Finite-Time Partial Stability Theory and Fractional Lyapunov Differential Inequalities

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Abstract— **In many practical applications, stability with respect to part of the system's states is often necessary with finite-time convergence to the equilibrium state of interest. Finite-time partial stability involves dynamical systems whose part of the trajectory converges to an equilibrium state in finite time. Since finite-time convergence implies non-uniqueness of system solutions in backward time, such systems possess non-Lipschitzian dynamics. In this paper, we address finite-time partial stability and uniform finite-time partial stability for nonlinear dynamical systems. Specifically, we provide Lyapunov conditions involving a Lyapunov function that is positive definite and decrescent with respect to part of the system state, and satisfies a differential inequality involving fractional powers for guaranteeing finite-time partial stability. In addition, we show that finite-time partial stability leads to uniqueness of solutions in forward time and we establish necessary and sufficient conditions for continuity of the settling-time function of the nonlinear dynamical system.**

I. Introduction

In many engineering applications *partial stability*, that is, stability with respect to part of the system's states, is often necessary. In particular, partial stability arises in the study of electromagnetics [1], inertial navigation systems [2], spacecraft stabilization via gimballed gyroscopes and/or flywheels [3], combustion systems [4], vibrations in rotating machinery [5], and biocenology [6], to cite but a few examples. For example, in the field of biocenology involving Lotka-Volterra predator-prey models of population dynamics with age structure, if some of the species preyed upon are left alone, then the corresponding population increases without bound while a subset of the prey species remains stable [6, pp. 260-269]. The need to consider partial stability in the aforementioned systems arises from the fact that stability notions involve equilibrium coordinates as well as a hyperplane of coordinates that is closed but *not* compact. Hence, partial stability involves motion lying in a subspace instead of an equilibrium point.

Another important notion in engineering applications is *finite-time stability*, wherein it is desirable for a dynamical system to possess the property that trajectories that converge to a Lyapunov stable equilibrium state must do so in finite time rather than merely asymptotically. In the case when the dynamics of a time-varying system are Lipschitz continuous in the state and piecewise continuous in time, the dynamical system possesses a unique solution in forward and backward times for a given pair of initial time and state [7]. However, if a dynamical system possesses trajectories that converge to an equilibrium point in finite time, then clearly this system has multiple solutions starting at the equilibrium point with time running backwards. Hence, such systems cannot be Lipschitz continuous at the equilibrium point.

The absence of Lipschitz continuity is only a necessary condition for non-uniqueness of the system trajectories, and uniqueness of solutions in forward time can be preserved in the case of finite-time convergence. Sufficient conditions that ensure uniqueness of solutions in forward time in the absence of Lipschitz continuity are given in [8]–[11]. In addition, it is shown in [12, Th. 4.3, p. 59] that uniqueness of solutions in forward time along with continuity of the system dynamics ensure that the system solutions are continuous functions of the system initial conditions even when the dynamics are not Lipschitz continuous.

In [13], a rigorous foundation for the theory of finite-time stability for autonomous nonlinear systems was developed. In [14], the authors extended the results of [13] to address finite-time and uniform finite-time stability for time-varying systems. In this paper, we merge the notions of finite-time stability and partial stability to develop analysis results for finite-time partial stability. Specifically, we establish necessary and sufficient conditions for continuity of the settling-time function, that is, the time at which a system trajectory reaches a partial equilibrium state. In addition, using the comparison principle we provide a Lyapunov theorem for finite-time partial stability in terms of a Lyapunov function that is positive definite and decrescent with respect to part of the system state, and satisfies a differential inequality involving fractional powers. Finally, since partial stability theory provides a unification between time-invariant stability theory and stability theory for time-varying systems [15], [16], we specialize our results to address finite-time stability for nonlinear time-varying systems and provide connections to [14]. A numerical example is presented to demonstrate the efficacy of the proposed finite-time partial stability framework. Due to space limitations, we omit all proofs in this paper. Detailed proofs of our results are provided in [17].

II. Notation, Definitions, and Mathematical Preliminaries

In this section, we establish notation, definitions, and introduce the notion of finite-time partial stability. Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}_+$ denote the set of positive real numbers, $\overline{\mathbb{R}}_+$ denote the set of nonnegative numbers, $\mathbb{R}^n$ denote the set of $n \times 1$ real column vectors, $\mathbb{R}^{n\times m}$ denote the set of $n \times m$ real matrices, and $\mathcal{B}_\varepsilon(x)$ denote the *open ball centered* at x with *radius* ε. We write $V'(x) \triangleq \frac{\partial V(x)}{\partial x}$ for the Fréchet derivative of V at x, $\|\cdot\|$ for the Euclidean

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vector norm, and I_n or I for the $n \times n$ identity matrix.

In this paper, we consider nonlinear dynamical systems of the form

$$\dot{x}_1(t) = f_1(x_1(t), x_2(t)), \quad x_1(0) = x_{10}, \quad t \in \mathcal{I}_{x_0}, \qquad (1)$$

$$\dot{x}_2(t) = f_2(x_1(t), x_2(t)), \quad x_2(0) = x_{20}, \qquad (2)$$

where, for every $t \in \mathcal{I}_{x_0}$, $x_1(t) \in \mathcal{D} \subseteq \mathbb{R}^{n_1}$ and $x_2(t) \in \mathbb{R}^{n_2}$, $\mathcal{I}_{x_0} \subset \mathbb{R}$ is the maximal interval of existence of a solution $x(t) \triangleq [x_1^{\mathrm{T}}(t), x_2^{\mathrm{T}}(t)]^{\mathrm{T}}$ of (1) and (2) with initial condition $x_0 \triangleq [x_{10}^{\mathrm{T}}, x_{20}^{\mathrm{T}}]^{\mathrm{T}}$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $f_1 : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1}$ is such that, for every $(x_1, x_2) \in \mathcal{D} \times \mathbb{R}^{n_2}$, $f_1(0, x_2) = 0$ and $f_1(\cdot, \cdot)$ is jointly continuous in x_1 and x_2, and $f_2 : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$ is such that, for every $(x_1, x_2) \in \mathcal{D} \times \mathbb{R}^{n_2}$, $f_2(\cdot, \cdot)$ is jointly continuous in x_1 and x_2. A continuously differentiable function $x : \mathcal{I}_{x_0} \to \mathcal{D} \times \mathbb{R}^{n_2}$ is said to be a *solution* of (1) and (2) on the interval $\mathcal{I}_{x_0} \subset \mathbb{R}$ if $x(\cdot) = [x_1^{\mathrm{T}}(\cdot), x_2^{\mathrm{T}}(\cdot)]^{\mathrm{T}}$ satisfies (1) and (2) for all $t \in \mathcal{I}_{x_0}$. If $x(\cdot) = [x_1^{\mathrm{T}}(\cdot), x_2^{\mathrm{T}}(\cdot)]^{\mathrm{T}}$ is a solution of (1) and (2) on the interval $\mathcal{I}_{x_0} \subset \mathbb{R}$, then $x_1(\cdot)$ is the solution of (1) and $x_2(\cdot)$ is the solution of (2).

