{1,\alpha,\rm tran}, \hat{t}_{2,\alpha,\rm tran}], \tag{32}$$

and

$$\begin{aligned}&\begin{bmatrix}\hat{p}_{\rm dyn}^{*\rm T}(t)\frac{\partial^2 r_{\alpha,1}(q^*)}{\partial q^{*2}}\\ \hat{p}_{\rm dyn}^{*\rm T}(t)\frac{\partial^2 r_{\alpha,2}(q^*)}{\partial q^{*2}}\\ \hat{p}_{\rm dyn}^{*\rm T}(t)\frac{\partial^2 r_{\alpha,3}(q^*)}{\partial q^{*2}}\end{bmatrix} F_\alpha(q^*(t)) \\ &\quad + \frac{\partial r_\alpha(q^*)}{\partial q^*}\begin{bmatrix}\sum_{\lambda=1}^{\gamma-\zeta} p_{{\rm dyn},\lambda}^*(t)\frac{\partial}{\partial q^*}\big(\mathcal{O}N_{\rm nh}\big(D^{-1}(q^*)\big)Z(q^*)\big)_{(1,\lambda)}\\ \cdots\\ \sum_{\lambda=1}^{\gamma-\zeta} p_{{\rm dyn},\lambda}^*(t)\frac{\partial}{\partial q^*}\big(\mathcal{O}N_{\rm nh}\big(D^{-1}(q^*)\big)Z(q^*)\big)_{(\gamma,\lambda)}\end{bmatrix} F_\alpha(q^*(t)) \le 0_{3\times 3}, \\ &\hspace{20em} t \in [\hat{t}_{1,\alpha,\rm tran}, \hat{t}_{2,\alpha,\rm tran}],\end{aligned} \tag{33}$$

where

$$F_\alpha(q) \triangleq D^{-1}(q)\begin{bmatrix}0_{\zeta\times\zeta} & 0_{\zeta\times(\gamma-\zeta)}\\ 0_{(\gamma-\zeta)\times\zeta} & Z(q)\end{bmatrix}D^{-\mathrm{T}}(q)\left(\frac{\partial r_\alpha(q)}{\partial q}\right)^{\mathrm{T}}. \tag{34}$$

Furthermore, given some neighborhood $\mathcal{I}(\hat{t}_{1,\alpha,\mathrm{tran}})$ and $\mathcal{I}(\hat{t}_{2,\alpha,\mathrm{tran}})$ of $\hat{t}_{1,\alpha,\mathrm{tran}}$ and $\hat{t}_{1,\alpha,\mathrm{tran}}$, respectively, if $u^*_{\alpha,\mathrm{tran}}(t)$ is piecewise analytic and (33) is verified for $t \in \mathcal{I}(\hat{t}_{1,\alpha,\mathrm{tran}})$ and $t \in \mathcal{I}(\hat{t}_{2,\alpha,\mathrm{tran}})$ with a strict inequality, then $u^*_{\alpha,\mathrm{tran}}(\cdot) \in C^{\phi_{1,\alpha,\mathrm{tran}}}(\mathcal{I}(\hat{t}_{1,\alpha,\mathrm{tran}}))$ and $u^*_{\alpha,\mathrm{tran}}(\cdot) \in C^{\phi_{2,\alpha,\mathrm{tran}}}(\mathcal{I}(\hat{t}_{2,\alpha,\mathrm{tran}}))$, where $\phi_{1,\alpha,\mathrm{tran}}$, $\phi_{2,\alpha,\mathrm{tran}} \in \mathbb{N}$ are odd numbers.

Theorem V.3 *Consider the performance measure (9) subject to (5), (14), (10), (11), and conditions $iii)$ and $iv)$ of Theorem IV.1. If $p^*_0 = 0$ and $\hat{p}^*_{\mathrm{dyn}}(t) \in \mathcal{N}\big(R^{-1}_{\mathrm{rod}}(\sigma_\beta(q^*(t)))\frac{\partial\sigma_\beta(q^*)}{\partial q^*}\big)$, $\beta = 1,\ldots,\kappa$, $t \in [\hat{t}_{1,\beta,\mathrm{rot}}, \hat{t}_{2,\beta,\mathrm{rot}}] \subset [t_1, t_2]$, then the candidate optimal singular rotational control $u^*_{\beta,\mathrm{rot}}(\cdot)$ is such that*

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(R^{-1}_{\mathrm{rod}}(\sigma_\beta(q^*(t)))\frac{\partial\sigma_\beta(q^*)}{\partial q^*}\hat{p}^*_{\mathrm{dyn}}(t)\right) = 0_3, \qquad t \in [\hat{t}_{1,\beta,\mathrm{rot}}, \hat{t}_{2,\beta,\mathrm{rot}}], \tag{35}$$