The joint continuity of $f(\cdot, \cdot) = [f_1^{\mathrm{T}}(\cdot, \cdot), f_2^{\mathrm{T}}(\cdot, \cdot)]^{\mathrm{T}}$ implies that, for every $(x_1, x_2) \in \mathcal{D} \times \mathbb{R}^{n_2}$, there exists $\tau_0 < 0 < \tau_1$ and a solution $[x_1^{\mathrm{T}}(\cdot), x_2^{\mathrm{T}}(\cdot)]^{\mathrm{T}}$ of (1) and (2) defined on the open interval (τ_0, τ_1) such that $[x_1^{\mathrm{T}}(0), x_2^{\mathrm{T}}(0)]^{\mathrm{T}} = [x_1^{\mathrm{T}}, x_2^{\mathrm{T}}]^{\mathrm{T}}$ [15, Th. 2.24]. A solution $t \mapsto [x_1^{\mathrm{T}}(t), x_2^{\mathrm{T}}(t)]^{\mathrm{T}}$ is said to be *right maximally* defined if $[x_1^{\mathrm{T}}, x_2^{\mathrm{T}}]^{\mathrm{T}}$ cannot be extended (either uniquely or nonuniquely) forward in time. We assume that all right maximal solutions to (1) and (2) exist on $[0, \infty)$, and hence, we assume that (1) and (2) is *forward complete*. Recall that every bounded solution to (1) and (2) can be extended on a semi-infinite interval $[0, \infty)$ [15]. That is, if $x : [0, \tau_{x_0}) \to \mathcal{D} \times \mathbb{R}^{n_2}$ is the right maximally defined solution of (1) and (2) such that $x(t) = [x_1^{\mathrm{T}}(t), x_2^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathcal{D}_{\mathrm{c}} \times \mathcal{Q}_{\mathrm{c}}$ for all $t \in [0, \tau_{x_0})$, where $\mathcal{D}_{\mathrm{c}} \subset \mathcal{D}$ and $\mathcal{Q}_{\mathrm{c}} \subset \mathbb{R}^{n_2}$ are compact, then $\tau_{x_0} = \infty$ [15, Cor. 2.5].

We assume that the nonlinear dynamical system given by (1) and (2) possesses unique solutions in forward time for all initial conditions except possibly at $x_1 = 0$ in the following sense. For every $(x_1, x_2) \in \mathcal{D} \backslash \{0\} \times \mathbb{R}^{n_2}$ there exists $\tau_x > 0$, where $x = [x_1^{\mathrm{T}}, x_2^{\mathrm{T}}]^{\mathrm{T}}$, such that, if $y_{\mathrm{I}} : [0, \tau_1) \to \mathcal{D} \times \mathbb{R}^{n_2}$ and $y_{\mathrm{II}} : [0, \tau_2) \to \mathcal{D} \times \mathbb{R}^{n_2}$ are two solutions of (1) and (2) with $y_{\mathrm{I}}(0) = y_{\mathrm{II}}(0) = x$, then $\tau_x \le \min\{\tau_1, \tau_2\}$ and $y_{\mathrm{I}}(t) = y_{\mathrm{II}}(t)$ for all $t \in [0, \tau_x)$. Without loss of generality, we assume that, for every (x_1, x_2), τ_x is chosen to be the largest such number in $\overline{\mathbb{R}}_+$. In this case, given $x = [x_1^{\mathrm{T}}, x_2^{\mathrm{T}}]^{\mathrm{T}} \in \mathcal{D} \times \mathbb{R}^{n_2}$, we denote by the continuously differentiable map $s^x(\cdot) \triangleq s(\cdot, x_1, x_2)$ the *trajectory* or the unique *solution curve* of (1) and (2) on $[0, \tau_x)$ satisfying $s(0, x_1, x_2) = [x_1^{\mathrm{T}}, x_2^{\mathrm{T}}]^{\mathrm{T}}$ and we denote by $s_1^x(\cdot)$ the *partial trajectory* or the unique *solution curve* of (1) on $[0, \tau_x)$. Sufficient conditions for forward uniqueness in the absence of Lipschitz continuity can be found in [8] [9, Section 10], [10], and [11, Section 1]. Finally, we assume that given a continuously differentiable function $x_1 : [0, \infty) \to \mathbb{R}^{n_1}$, the solution $x_2(t)$, $t \ge 0$, to (2) is unique.

The following definitions introduce the notion of finite-time partial stability.

Definition 2.1: The nonlinear dynamical system (1) and (2) is *finite-time stable with respect to* x_1 if there exist an open neighborhood $\mathcal{D}_0 \subseteq \mathcal{D}$ of $x_1 = 0$ and a function $T : \mathcal{D}_0 \setminus \{0\} \times \mathbb{R}^{n_2} \to (0, \infty)$, called the *settling-time function*, such that the following statements hold:

i) *Finite-time partial convergence.* For every $(x_{10}, x_{20}) \in \mathcal{D}_0 \backslash \{0\} \times \mathbb{R}^{n_2}$, $s^{x_0}(t)$ is defined on $[0, T(x_{10}, x_{20}))$, where $x_0 = [x_{10}^{\mathrm{T}}, x_{20}^{\mathrm{T}}]^{\mathrm{T}}$, $s_1^{x_0}(t) \in \mathcal{D}_0 \backslash \{0\}$ for all $t \in [0, T(x_{10}, x_{20}))$, and $s_1^{x_0}(t) \to 0$ as $t \to T(x_{10}, x_{20})$.

ii) *Partial Lyapunov stability.* For every $\varepsilon > 0$ and $x_{20} \in \mathbb{R}^{n_2}$ there exists $\delta = \delta(\varepsilon, x_{20}) > 0$ such that $\mathcal{B}_\delta(0) \subset \mathcal{D}_0$ and, for every $x_{10} \in \mathcal{B}_\delta(0) \backslash \{0\}$, $s_1^{x_0}(t) \in \mathcal{B}_\varepsilon(0)$ for all $t \in [0, T(x_{10}, x_{20}))$.

The nonlinear dynamical system (1) and (2) is *finite-time stable with respect to* x_1 *uniformly in* x_{20} if (1) and (2) is finite-time stable with respect to x_1 and the following statment holds:

iii) *Partial uniform Lyapunov stability.* For every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\mathcal{B}_\delta(0) \subset \mathcal{D}_0$ and, for every $x_{10} \in \mathcal{B}_\delta(0) \setminus \{0\}$, $s_1^{x_0}(t) \in \mathcal{B}_\varepsilon(0)$ for all $t \in [0, T(x_{10}, x_{20}))$ and for all $x_{20} \in \mathbb{R}^{n_2}$.

The nonlinear dynamical system (1) and (2) is *strongly finite-time stable with respect to* x_1 *uniformly in* x_{20} if (1) and (2) is uniformly finite-time stable with respect to x_1 and the following statment holds:

iv) *Finite-time partial uniform convergence.* For every $(x_{10}, x_{20}) \in \mathcal{D}_0 \setminus \{0\} \times \mathbb{R}^{n_2}$, $s^{x_0}(t)$ is defined on $[0, T(x_{10}, x_{20}))$, $s_1^{x_0}(t) \in \mathcal{D}_0 \setminus \{0\}$ for all $t \in [0, T(x_{10}, x_{20}))$, and $s_1^{x_0}(t) \to 0$ as $t \to T(x_{10}, x_{20})$ uniformly in x_{20} for all $x_{20} \in \mathbb{R}^{n_2}$.

The nonlinear dynamical system (1) and (2) is *globally finite-time stable with respect to* x_1 (respectively, *globally finite-time stable with respect to* x_1 *uniformly in* x_{20} or *globally strongly finite-time stable with respect to* x_1 *uniformly in* x_{20}) if it is finite-time stable with respect to x_1 (respectively, finite-time stable with respect to x_1 uniformly in x_{20} or strongly finite-time stable with respect to x_1 uniformly in x_{20}) with $\mathcal{D}_0 = \mathbb{R}^{n_1}$.

Remark 2.1: It is important to note that there is a key difference between the partial stability definitions given in Definition 2.1 and the definitions of partial stability given in [18]. In particular, the partial stability definitions given in [18] require that both initial conditions x_{10} and x_{20} lie in a neighborhood of the origin, whereas in Definition 2.1, x_{20} can be arbitrary. Furthermore, in the definition of partial stability given in [18], the state $x_1(t)$, $t \ge 0$, converges to zero and the state $x_2(t)$, $t \ge 0$, is bounded and converges to a constant that possibly depends on the system initial conditions. In contrast, in Definition 2.1 the state $x_2(t)$ can diverge as $t \to \infty$. As will be seen below, this difference allows us to unify autonomous partial stability theory with time-varying stability theory.

As shown in [15] and [16], an important application of partial stability theory is the unification it provides between time-invariant stability theory and stability theory for time-varying systems. Specifically, consider the nonlinear time-varying dynamical system given by

$$\dot{x}(t) = f(t, x(t)), \qquad x(t_0) = x_0, \qquad t \in \mathcal{I}_{x_0, t_0}, \qquad (3)$$

where, for every $t \in \mathcal{I}_{t_0, x_0}$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $\mathcal{I}_{t_0, x_0} \subseteq [t_0, \infty)$ is the maximal interval of existence of a solution $x(t)$

of (3), $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, and $f : \mathcal{I}_{t_0,x_0} \times \mathcal{D} \to \mathbb{R}^n$ is such that, for every $(t,x) \in \mathcal{I}_{t_0,x_0} \times \mathcal{D}$, $f(t,0) = 0$ and $f(\cdot,\cdot)$ is jointly continuous in t and x. In this paper, we assume that the nonlinear time-varying dynamical system (3) possesses unique solutions in forward time for all initial conditions except possibly $x = 0$ and, given $x_0 \in \mathcal{D}$, we denote by the continuously differentiable map $s^{t_0,x_0}(\cdot) \triangleq s(\cdot, t_0, x_0)$ the *trajectory* or the unique *solution curve* of (3) on $\mathcal{I}_{t_0,x_0}$ satisfying $s(0,t_0,x_0) = x_0$. Now, defining $x_1(\tau) \triangleq x(t)$ and $x_2(\tau) \triangleq t$, where $\tau \triangleq t - t_0$, it follows that the solution $x(t)$, $t \in \mathcal{I}_{t_0,x_0}$, to the nonlinear time-varying dynamical system (3) can be equivalently characterized by the solution $x_1(\tau)$, $\tau \in \mathcal{T}_{t_0,x_0}$, to the nonlinear autonomous dynamical system

$$\dot{x}_1(\tau) = f(x_2(\tau), x_1(\tau)), \quad x_1(0) = x_0, \quad \tau \in \mathcal{T}_{x_0,t_0}, \quad (4)$$

$$\dot{x}_2(\tau) = 1, \quad x_2(0) = t_0, \quad (5)$$

where $\mathcal{T}_{t_0,x_0} \triangleq \{\tau \in \overline{\mathbb{R}}_+ : \tau = t - t_0, t \in \mathcal{I}_{t_0,x_0}\}$. Note that (4) and (5) are in the same form as the system given by (1) and (2), and hence, Definition 2.1 applied to (4) and (5) specializes to the following definition.