and

$$\begin{aligned}\Bigg[&\begin{bmatrix}\sum_{\lambda=1}^{3}\left(\frac{\partial\sigma_\beta(q^*)}{\partial q^*}\hat{p}^*_{\mathrm{dyn}}(t)\right)_\lambda \frac{\partial(R^{-1}_{\mathrm{rod}}(\sigma^*_\beta))_{(1,\lambda)}}{\partial\sigma^*_\beta}\\ \sum_{\lambda=1}^{3}\left(\frac{\partial\sigma_\lambda(q^*)}{\partial q^*}\hat{p}^*_{\mathrm{dyn}}(t)\right)_\lambda \frac{\partial(R^{-1}_{\mathrm{rod}}(\sigma^*_\beta))_{(2,\lambda)}}{\partial\sigma^*_\beta}\\ \sum_{\lambda=1}^{3}\left(\frac{\partial\sigma_\lambda(q^*)}{\partial q^*}\hat{p}^*_{\mathrm{dyn}}(t)\right)_\lambda \frac{\partial(R^{-1}_{\mathrm{rod}}(\sigma^*_\beta))_{(3,\lambda)}}{\partial\sigma^*_\beta}\end{bmatrix}\frac{\partial\sigma_\beta(q^*)}{\partial q^*}\\ &+ R^{-1}_{\mathrm{rod}}(\sigma_\beta(q^*(t)))\Bigg(\begin{bmatrix}\hat{p}^{*\mathrm{T}}_{\mathrm{dyn}}(t)\frac{\partial^2\sigma_{\beta,1}(q^*)}{\partial^2 q^*}\\ \hat{p}^{*\mathrm{T}}_{\mathrm{dyn}}(t)\frac{\partial^2\sigma_{\beta,2}(q^*)}{\partial^2 q^*}\\ \hat{p}^{*\mathrm{T}}_{\mathrm{dyn}}(t)\frac{\partial^2\sigma_{\beta,3}(q^*)}{\partial^2 q^*}\end{bmatrix}\\ &+ \frac{\partial\sigma_\beta(q^*)}{\partial q^*}\begin{bmatrix}\sum_{\lambda=1}^{\gamma-\zeta} p^*_{\mathrm{dyn},\lambda}(t)\frac{\partial}{\partial q^*}\big(\mathcal{O}N_{\mathrm{nh}}\big(D^{-1}(q^*)\big)Z(q^*)\big)_{(1,\lambda)}\\ \cdots\\ \sum_{\lambda=1}^{\gamma-\zeta} p^*_{\mathrm{dyn},\lambda}(t)\frac{\partial}{\partial q^*}\big(\mathcal{O}N_{\mathrm{nh}}\big(D^{-1}(q^*)\big)Z(q^*)\big)_{(\gamma,\lambda)}\end{bmatrix}\Bigg)\Bigg]H_\beta(q^*(t)) \leq 0_{3\times3},\\ &\qquad t \in [\hat{t}_{1,\beta,\mathrm{rot}}, \hat{t}_{2,\beta,\mathrm{rot}}],\end{aligned} \tag{36}$$

where

$$H_\beta(q) \triangleq D^{-1}(q)\begin{bmatrix}0_{\zeta\times\zeta} & 0_{\zeta\times(\gamma-\zeta)}\\ 0_{(\gamma-\zeta)\times\zeta} & Z(q)\end{bmatrix}D^{-\mathrm{T}}(q)\left(R^{-1}_{\mathrm{rod}}(\sigma_\beta)\frac{\partial\sigma_\beta(q)}{\partial q}\right)^{\mathrm{T}}. \tag{37}$$

Furthermore, given some neighborhood $\mathcal{I}(\hat{t}_{1,\beta,\mathrm{rot}})$ and $\mathcal{I}(\hat{t}_{2,\beta,\mathrm{rot}})$ of $\hat{t}_{1,\beta,\mathrm{rot}}$ and $\hat{t}_{2,\beta,\mathrm{rot}}$, respectively, if $u^*_{\beta,\mathrm{rot}}(t)$ is piecewise analytic and (36) is verified for $t \in \mathcal{I}(\hat{t}_{1,\beta,\mathrm{rot}})$ and $t \in \mathcal{I}(\hat{t}_{2,\beta,\mathrm{rot}})$ with a strict inequality, then $u^*_{\beta,\mathrm{rot}}(\cdot) \in C^{\phi_{1,\beta,\mathrm{rot}}}(\mathcal{I}(\hat{t}_{1,\beta,\mathrm{rot}}))$ and $u^*_{\beta,\mathrm{rot}}(\cdot) \in C^{\phi_{2,\beta,\mathrm{rot}}}(\mathcal{I}(\hat{t}_{2,\beta,\mathrm{rot}}))$, where $\phi_{1,\beta,\mathrm{rot}}$, $\phi_{2,\beta,\mathrm{rot}} \in \mathbb{N}$ are odd numbers.

Example V.3 *Consider the dynamical systems presented in Example III.1. In this case, $H_1(\sigma_1) = R_{\mathrm{rod}}(\sigma_1)\mathrm{I}^{-1}_{\mathrm{in},1}$ and if $p^*_0 = 0$ and $p^*_{\mathrm{dyn}} = 0_3$, then it follows from Theorem V.3 that (35) and (36) are satisfied. In fact, the left-hand-side* *of* (35) *reduces to* (31)*, which is identically equal to zero, and the left-hand-side of* (36) *reduces to*

$$\left(\begin{bmatrix}\sum_{\lambda=1}^{3}\frac{\partial(R_{\text{rod}}^{-1}(\sigma_1^*))_{(1,\lambda)}}{\partial\sigma_1^*}\hat{p}^*_{\text{dyn},\lambda}\\ \sum_{\lambda=1}^{3}\frac{\partial(R_{\text{rod}}^{-1}(\sigma_1^*))_{(2,\lambda)}}{\partial\sigma_1^*}\hat{p}^*_{\text{dyn},p}\\ \sum_{\lambda=1}^{3}\frac{\partial(R_{\text{rod}}^{-1}(\sigma_1^*))_{(3,\lambda)}}{\partial\sigma_1}\hat{p}^*_{\text{dyn},\lambda}\end{bmatrix}\right.$$