Definition 2.2: The nonlinear dynamical system (3) is *finite-time stable* if there exist an open neighborhood $\mathcal{D}_0 \subseteq \mathcal{D}$ of the origin and a function $T : [0,\infty) \times \mathcal{D}_0 \backslash \{0\} \to (t_0, \infty)$, called the *settling-time function*, such that the following statements hold:

$i)$ *Finite-time convergence.* For every $(t_0, x_0) \in [0,\infty) \times \mathcal{D}_0 \setminus \{0\}$, $s^{t_0,x_0}(t)$ is defined on $[t_0, T(t_0,x_0))$, $s^{t_0,x_0}(t) \in \mathcal{D}_0 \backslash \{0\}$ for all $t \in [t_0, T(t_0,x_0))$, and $s^{t_0,x_0}(t) \to 0$ as $t \to T(t_0,x_0)$.

$ii)$ *Lyapunov stability.* For every $\varepsilon > 0$ and $t_0 \in [0,\infty)$ there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that $\mathcal{B}_\delta(0) \subset \mathcal{D}_0$ and, for every $x_0 \in \mathcal{B}_\delta(0) \backslash \{0\}$, $s^{t_0,x_0}(t) \in \mathcal{B}_\varepsilon(0)$ for all $t \in [t_0, T(t_0,x_0))$.

The nonlinear dynamical system (3) is *uniformly finite-time stable* if (3) is finite-time stable and the following statement holds:

$iii)$ *Uniform Lyapunov stability.* For every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\mathcal{B}_\delta(0) \subset \mathcal{D}_0$ and, for every $x_0 \in \mathcal{B}_\delta(0) \backslash \{0\}$, $s^{t_0,x_0}(t) \in \mathcal{B}_\varepsilon(0)$ for all $t \in [t_0, T(t_0,x_0))$ and for all $t_0 \in [0,\infty)$.

The nonlinear dynamical system (3) is *strongly uniformly finite-time stable* if (3) is uniformly finite-time stable and the following statement holds:

$iv)$ *Uniform finite-time convergence.* For every $(t_0, x_0) \in [0,\infty) \times \mathcal{D}_0 \backslash \{0\}$, $s^{t_0,x_0}(t)$ is defined on $[t_0, T(t_0,x_0))$, $s^{t_0,x_0}(t) \in \mathcal{D}_0 \setminus \{0\}$ for all $t \in [t_0, T(t_0,x_0))$, and $s^{t_0,x_0}(t) \to 0$ as $t \to T(t_0,x_0)$ uniformly in t_0 for all $t_0 \in [0,\infty)$.

The nonlinear dynamical system (3) is *globally finite-time stable* (respectively, *globally uniformly finite-time stable* or *globally strongly uniformly finite-time stable*) if it is finite-time stable (respectively, uniformly finite-time stable or strongly uniformly finite-time stable) with $\mathcal{D}_0 = \mathbb{R}^n$.

III. Finite-Time Partial Stability Theory

In this section, we present sufficient conditions for finite-time partial stability using a Lyapunov function satisfying a differential inequality involving fractional powers. The following proposition shows that if the nonlinear dynamical system (1) and (2) is finite-time stable with respect to x_1, then it possesses a unique solution $s(\cdot, x_{10}, x_{20})$ defined on $\overline{\mathbb{R}}_+ \times \mathcal{D}_0 \times \mathbb{R}^{n_2}$ for every x_{10} in a neighborhood of $x_1 = 0$, including $x_1 = 0$, and, for every $x_{20} \in \mathbb{R}^{n_2}$, $s_1(t, x_{10}, x_{20}) = 0$ for all $t \geq T(x_{10}, x_{20})$, where $T(0, x_{20}) \triangleq 0$.

Proposition 3.1: Consider the nonlinear dynamical system $\mathcal{G}$ given by (1) and (2). Assume $\mathcal{G}$ is finite-time stable with respect to x_1 and let $\mathcal{D}_0 \subseteq \mathcal{D}$ and $T : \mathcal{D}_0 \backslash \{0\} \times \mathbb{R}^{n_2} \to (0, \infty)$ be defined as in Definition 2.1. Then, for every $(x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}$, there exists a unique solution $s(t, x_{10}, x_{20}) = [s_1^{\mathrm{T}}(t, x_{10}, x_{20}), s_2^{\mathrm{T}}(t, x_{10}, x_{20})]^{\mathrm{T}}$, $t \geq 0$, to (1) and (2) defined on $\overline{\mathbb{R}}_+ \times \mathcal{D}_0 \times \mathbb{R}^{n_2}$ such that $s_1(t, x_{10}, x_{20}) \in \mathcal{D}_0$, $t \in [0, T(x_{10}, x_{20}))$, and such that $s_1(t, x_{10}, x_{20}) = 0$, $t \geq T(x_{10}, x_{20})$, where $T(0, x_{20}) \triangleq 0$.

It follows from Proposition 3.1 and the assumptions on $f_2(\cdot,\cdot)$ that if the nonlinear dynamical system (1) and (2) is finite-time stable with respect to x_1, then it defines a *global semiflow* on $\mathcal{D}_0 \times \mathbb{R}^{n_2}$; that is, the solution curve $s(\cdot,\cdot,\cdot)$ of (1) and (2) satisfies the consistency property $s(0, x_1, x_2) = [x_1^{\mathrm{T}}, x_2^{\mathrm{T}}]^{\mathrm{T}}$ and the semigroup property $s(t, s_1(\tau, x_1, x_2), s_2(\tau, x_1, x_2)) = s(t + \tau, x_1, x_2)$ for every $(x_1, x_2) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}$ and t, $\tau \in \overline{\mathbb{R}}_+$. Furthermore, $s(\cdot,\cdot,\cdot)$ satisfies

$$s_1(T(x_{10}, x_{20}) + t_1, x_{10}, x_{20}) = 0 \quad (6)$$

for all $(x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}$ and $t_1 \geq 0$.

In general, finite-time partial stability does not imply that the settling-time function $T(\cdot,\cdot)$ is continuous [13]. The following proposition generalizes Proposition 2.4 of [13] to show that the settling-time function $T(\cdot,\cdot)$ of a finite-time partially stable system is jointly continuous on $\mathcal{D}_0 \times \mathbb{R}^{n_2}$ if and only if it is continuous at $(0, \cdot)$.

Proposition 3.2: Consider the nonlinear dynamical system $\mathcal{G}$ given by (1) and (2). Assume $\mathcal{G}$ is finite-time stable with respect to x_1, let $\mathcal{D}_0 \subseteq \mathcal{D}$ be as defined in Definition 2.1, and let $T : \mathcal{D}_0 \backslash \{0\} \times \mathbb{R}^{n_2} \to [0, \infty)$ be the settling-time function of $\mathcal{G}$. Then the following statements hold:

$i)$ If $t_1 \geq 0$ and $(x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}$, then

$$T(s_1(t_1, x_{10}, x_{20}), s_2(t_1, x_{10}, x_{20})) = \max\{T(x_{10}, x_{20}), t_1\}. \quad (7)$$

$ii)$ $T(\cdot,\cdot)$ is jointly continuous on $\mathcal{D}_0 \times \mathbb{R}^{n_2}$ if and only if $T(\cdot,\cdot)$ is jointly continuous at $(0, x_2)$, $x_2 \in \mathbb{R}^{n_2}$.

Next, we present sufficient conditions for finite-time partial stability using a Lyapunov function involving a scalar differential inequality. Given the nonlinear dynamical system (1) and (2), for the statement of the following result define

$$\dot{V}(x_1, x_2) \triangleq V'(x_1, x_2) f(x_1, x_2),$$

where $f(x_1, x_2) \triangleq [f_1^{\mathrm{T}}(x_1, x_2), f_2^{\mathrm{T}}(x_1, x_2)]^{\mathrm{T}}$ and $V : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$ is a continuously differentiable function, and recall the definitions of class $\mathcal{K}$ and $\mathcal{K}_\infty$ functions given in [15, Def. 3.3].

Theorem 3.1: Consider the nonlinear dynamical system $\mathcal{G}$ given by (1) and (2). Then the following statements hold:

$i)$ If there exist a continuously differentiable function $V : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$, a class $\mathcal{K}$ function $\alpha(\cdot)$, a continuous function $k : [0, \infty) \to \mathbb{R}_+$, a real number $\theta \in (0, 1)$, and an open neighborhood $\mathcal{M} \subseteq \mathcal{D}$ of $x_1 = 0$ such that

$$V(0, x_2) = 0, \qquad x_2 \in \mathbb{R}^{n_2}, \tag{8}$$

$$\alpha(\|x_1\|) \le V(x_1, x_2), \quad (x_1, x_2) \in \mathcal{M} \times \mathbb{R}^{n_2}, \tag{9}$$

$$\dot{V}(x_1, x_2) \le -k(\|x_2\|)(V(x_1, x_2))^\theta, \quad (x_1, x_2) \in \mathcal{M} \times \mathbb{R}^{n_2}, \tag{10}$$

then $\mathcal{G}$ is finite-time stable with respect to x_1. Moreover, there exist a neighborhood $\mathcal{D}_0$ of $x_1 = 0$ and a settling-time function $T : \mathcal{D}_0 \times \mathbb{R}^{n_2} \to [0, \infty)$ such that

$$T(x_{10}, x_{20}) \le q^{-1}\left(\frac{(V(x_{10}, x_{20}))^{1-\theta}}{1-\theta}\right), \quad (x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}, \tag{11}$$

where $q : [0, \infty) \to \mathbb{R}$ is continuously differentiable and satisfies

$$\dot{q}(t) = k(\|x_2(t)\|), \qquad q(0) = 0, \qquad t \ge 0, \tag{12}$$

and $T(\cdot, \cdot)$ is jointly continuous on $\mathcal{D}_0 \times \mathbb{R}^{n_2}$.

ii) If $\mathcal{M} = \mathcal{D} = \mathbb{R}^{n_1}$ and there exist a continuously differentiable function $V : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$, a class $\mathcal{K}_\infty$ function $\alpha(\cdot)$, a continuous function $k : [0, \infty) \to \mathbb{R}_+$, and a real number $\theta \in (0, 1)$ such that (8)–(10) hold, then $\mathcal{G}$ is globally finite-time stable with respect to x_1. Moreover, there exists a settling-time function $T : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to [0, \infty)$ such that (11) holds with $\mathcal{D}_0 = \mathbb{R}^{n_1}$ and $T(\cdot, \cdot)$ is jointly continuous on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$.

iii) If there exist a continuously differentiable function $V : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$, class $\mathcal{K}$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, a continuous function $k : [0, \infty) \to \mathbb{R}_+$, a real number $\theta \in (0, 1)$, and an open neighborhood $\mathcal{M} \subseteq \mathcal{D}$ of $x_1 = 0$ such that (9) and (10) hold, and

$$V(x_1, x_2) \le \beta(\|x_1\|), \qquad (x_1, x_2) \in \mathcal{M} \times \mathbb{R}^{n_2}, \tag{13}$$

then $\mathcal{G}$ is finite-time stable with respect to x_1 uniformly in x_{20}. Moreover, there exist a neighborhood $\mathcal{D}_0$ of $x_1 = 0$ and a settling-time function $T : \mathcal{D}_0 \times \mathbb{R}^{n_2} \to [0, \infty)$ such that (11) holds and $T(\cdot, \cdot)$ is jointly continuous on $\mathcal{D}_0 \times \mathbb{R}^{n_2}$.

iv) If $\mathcal{M} = \mathcal{D} = \mathbb{R}^{n_1}$ and there exist a continuously differentiable function $V : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$, class $\mathcal{K}_\infty$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, a continuous function $k : [0, \infty) \to \mathbb{R}_+$, and a real number $\theta \in (0, 1)$ such that (9), (10), and (13) hold, then $\mathcal{G}$ is globally finite-time stable with respect to x_1 uniformly in x_{20}. Moreover, there exists a settling-time function $T : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to [0, \infty)$ such that (11) holds with $\mathcal{D}_0 = \mathbb{R}^{n_1}$ and $T(\cdot, \cdot)$ is jointly continuous on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$.