$$\left.+R_{\text{rod}}^{-1}(\sigma_1^*(t))\begin{bmatrix}\sum_{\lambda=1}^{3}\frac{\partial(R_{\text{rod}}(\sigma_1^*))_{(1,\lambda)}}{\partial\sigma_1^*}\left(I_{\text{in},1}^{-1}p^*_{\text{dyn}}\right)_\lambda\\ \sum_{\lambda=1}^{3}\frac{\partial(R_{\text{rod}}(\sigma_1^*))_{(2,\lambda)}}{\partial\sigma_1^*}\left(I_{\text{in},1}^{-1}p^*_{\text{dyn}}\right)_\lambda\\ \sum_{\lambda=1}^{3}\frac{\partial(R_{\text{rod}}(\sigma_1^*))_{(3,\lambda)}}{\partial\sigma_1^*}\left(I_{\text{in},1}^{-1}p^*_{\text{dyn}}\right)_\lambda\end{bmatrix}\right)R_{\text{rod}}(\sigma_1^*(t))I_{\text{in},1}^{-1}, \tag{38}$$

which is identically equal to the zero matrix. Alternatively, the same result can be achieved directly applying Theorem IV.2 and Corollary IV.1 to the problem of minimizing (9) *subject to* (8), (14), (10), (11)*, and conditions iii) and iv) of Theorem IV.1 with $N=1$, $q_{\text{dot}}=\omega_1$, $D(q)=R_{\text{rod}}^{-1}(\sigma_1)$, $r=0$, $a(\tilde{x}_1)=0$, and $m(\tilde{x}_1)=0$.* △

B. The Normal Optimization Problem

In this section, we provide necessary conditions to solve the normal control effort minimization problem, that is, we assume that $p_0^*=1$ in Theorem IV.1. To this goal, recall that $\mu_\alpha\in(0,1)$, $\alpha=1,\ldots,N$, is a measure of the relative importance of minimizing the control effort of the αth body with respect to the others.

The next lemma is a necessary condition to minimize the performance measure (9) subject to the dynamic equations (5) and the constraints (10) and (11). This result also proves some necessary condition for the existence of singular translational and rotational controls.

Lemma V.2 *Consider the performance measure* (9) *subject to* (5), (14), (10), (11)*, and conditions iii) and iv) of Theorem IV.1. Let $p_0^*=1$ and*

$$\hat{p}^*_{\text{dyn}}(t)\in\left(\bigcap_{\alpha=1}^{\nu}\mathcal{N}\left(\frac{\partial r_\alpha(q^*)}{\partial q^*}\right)\right)\cap\left(\bigcap_{\beta=1}^{\kappa}\mathcal{N}\left(\frac{\partial\sigma_\beta(q^*)}{\partial q^*}\right)\right),\qquad t\in[\hat{t}_1,\hat{t}_2]\subset[t_1,t_2].$$

Then, the performance measure (9) *is minimized if, for all $t\in[\hat{t}_1,\hat{t}_2]$, the following three conditions hold:*
a) $u_{\delta_1}(t)=0_6$, $\delta_1=1,\ldots,\min\{\nu,\kappa\}$,
b) if $\nu\leq\kappa$, (respectively, $\kappa\leq\nu$), then $u^*_{\delta_3,\text{rot}}(t)=0_3$ (respectively, $u^*_{\delta_3,\text{tran}}(t)=0_3$), $\delta_3=\nu+1,\ldots,\kappa$ (respectively, $\delta_3=\kappa+1,\ldots,\nu$), $\frac{u^*_{\delta_3,\text{tran}}(t)}{\|u^*_{\delta_3,\text{tran}}(t)\|_2}$ is parallel to $-\frac{\partial r_{\delta_3}(q^*)}{\partial q^*}\hat{p}^*_{\text{dyn}}(t)$, (respectively, $\frac{u^*_{\delta_3,\text{rot}}(t)}{\|u^*_{\delta_3,\text{rot}}(t)\|_2}$ is parallel to $-R_{\text{rod}}^{-1}(\sigma_{\delta_3}(q^*(t)))\frac{\partial\sigma_{\delta_3}(q^*)}{\partial q^*}\hat{p}^*_{\text{dyn}}(t)$), and,

i) *if $\mu_{\delta_3}>\left\|\frac{\partial r_{\delta_3}(q^*)}{\partial q^*}\hat{p}^*_{\text{dyn}}(t)\right\|_2$, then $u^*_{\delta_3,\text{tran}}(t)=0_3$,*

ii) *if $\mu_{\delta_3}<\left\|\frac{\partial r_{\delta_3}(q^*)}{\partial q^*}\hat{p}^*_{\text{dyn}}(t)\right\|_2$, then $\|u^*_{\delta_3,\text{tran}}(t)\|_2=\rho_{\delta_3,2}$,*

(respectively,

i) *if $\mu_{\delta_3}>\left\|R_{\text{rod}}^{-1}(\sigma_{\delta_3}(q^*(t)))\frac{\partial\sigma_{\delta_3}(q^*)}{\partial q^*}\hat{p}^*_{\text{dyn}}(t)\right\|_2$, then $u^*_{\delta_3,\text{rot}}(t)=0_3$,*

ii) *if* $\mu_{\delta_3} < \left\| R_{\rm rod}^{-1}(\sigma_{\delta_3}(q^*(t)))\frac{\partial\sigma_{\delta_3}(q^*)}{\partial q^*}\hat{p}_{\rm dyn}^*(t)\right\|_2$, *then* $\|u_{\delta_3,{\rm rot}}^*(t)\|_2 = \rho_{\delta_3,4}$*.)*