v) If there exist a continuously differentiable function $V : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$, class $\mathcal{K}$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, a real number $\theta \in (0, 1)$, and an open neighborhood $\mathcal{M} \subseteq \mathcal{D}$ of $x_1 = 0$ such that (9), (10), and (13) hold with $k(\|x_2\|) = k \in \mathbb{R}_+$, $x_2 \in \mathbb{R}^{n_2}$, then $\mathcal{G}$ is strongly finite-time stable with respect to x_1 uniformly in x_{20}. Moreover, there exist a neighborhood $\mathcal{D}_0$ of $x_1 = 0$ and a settling-time function $T : \mathcal{D}_0 \times \mathbb{R}^{n_2} \to [0, \infty)$ such that

$$T(x_{10}, x_{20}) \le \frac{(V(x_{10}, x_{20}))^{1-\theta}}{k(1-\theta)}, \quad (x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}, \tag{14}$$

and $T(\cdot, \cdot)$ is jointly continuous on $\mathcal{D}_0 \times \mathbb{R}^{n_2}$.

vi) If $\mathcal{M} = \mathcal{D} = \mathbb{R}^{n_1}$ and there exist a continuously differentiable function $V : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$, class $\mathcal{K}_\infty$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, and a real number $\theta \in (0, 1)$ such that (9), (10), and (13) hold with $k(\|x_2\|) = k \in \mathbb{R}_+$, $x_2 \in \mathbb{R}^{n_2}$, then $\mathcal{G}$ is globally strongly finite-time stable with respect to x_1 uniformly in x_{20}. Moreover, there exists a settling-time function $T : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to [0, \infty)$ such that (14) holds with $\mathcal{D}_0 = \mathbb{R}^{n_1}$ and $T(\cdot, \cdot)$ is jointly continuous on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$.

Example 3.1: Consider the nonlinear dynamical system given by

$$\dot{x}_1(t) = -x_2(t)\,(x_1(t))^{\frac{1}{3}}, \quad x_1(0) = x_{10}, \quad t \ge t_0, \tag{15}$$

$$\dot{x}_2(t) = x_2(t), \qquad x_2(0) = x_{20}, \tag{16}$$

where $x_{20} > 0$, and hence, $x_2(t) > 0$, $t \ge 0$. To show that (15) and (16) is globally finite-time stable with respect to x_1, consider the Lyapunov function candidate $V(x_1, x_1) = x_1^{\frac{4}{3}}$ and let $\mathcal{D} = \mathbb{R}$. Clearly, (9) and (13) hold, and

$$\dot{V}(x_1, x_2) = \frac{4}{3}x_1^{\frac{1}{3}}\left(-x_2 x_1^{\frac{1}{3}}\right) \le -k(x_2)\,(V(x_1, x_2))^{\frac{1}{2}}, \tag{17}$$

where $k(x_2) = \frac{4}{3}x_2 > 0$ and $x_2 > 0$. Hence, it follows from *iv*) of Theorem 3.1 that (15) and (16) is globally finite-time stable with respect to x_1. △

The following results specialize Propositions 3.1 and 3.2, and Theorem 3.1 to nonlinear time-varying dynamical systems.

Proposition 3.3: Consider the nonlinear dynamical system $\mathcal{G}$ given by (3). Assume $\mathcal{G}$ is finite-time stable and let $\mathcal{D}_0 \subseteq \mathcal{D}$ and $T : [0, \infty) \times \mathcal{D}_0 \backslash \{0\} \to (t_0, \infty)$ be defined as in Definition 2.2. Then, for every $(t_0, x_0) \in [0, \infty) \times \mathcal{D}_0$, there exists a unique solution $s(t, t_0, x_0)$, $t \ge t_0$, to (3) such that $s(t, t_0, x_0) \in \mathcal{D}_0$, $t \in [t_0, T(t_0, x_0))$, and such that $s(t, t_0, x_0) = 0$, $t \ge T(t_0, x_0)$, where $T(t_0, 0) \triangleq t_0$.

Proposition 3.4: Consider the nonlinear dynamical system $\mathcal{G}$ given by (3). Assume $\mathcal{G}$ is finite-time stable, let $\mathcal{D}_0 \subseteq \mathcal{D}$ be as defined in Definition 2.2, and let $T : [0, \infty) \times \mathcal{D}_0 \backslash \{0\} \to [t_0, \infty)$ be the settling-time function of $\mathcal{G}$. Then the following statements hold:

i) If $t_1 \ge t_0$ and $(t_0, x_0) \in [0, \infty) \times \mathcal{D}_0$, then

$$T(t_1, s(t_1, t_0, x_0)) = \max\{T(t_0, x_0), t_1\}. \tag{18}$$

ii) $T(\cdot, \cdot)$ is jointly continuous on $\mathbb{R}_+ \times \mathcal{D}_0$ if and only if $T(\cdot, \cdot)$ is jointly continuous at $(t, 0)$, $t \in [t_0, \infty)$.

Given the nonlinear time-varying dynamical system (3), for the statement of the following result define

$$\dot{V}(t, x) \triangleq \frac{\partial V(t, x)}{\partial t} + \frac{\partial V(t, x)}{\partial x} f(t, x),$$

where $V : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function.

Theorem 3.2: Consider the nonlinear dynamical system $\mathcal{G}$ given by (3). Then the following statements hold:

i) If there exist a continuously differentiable function $V : [t_0, \infty) \times \mathcal{D} \to \mathbb{R}$, a class $\mathcal{K}$ function $\alpha(\cdot)$, a continuous function $k : [t_0, \infty) \to \mathbb{R}_+$, a real number $\theta \in (0, 1)$, and an open neighborhood $\mathcal{M} \subseteq \mathcal{D}$ of the origin such that

$$V(t, 0) = 0, \qquad t \in [t_0, \infty), \tag{19}$$

$$\alpha(\|x\|) \le V(t, x), \qquad (t, x) \in [t_0, \infty) \times \mathcal{M}, \tag{20}$$

$$\dot{V}(t, x) \le -k(t)(V(t, x))^{\theta}, \qquad (t, x) \in [t_0, \infty) \times \mathcal{M}, \tag{21}$$

then $\mathcal{G}$ is finite-time stable. Moreover, there exist a neighborhood $\mathcal{D}_0$ of the origin and a settling-time function $T : [t_0, \infty) \times \mathcal{D}_0 \to [t_0, \infty)$ such that

$$T(t_0, x_0) \le q^{-1}\left(\frac{(V(t_0, x_0))^{1-\theta}}{1-\theta}\right), \qquad (t_0, x_0) \in [0, \infty) \times \mathcal{D}_0, \tag{22}$$

where $q : [0, \infty) \to \mathbb{R}$ is continuously differentiable and

$$\dot{q}(t) = k(t), \qquad q(0) = 0, \qquad t \ge 0, \tag{23}$$

and $T(\cdot, \cdot)$ is jointly continuous on $[t_0, \infty) \times \mathcal{D}_0$.

ii) If $\mathcal{M} = \mathcal{D} = \mathbb{R}^n$ and there exist a continuously differentiable function $V : [t_0, \infty) \times \mathcal{D} \to \mathbb{R}$, a class $\mathcal{K}_\infty$ function $\alpha(\cdot)$, a continuous function $k : [t_0, \infty) \to \mathbb{R}_+$, and a real number $\theta \in (0, 1)$ such that (19)–(21) hold, then $\mathcal{G}$ is globally finite-time stable. Moreover, there exists a settling-time function $T : [0, \infty) \times \mathbb{R}^n \to [t_0, \infty)$ such that (22) holds with $\mathcal{D}_0 = \mathbb{R}^n$ and $T(\cdot, \cdot)$ is jointly continuous on $[0, \infty) \times \mathbb{R}^n$.

iii) If there exist a continuously differentiable function $V : [t_0, \infty) \times \mathcal{D} \to \mathbb{R}$, class $\mathcal{K}$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, a continuous function $k : [t_0, \infty) \to \mathbb{R}_+$, a real number $\theta \in (0, 1)$, and an open neighborhood $\mathcal{M} \subseteq \mathcal{D}$ of the origin such that (20) and (21) hold and

$$V(t, x) \le \beta(\|x\|), \qquad (t, x) \in [t_0, \infty) \times \mathcal{M}, \tag{24}$$

then $\mathcal{G}$ is uniformly finite-time stable. Moreover, there exist a neighborhood $\mathcal{D}_0$ of the origin and a settling-time function $T : [0, \infty) \times \mathcal{D}_0 \to [t_0, \infty)$ such that (22) holds and $T(\cdot, \cdot)$ is jointly continuous on $[0, \infty) \times \mathcal{D}_0$.

iv) If $\mathcal{M} = \mathcal{D} = \mathbb{R}^n$ and there exist a continuously differentiable function $V : [t_0, \infty) \times \mathcal{D} \to \mathbb{R}$, class $\mathcal{K}_\infty$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, a continuous function $k : [t_0, \infty) \to \mathbb{R}_+$, and a real number $\theta \in (0, 1)$ such that (20), (21), and (24) hold, then $\mathcal{G}$ is globally uniformly finite-time stable. Moreover, there exists a settling-time function $T : [0, \infty) \times \mathbb{R}^n \to [t_0, \infty)$ such that (22) holds with $\mathcal{D}_0 = \mathbb{R}^n$ and $T(\cdot, \cdot)$ is jointly continuous on $[0, \infty) \times \mathbb{R}^{n_2}$.

v) If there exist a continuously differentiable function $V : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}$, class $\mathcal{K}$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, a real number $\theta \in (0, 1)$, and an open neighborhood $\mathcal{M} \subseteq \mathcal{D}$ of the origin such that (20), (21), and (24) hold with $k(t) = k \in \mathbb{R}_+$, $t \ge t_0$, then $\mathcal{G}$ is strongly uniformly finite-time stable. Moreover, there exist a neighborhood $\mathcal{D}_0$ of the origin and a settling-time function $T : [0, \infty) \times \mathcal{D}_0 \to [t_0, \infty)$ such that

$$T(t_0, x_0) \le \frac{(V(t_0, x_0))^{1-\theta}}{k(1-\theta)}, \quad (t_0, x_0) \in [0, \infty) \times \mathcal{D}_0, \tag{25}$$

and $T(\cdot, \cdot)$ is jointly continuous on $[0, \infty) \times \mathcal{D}_0$.

vi) If $\mathcal{M} = \mathcal{D} = \mathbb{R}^n$ and there exist a continuously differentiable function $V : \mathcal{D} \times \mathbb{R}^n \to \mathbb{R}$, class $\mathcal{K}_\infty$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, and a real number $\theta \in (0, 1)$ such that (20), (21), and (24) hold with $k(t) = k \in \mathbb{R}_+$, $t \ge t_0$, then $\mathcal{G}$ is globally strongly uniformly finite-time stable. Moreover, there exists a settling-time function $T : [0, \infty) \times \mathbb{R}^n \to [t_0, \infty)$ such that (25) holds with $\mathcal{D}_0 = \mathbb{R}^n$ and $T(\cdot, \cdot)$ is jointly continuous on $[0, \infty) \times \mathbb{R}^{n_1}$.

Remark 3.1: Propositions 3.3 and 3.4 along with Statements *i*)–*iv*) of Theorem 3.2 appear in [14]. See also [19].

Example 3.2: Consider the nonlinear time-varying dynamical system given by

$$\dot{x}(t) = -t\,(x(t))^{\frac{1}{3}} - t\,(x(t))^{\frac{1}{5}}, \quad x(0) = x_0, \quad t \ge t_0. \tag{26}$$

To show that the zero solution $x(t) \equiv 0$ to (26) is globally uniformly finite-time stable, consider the Lyapunov function candidate $V(t, x) = x^{\frac{4}{3}}$ and let $\mathcal{D} = \mathbb{R}$. Clearly, (20) and (24) hold, and

$$\dot{V}(t, x) = -\frac{4}{3}t\left(x^{\frac{2}{3}} + x^{\frac{8}{15}}\right) \le -k(t)\,(V(t, x))^{\frac{1}{2}}, \tag{27}$$

where $k(t) = 2t > 0$, $t \ge t_0$. Hence, it follows from *iv*) of Theorem 3.2 that the zero solution $x(t) \equiv 0$ to (26) is globally uniformly finite-time stable. △