c) $u_{\delta_2,{\rm tran}}^*(t)$ *is parallel to* $-\frac{\partial r_{\delta_2}(q^*)}{\partial q^*}\hat{p}_{\rm dyn}^*(t)$*,* $u_{\delta_2,{\rm rot}}^*(t)$ *is parallel to* $-R_{\rm rod}^{-1}(\sigma_{\delta_2}(q^*(t)))\frac{\partial\sigma_{\delta_2}(q^*)}{\partial q^*}\hat{p}_{\rm dyn}^*(t)$*,* $\delta_2 = \max\{\nu, \kappa\}, \ldots, N$*, and*

i) *if* $\mu_{\delta_2} > \left\|\frac{\partial r_{\delta_2}(q^*)}{\partial q^*}\hat{p}_{\rm dyn}^*(t)\right\|_2$, *then* $u_{\delta_2,{\rm tran}}^*(t) = 0_3$*,*

ii) *if* $\mu_{\delta_2} > \left\| R_{\rm rod}^{-1}(\sigma_{\delta_2}(q^*(t)))\frac{\partial\sigma_{\delta_2}(q^*)}{\partial q^*}\hat{p}_{\rm dyn}^*(t)\right\|_2$, *then* $u_{\delta_2,{\rm rot}}^*(t) = 0_3$*,*

iii) *if* $\mu_{\delta_2} < \left\|\frac{\partial r_{\delta_2}(q^*)}{\partial q^*}\hat{p}_{\rm dyn}^*(t)\right\|_2$, *then* $\|u_{\delta_2,{\rm tran}}^*(t)\|_2 = \rho_{\delta_2,2}$*,*

iv) *if* $\mu_{\delta_2} < \left\| R_{\rm rod}^{-1}(\sigma_{\delta_2}(q^*(t)))\frac{\partial\sigma_{\delta_2}(q^*)}{\partial q^*}\hat{p}_{\rm dyn}^*(t)\right\|_2$, *then* $\|u_{\delta_2,{\rm rot}}^*(t)\|_2 = \rho_{\delta_2,4}$*.*

Furthermore, if $\nu \leq \kappa$ *(respectively,* $\kappa \leq \nu$*) and*

$$\mu_{\delta_3} = \left\|\frac{\partial r_{\delta_3}(q^*)}{\partial q^*}\hat{p}_{\rm dyn}^*(t)\right\|_2, \qquad \delta_3 = \nu+1, \ldots, \kappa, \tag{39}$$

(respectively,

$$\mu_{\delta_3} = \left\| R_{\rm rod}^{-1}(\sigma_{\delta_3}(q^*(t)))\frac{\partial\sigma_{\delta_3}(q^*)}{\partial q^*}\hat{p}_{\rm dyn}^*(t)\right\|_2, \qquad \delta_3 = \kappa+1, \ldots, \nu,) \tag{40}$$

then $u_{\delta_3,{\rm tran}}^*(t)$ *(respectively,* $u_{\delta_3,{\rm rot}}^*(t)$*) is a candidate optimal singular translational (respectively, rotational) control. Finally, if*

$$\mu_{\delta_2} = \left\|\frac{\partial r_{\delta_2}(q^*)}{\partial q^*}\hat{p}_{\rm dyn}^*(t)\right\|_2, \qquad \delta_2 = \max\{\nu, \kappa\}, \ldots, N, \tag{41}$$

then $u_{\delta_2,{\rm tran}}^*(t)$ *is unspecified and, if*

$$\mu_{\delta_2} = \left\| R_{\rm rod}^{-1}(\sigma_{\delta_2}(q^*(t)))\frac{\partial\sigma_{\delta_2}(q^*)}{\partial q^*}\hat{p}_{\rm dyn}^*(t)\right\|_2, \qquad \delta_2 = \max\{\nu, \kappa\}, \ldots, N, \tag{42}$$

then $u_{\delta_2,{\rm rot}}^*(t)$ *is unspecified.*

The following result proves that the order of singularity of candidate optimal singular translational and rotational controls is equal to one in the normal optimization problem.

Theorem V.4 *Assume the conditions of Lemma V.2 are verified. If* $\nu \leq \kappa$ *(respectively,* $\kappa \leq \nu$*) and (39) (respectively, (40)) holds, then the order of the singularity of the candidate optimal singular translational (respectively, rotational) control* $u_{\delta_3,{\rm tran}}^*(t)$*,* $\delta_3 = \nu+1, \ldots, \kappa$*, (respectively,* $u_{\delta_3,{\rm rot}}^*(t)$*,* $\delta_3 = \kappa+1, \ldots, \nu$*),* $t \in [\hat{t}_1, \hat{t}_2] \subset [t_1, t_2]$*, is one, that is,* $\nu_{\delta_3,{\rm tran}} = 1$ *(respectively,* $\nu_{\delta_3,{\rm rot}} = 1$*.)*

For the statement of the next result, the left-hand-sides of (33) and (36) are denoted by ((33)) and ((36)), respectively.

Theorem V.5 *Assume the conditions of Lemma V.2 are verified. If* $\nu \leq \kappa$ *(respectively,* $\kappa \leq \nu$*), (39) (respectively, (40)) is satisfied, and* $\|u_{\delta_3,{\rm tran}}^*(t)\|_2 \neq 0$ *(respectively,* $\|u_{\delta_3,{\rm rot}}^*(t)\|_2 \neq 0$*),* $t \in [\hat{t}_1, \hat{t}_2] \subset [t_1, t_2]$ *and* $\delta_3 = \nu+1, \ldots, \kappa$ *(respectively,* $\delta_3 = \kappa+1, \ldots, \nu$*), then the candidate optimal singular translational (respectively, rotational) optimal control* $u_{\delta_3,{\rm tran}}^*(t)$ *(respectively,* $u_{\delta_3,{\rm rot}}^*(t)$*) is such that*

$$\frac{{\rm d}^2}{{\rm d}t^2}\left\|\frac{\partial r_{\delta_3}(q^*)}{\partial q^*}\hat{p}_{\rm dyn}^*(t)\right\|_2 = 0, \qquad t \in [\hat{t}_1, \hat{t}_2], \tag{43}$$

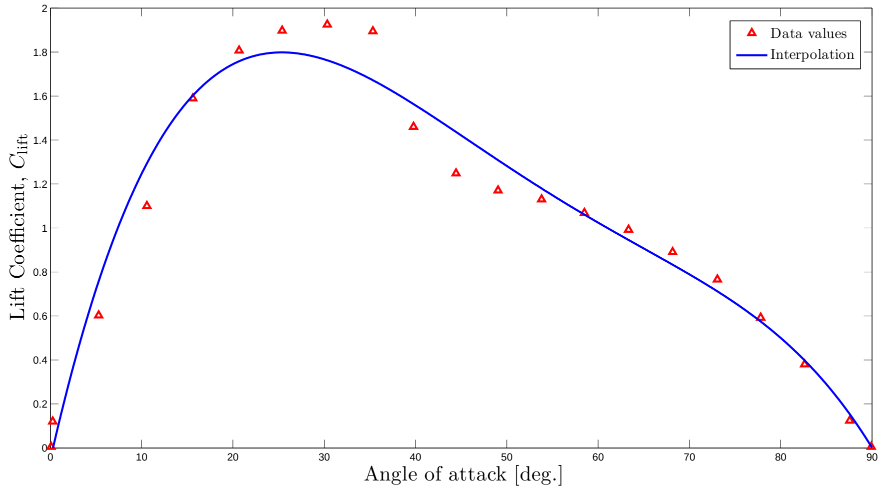

Figure 1. Lift coefficient as function of the angle of attack.

(respectively,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left\|R_{\rm rod}^{-1}(\sigma_{\delta_3}(q^*))\frac{\partial\sigma_{\delta_3}(q^*)}{\partial q^*}\hat{p}_{\rm dyn}^*(t)\right\|_2=0,\qquad t\in[\hat{t}_1,\hat{t}_2],) \tag{44}$$

and

$$\left(\frac{\partial r_{\delta_3}(q^*)}{\partial q^*}\hat{p}_{\rm dyn}^*(t)\right)^{\rm T}((33))\frac{u_{\delta_3,{\rm tran}}^*(t)}{\|u_{\delta_3,{\rm tran}}^*(t)\|_2}\geq 0,,\qquad t\in[\hat{t}_1,\hat{t}_2], \tag{45}$$

(respectively,

$$\left(R_{\rm rod}^{-1}(\sigma_{\delta_3}(q^*))\frac{\partial\sigma_{\delta_3}(q^*)}{\partial q^*}\hat{p}_{\rm dyn}^*(t)\right)^{\rm T}((36))\frac{u_{\delta_3,{\rm rot}}^*(t)}{\|u_{\delta_3,{\rm rot}}^*(t)\|_2}\geq 0,\qquad t\in[\hat{t}_1,\hat{t}_2].) \tag{46}$$

Furthermore, given some neighborhood $\mathcal{I}(\hat{t}_1)$ and $\mathcal{I}(\hat{t}_2)$ of $\hat{t}_1$ and $\hat{t}_2$, respectively, if $u_{\alpha,{\rm tran}}^*(t)$ (respectively, $u_{\alpha,{\rm rot}}^*(t)$) is piecewise analytic and (45) (respectively, (46)) is verified for $t\in\mathcal{I}(\hat{t}_1)$ and $t\in\mathcal{I}(\hat{t}_2)$ with a strict inequality, then $u_{\delta_3,{\rm tran}}^*(\cdot)\in C^{\phi_{1,\delta_3,{\rm tran}}}(\mathcal{I}(\hat{t}_1))$ and $u_{\delta_3,{\rm tran}}^*(\cdot)\in C^{\phi_{2,\delta_3,{\rm tran}}}(\mathcal{I}(\hat{t}_2))$, (respectively, $u_{\delta_3,{\rm rot}}^*(\cdot)\in C^{\phi_{1,\delta_3,{\rm rot}}}(\mathcal{I}(\hat{t}_1))$ and $u_{\delta_3,{\rm rot}}^*(\cdot)\in C^{\phi_{2,\delta_3,{\rm rot}}}(\mathcal{I}(\hat{t}_2))$), where $\phi_{1,\delta_3,{\rm tran}}$ and $\phi_{2,\delta_3,{\rm tran}}$ (respectively, $\phi_{1,\delta_3,{\rm rot}}$ and $\phi_{2,\delta_3,{\rm rot}}\in\mathbb{N}$) are odd numbers.