IV. Illustrative Numerical Example

Consider the nonlinear dynamical system adopted from [15], [20] given by

$$\dot{q}_1(t) = -\alpha_1 q_1(t) - \beta q_1(t) q_2(t) \cos q_3(t) + u_1(t), \qquad q_1(0) = q_{10}, \qquad t \ge 0, \tag{28}$$

$$\dot{q}_2(t) = -\alpha_2 q_2(t) + \beta q_1^2(t) \cos q_3(t) + u_2(t), \qquad q_2(0) = q_{20}, \tag{29}$$

$$\dot{q}_3(t) = 2\theta_1 - \theta_2 - \beta\left(\frac{q_1^2(t)}{q_2(t)} - 2q_2(t)\right)\sin q_3(t), \qquad q_3(0) = q_{30}, \tag{30}$$

representing a time-averaged, two-mode thermoacoustic combustion model where q_1, q_2, and q_3 represent modal shapes, $\alpha_1 > 0$ and $\alpha_2 > 0$ represent decay constants, θ_1 and $\theta_2 \in \mathbb{R}$ represent frequency shift constants, $\beta = \big((\gamma+1)/8\gamma\big)\omega_1$, where γ denotes the ratio of specific heats and ω_1 is the frequency of the fundamental mode, and $u = [u_1, u_2, u_3]^{\mathrm{T}}$ is the control input signal. As shown in [20] and [21], only the first two states q_1 and q_2 representing the modal amplitudes of a two-mode thermoacoustic combustion model are relevant in characterizing system instabilities since the third state q_3 represents the phase difference between the two modes [22]. Hence, we require asymptotic stability of $q_1(t)$, $t \ge 0$, and $q_2(t)$, $t \ge 0$, which necessitates partial stabilization.

For the state feedback controller given by

$$u = \phi(x_1, x_2) = -f(x_1, x_2) - 2^{-\frac{3}{4}} g(x_1, x_2), \qquad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \tag{31}$$

where $x_1 = [q_1, q_2]^{\mathrm{T}}$, $x_2 = q_3$, $f(x_1, x_2) = \big[-\alpha_1 q_1 - \beta q_1 q_2 \cos q_3,\ -\alpha_2 q_2 + \beta q_1^2 \cos q_3\big]^{\mathrm{T}}$, and $g(x_1, x_2) =$

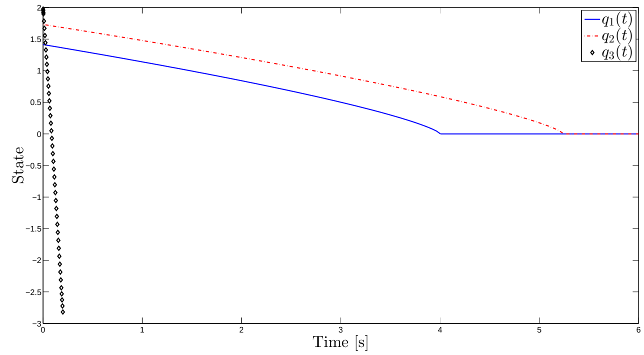

Fig. 1. Closed-loop system trajectories versus time.

$\left[q_1^{\frac{1}{3}}, q_2^{\frac{1}{3}}\right]^{\mathrm{T}}$, we show that the closed-loop dynamical system (28)–(31) is globally strongly finite-time stable with respect to x_1 uniformly in $x_2(0)$. To see this, we apply Theorem 3.1 with $n_1 = 2$ and $n_2 = 1$. Let

$$V(x_1, x_2) = \frac{1}{2} x_1^{\mathrm{T}} x_1, \qquad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \tag{32}$$

which is positive definite and radially unbounded with respect to x_1, and hence, (9) and (13) hold.

Next, note that

$$\begin{aligned} V'(x_1, x_2)[f(x_1, x_2) + \phi(x_1, x_2)] &= -2^{-\frac{3}{4}} \left(q_1^{\frac{4}{3}} + q_2^{\frac{4}{3}}\right) \\ &\leq -2^{-\frac{3}{4}} \left(q_1^{\frac{4}{3}} + q_2^{\frac{4}{3}}\right)^{\frac{3}{4}} \\ &= -\left(V(x_1, x_2)\right)^{\frac{3}{4}}, \end{aligned} \tag{33}$$

for all $(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Since all of the conditions of $vi)$ of Theorem 3.1 hold, it follows that the closed-loop system (28)–(31) is globally strongly finite-time stable with respect to x_1 uniformly in $x_2(0)$. Furthermore, there exists a settling-time function $T : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to [0, \infty)$, jointly continuous on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, such that

$$T(x_1(0), x_2(0)) \leq 2^{\frac{7}{4}} \|x_1(0)\|, \qquad (x_1(0), x_2(0)) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}. \tag{34}$$

Let $\alpha_1 = 5\,\mathrm{Hz}$, $\alpha_2 = 45\,\mathrm{Hz}$, $\gamma = 1.4$, $\omega_1 = 1\,\mathrm{Hz}$, $\theta_1 = 4\,\mathrm{Hz}$, $\theta_2 = 32\,\mathrm{Hz}$, $q_{10} = 2$, $q_{20} = 3$, and $q_{30} = 2$. Figure 1 shows the state trajectories of the controlled system versus time. Note that $x_1(t) = [q_1(t), q_2(t)]^{\mathrm{T}} \to 0$ as $t \to 5.2474\,\mathrm{s} < T(x_1(0), x_2(0)) = 12.127\,\mathrm{s}$, whereas $x_2(t) = q_3(t)$ is unstable.

V. Conclusion

In this paper, we addressed finite-time partial stability for nonlinear dynamical systems. Specifically, we provided Lyapunov conditions involving a Lyapunov function that is positive definite and decrescent with respect to part of the system state, and satisfies a differential inequality involving fractional powers for guarnateeing finite-time partial stability. In addition, we showed that finite-time partial stability leads to uniqueness of solutions in forward time. Finally, we established necessary and sufficient conditions for continuity of the settling-time function of a nonlinear dynamical system. Extensions of this framework for addressing finite-time partial stabilization and optimal feedback control of nonlinear dynamical systems is currently under development [17].

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