Example V.4 *Consider the dynamical system presented in Example III.1. If $p_0^*=1$ and $p_{\rm dyn}^*(t)=0_3$, $t\in[\hat{t}_1,\hat{t}_2]\subset[t_1,t_2]$, then it follows from condition a) of Lemma V.2 that $u_{1,{\rm rot}}^*(t)=0_3$. Alternatively, suppose that $p_0^*=1$ and $p_{\rm dyn}^*(t)\neq 0_3$, $t\in[\hat{t}_1,\hat{t}_2]$. Then, it follows from condition b) of Lemma V.2 that if $1>\left\|I_{{\rm in},1}^{-1}p_{\rm dyn}^*(t)\right\|_2$, then $u_{1,{\rm rot}}^*(t)=0_3$, $t\in(\hat{t}_1,\hat{t}_2)$. Alternatively, if $1<\left\|I_{\rm in}^{-1}p_{\rm dyn}^*(t)\right\|_2$, then $\|u_{1,{\rm rot}}^*(t)\|_2=\rho_{1,4}$, and if $1=\left\|I_{\rm in}^{-1}p_{\rm dyn}^*(t)\right\|_2$, then $u_{1,{\rm rot}}^*(t)$ is parallel to $-I_{{\rm in},1}^{-1}p_{\rm dyn}^*(t)$ and $u_{1,{\rm rot}}^*(t)$ is a singular rotational control. Furthermore, it follows from Theorem V.5 that the candidate optimal rotational singular control $u_{1,{\rm rot}}^*(t)$ is such that $p_{\rm dyn}^{*{\rm T}}(t)I_{{\rm in},1}^{-1}((38))\frac{u_{1,{\rm rot}}^*(t)}{\|u_{1,{\rm rot}}^*(t)\|_2}\leq 0$, $t\in[\hat{t}_1,\hat{t}_2]$.* $\triangle$

VI. Illustrative Numerical Example

In this section, we present a numerical example to highlight the efficacy of the framework presented in this paper. In particular, given a formation of two F-16 Fighting Falcons [23], we find the controls that minimize (9) while performing an Immelmann turn [41], which consists in rolling the aircraft in inverted flight and simultaneously executing an ascending half-loop so that, at the end of the maneuver, the aircraft move in the opposite direction at a higher altitude and in level flight.

In this example, we impose that the two aircraft have the same attitude during the course of the maneuver and the

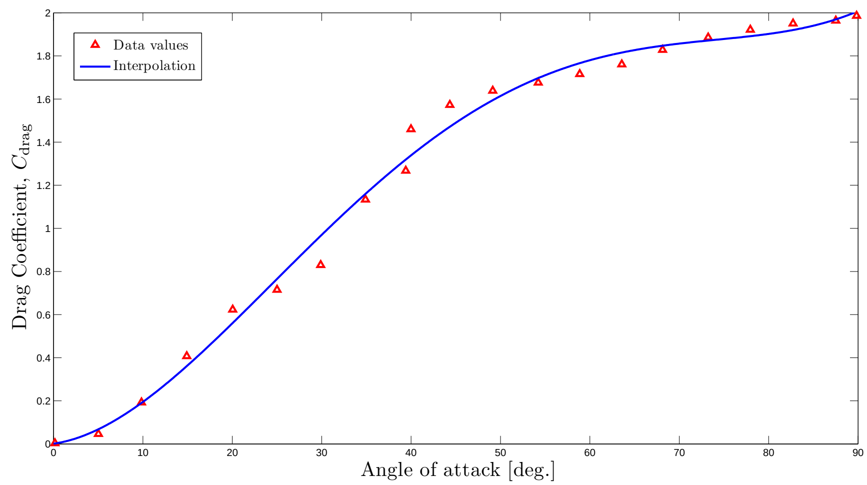

Figure 2. Drag coefficient as function of the angle of attack.

distance between the vehicles' centers of mass ranges between 100 and 200 meters, that is, $N = 2$,

$$0 = \sigma_1(t) - \sigma_2(t), \qquad t \in [t_1, t_2], \tag{47}$$
$$100 < \|r_1(t) - r_2(t)\| < 200. \tag{48}$$

In this case, we introduce two additional parameters $s_i : [t_1, t_2] \to \mathbb{R}$, $i = 1, 2$, known as *slack variables*, so that $\gamma = 9$ and $q = [s_1,\, r_{1,1},\, r_{1,2},\, \sigma_1^{\mathrm{T}},\, r_2^{\mathrm{T}}]^{\mathrm{T}}$. The choice of the independent generalized coordinates for a system of two rigid bodies subject to the constraints (47) and (48) is discussed in detail in [11].

The external moment acting on each aircraft is neglected, that is, $m(\tilde{x}_1) = m(\tilde{x}_2) = 0$, $\tilde{x}_1, \tilde{x}_2 \in \mathbb{R}^{12}$. The external force acting on each aircraft is the sum of the gravitational force, the aerodynamic lift, and the aerodynamic drag, that is, $a(\tilde{x}_\alpha) = m_\alpha g + a_{\text{lift},\alpha}(v_\alpha) + a_{\text{drag},\alpha}(v_\alpha)$, $\alpha = 1, 2$. Specifically, the gravitational force acting on the αth vehicle, $\alpha = 1, 2$, is modeled as $m_\alpha g$, where $m_1 = m_2 = 19 \cdot 10^3\,\mathrm{kg}$ and $g = [0,\, 0,\, 9.81]^{\mathrm{T}} \frac{\mathrm{m}}{\mathrm{s}^2}$ in an Earth-fixed reference frame. The aerodynamic lift is modeled as

$$a_{\text{lift},\alpha}(v_\alpha) = -\frac{1}{2}\rho S C_{\text{lift}}(v_\alpha)\|v_\alpha\|_2^2 z_{\text{wind},\alpha}(t) \tag{49}$$

and the aerodynamic drag is modeled as

$$a_{\text{drag},\alpha}(v_\alpha) = -\frac{1}{2}\rho S C_{\text{drag}}(v_\alpha)\|v_\alpha\|_2 v_\alpha(t), \tag{50}$$

where ρ denotes the air density, which we assume constant and equal to $1.27\,\frac{\mathrm{kg}}{\mathrm{m}^3}$, S denotes the the wing planform area, which is equal to $26\,\mathrm{m}^2$, C_{lift} and $C_{\text{drag}} : \mathbb{R}^3 \to \mathbb{R}$ denote the lift and drag coefficients, respectively, and $z_{\text{wind},\alpha} : [t_1, t_2] \to \mathbb{R}^3$ lays in the αth aircraft plane of symmetry, so that $z_{\text{wind},\alpha}^{\mathrm{T}}(t) v_\alpha(t) = 0$ and $\|z_{\text{wind},\alpha}(t)\|_2 = 1$, $t \in [t_1, t_2]$, $\alpha = 1, 2$. In [42], the authors provide the lift and drag coefficients C_{lift} and C_{drag} for an F-16 aircraft at 21 angles of attack and, as shown in Figure 1 and 2, in this paper we use fourth order polynomials to deduce a continuously differentiable dependence of the lift and drag coefficients on the angle of attack.

The vehicles' translational control is provided by the thrust force, which is applied along the first axis of the body reference frame and has magnitude ranging between zero and $1.9 \cdot 10^3\,\mathrm{N}$. The rotational control is the sum of the moments induced by the aileron, elevator, and rudder deflections, that is,

$$\begin{aligned} m_{\text{roll}}(v_\alpha, \delta_{\text{aileron}}) = {} & \frac{1}{8}\rho S \frac{\mathrm{d}C_{\text{lift}}(v_\alpha)}{\mathrm{d}\varepsilon} \delta_{\text{aileron}} \|v_\alpha\|_2^2 l_{\text{aileron}} x_{\text{wind},\alpha}(t) \\ & - \frac{1}{10}\rho S \frac{\mathrm{d}C_{\text{lift}}(v_\alpha)}{\mathrm{d}\varepsilon} \delta_{\text{rudder}} \|v_\alpha\|_2^2 h_{\text{rudder}} x_{\text{wind},\alpha}(t), \end{aligned} \tag{51}$$

$$m_{\text{pitch}}(v_\alpha, \delta_{\text{elevator}}) = -\frac{1}{12}\rho S \frac{\mathrm{d}C_{\text{lift}}(v_\alpha)}{\mathrm{d}\varepsilon} \delta_{\text{elevator}} \|v_\alpha\|_2^2 l_{\text{elevator}} y_{\text{wind},\alpha}(t), \tag{52}$$

$$m_{\text{yaw}}(v_\alpha, \delta_{\text{rudder}}) = \frac{1}{10}\rho S \frac{\mathrm{d}C_{\text{lift}}(v_\alpha)}{\mathrm{d}\varepsilon} \delta_{\text{rudder}} \|v_\alpha\|_2^2 l_{\text{elevator}} z_{\text{wind},\alpha}(t), \tag{53}$$

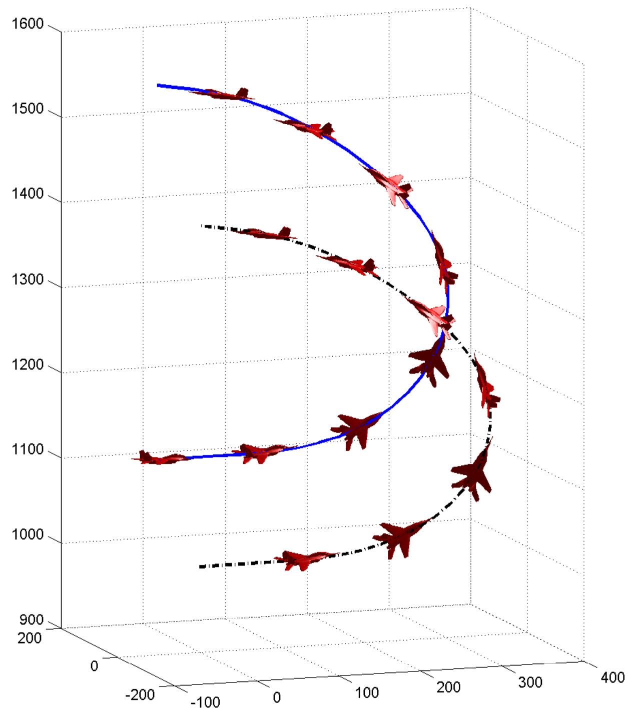

Figure 3. Immelmann turn for a formation of two F-16.

where $\delta_{\text{aileron}} \in [-40,\, 40]\,\text{deg}$ denotes the aileron deflection, $\delta_{\text{elevator}} \in [-40,\, 40]\,\text{deg}$ denotes the elevator deflection, $\delta_{\text{rudder}} \in [-40,\, 40]\,\text{deg}$ denotes the rudder deflection, $l_{\text{aileron}} = 4.98\,\text{m}$ denotes half of the F-16 wingspan, $l_{\text{elevator}} = 7.53\,\text{m}$ denotes the distance between the aircraft center of mass and the elevator aerodynamic center, $h_{\text{rudder}} = 2\text{m}$ denotes the distance between the aircraft center of mass and the stabilizer aerodynamic center, $x_{\text{wind},\alpha} : [t_1, t_2] \to \mathbb{R}^3$ lays in the plane of symmetry of the aircraft, so that $z^{\text{T}}_{\text{wind},\alpha}(t) x_{\text{wind},\alpha}(t) = 0$ and $\|x_{\text{wind},\alpha}(t)\|_2 = 1$, $t \in [t_1, t_2]$, $\alpha = 1, 2$, $y_{\text{wind},\alpha} : [t_1, t_2] \to \mathbb{R}^3$ is orthogonal to the plane of symmetry of the aircraft, $\|y_{\text{wind},\alpha}(t)\|_2 = 1$, $t \in [t_1, t_2]$, $\alpha = 1, 2$, and $\frac{\mathrm{d}C_{\text{lift}}(v_\alpha)}{\mathrm{d}\varepsilon}$ denotes the variation of the lift coefficient with respect to the angle of attack, which can be deduced from Figure 1. In this paper, we ignore forces and moments induced by the aerodynamic coupling, such as the yaw moment induced by the aileron deflection, with the exception of the roll moment induced by the rudder deflection.

For this numerical example, the control parameters are the thrust force and the aileron, elevator, and rudder deflections. Since the magnitude of the thrust is considerably greater than the magnitude of the moments induced by the deflection of the control surfaces and F-16 are propelled by conventional fuel-based engines, the performance measure (9) provides a good estimate of the fuel consumption for this aircraft formation. The proposed optimal control problem is normal and the optimality of numerical results computed using GPOPS [26], a commercial software for numerical optimization, is verified by applying Lemma V.2. In addition, Lemma V.2 allows identifying singular translational controls in the second vehicle's aileron deflection for $t \in [1.87, 2.02]$s. Therefore, as recommended in [27], in order to increase the accuracy of the numerical results, the second vehicle's optimal controls are computed in a dedicated numerical simulation in this interval. For this optimal control problem, the aircraft trajectories are shown in Figure 3 and the control inputs, the roll, pitch, and yaw angles, and the Hamiltonian function of the second vehicle are shown in Figures 4, 5, and 6, respectively.

VII. Conclusion

In this paper, we addressed the control effort minimization problem for systems of N rigid bodies. Specifically, this study has provided a complete discussion of the abnormal optimization problem, has given a further insight in the solution to the normal control effort minimization problem, has proven that singular controls have order of singularity equal to one, has shown that singular controls are always analytical and differentiable an odd number of times in the neighborhood of the junctions between singular and non-singular arcs, and has presented a set of second order

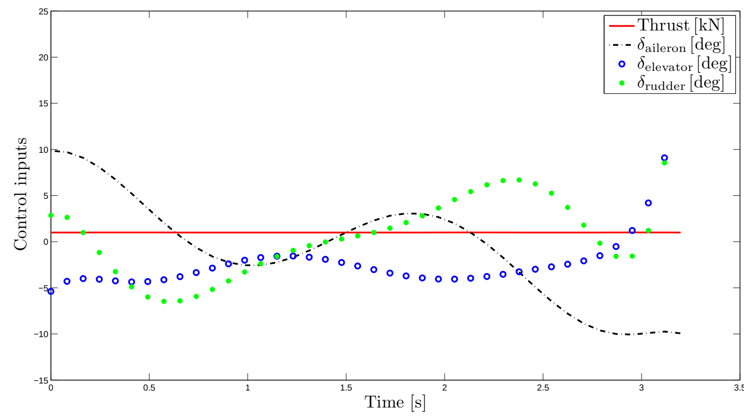

Figure 4. Control inputs for the second aircraft.

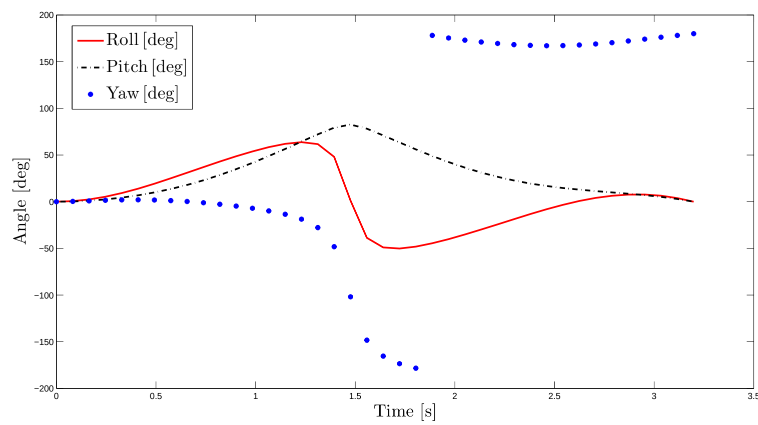

Figure 5. Roll, pitch, and yaw angles for the second aircraft.

differential equations that are verified by translational and rotational candidate optimal singular controls. Finally, analytical results achieved have been applied to verify the optimality of a maneuver for a formation of F-16.

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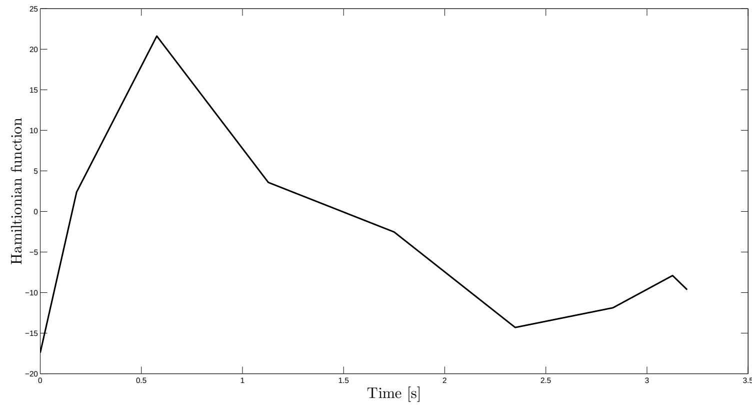

Figure 6. Hamiltonian function for the second aircraft.

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