Available online at www.sciencedirect.com

ScienceDirect

Journal of the Franklin Institute 354 (2017) 456–478

Journal
of The
Franklin Institute

www.elsevier.com/locate/jfranklin

Differential games, partial-state stabilization, and model reference adaptive control

Andrea L'Afflitto

School of Aerospace and Mechanical Engineering, The University of Oklahoma, Norman, OK73019-0390, USA

Received 28 September 2015; received in revised form 18 July 2016; accepted 8 October 2016
Available online 15 October 2016

Abstract

In this paper, we develop a unified framework to find state-feedback control laws that solve two-player zero-sum differential games over the infinite time horizon and guarantee partial-state asymptotic stability of the closed-loop system. Partial-state asymptotic stability is guaranteed by means of a Lyapunov function that is positive definite and decrescent with respect to part of the system state. The existence of a saddle point for the system's performance measure is guaranteed by the fact that this Lyapunov function satisfies a partial differential equation that corresponds to a steady-state form of the Hamilton–Jacobi–Isaacs equation. In the second part of this paper, we show how our differential game framework can be applied not only to solve pursuit-evasion and robust optimal control problems, but also to assess the effectiveness of a model reference adaptive control law. Specifically, the model reference adaptive control architecture is designed to guarantee satisfactory trajectory tracking for uncertain nonlinear dynamical systems, whose matched nonlinearities are captured by the regressor vector. By modeling matched and unmatched nonlinearities, which are not captured by the regressor vector, as the pursuer's and evader's control inputs in a differential game, we provide an explicit characterization of the system's uncertainties that do not disrupt the trajectory tracking capabilities of the adaptive controller. Two numerical examples illustrate the applicability of our results.

1. Introduction

Differential game theory [21] deals with dynamical systems endowed with multiple control inputs, some of which, usually named *evaders*, strive to maximize a given performance measure, whereas some others, usually named *pursuers*, concurrently try to minimize this performance

E-mail address: a.lafflitto@ou.edu

http://dx.doi.org/10.1016/j.jfranklin.2016.10.012

measure. Classic applications of differential games range from aerospace engineering [21,33] to marine engineering [37], electrical engineering [14], communication networks [2], and economics [12]. The study of zero-sum differential games over the *infinite* time interval has received considerably less attention, since it has been mostly explored for linear dynamical systems with quadratic performance measures [41] and to establish connections with the classic $\mathcal{H}_\infty$ control theory [7,13,22,29,31]. Differential games on the infinite horizon for nonlinear systems have been explored in [1,34], and in [4–7,23,32] within the context of the disturbance rejection problem; however, connections between the feedback stabilization problem and differential games involving nonlinear dynamical systems with nonlinear–nonquadratic performance measures have not been explored before.

Partial-state stabilization, that is, the problem of stabilizing a dynamical system with respect to a subset of the system state variables, arises in many engineering applications [30,36]. For example, in the control of rotating machinery with mass imbalance, spin stabilization about a nonprincipal axis of inertia requires motion stabilization with respect to a subspace instead of the origin [30]. In general, the need to consider partial stability arises in control problems involving equilibrium coordinates as well as a manifold of coordinates that is closed but *not* compact.

In this paper, we address a form of the differential game problem, which has not been considered before. Specifically, consider a two-player zero-sum differential game problem involving nonlinear dynamical systems with nonlinear–nonquadratic performance measures over the infinite time horizon. We provide sufficient conditions for the pursuer's and evader's state-feedback control laws to guarantee both partial-state asymptotic stability of the closed-loop dynamical system and the existence of a saddle point for the system's performance measure. Remarkably, if the end-of-game condition is given by the convergence to an equilibrium point of the trajectory of a subset of the system state variables, then closed-loop partial-state asymptotic stability is a key feature to guarantee that this condition is permanently enforced.

The framework presented in this paper is also suitable to address problems, in which the differential game ends when part of the system state trajectory enters a *given* neighborhood of an equilibrium point within some time interval that is finite and not assigned *a priori*. Possible applications for this framework concern, but are not limited to, games of degree [21, p. 12] such as the game of two cars [21, pp. 237–244], whereby the pursuer's goal is to reach some neighborhood of the evader irrespectively of the angle between the agents' velocity vectors.

The underlying idea of this paper is that if there exists a Lyapunov function that is positive definite and decrescent with respect to part of the system state, and satisfies a partial differential equation corresponding to a steady-state form of the Hamilton–Jacobi–Isaacs equation for the controlled system, then there exist pursuer's and evader's state feedback control policies that guarantee both partial-state asymptotic stability of the closed-loop dynamical system and the existence of a saddle point for the system's performance measure. In this case, we provide an explicit closed-form expression for the performance measure evaluated at the saddle point and characterize the optimal evader's and pursuer's control policies. No collaboration between the pursuer and the evader is assumed in this paper to achieve closed-loop partial-state asymptotic stability. Indeed, the pursuer's control policy is designed to guarantee closed-loop stability with respect to a class of evader's admissible controls, some of which may lead to system instability if applied in conjunction with other pursuer's admissible controls.

The optimal control problem for several forms of closed-loop stability has been addressed in [8,17,26,27]. In these works, the main idea is that a solution of the Hamilton–Jacobi–Bellman equation can serve as a Lyapunov function to prove close-loop asymptotic stability [8], partial-state stability [26], partial-state finite-time stability [17], and semistability [27].

The Hamilton–Jacobi–Isaacs equation involves the minimization with respect to the pursuer and maximization with respect to the evader of the Hamiltonian function, whereas the Hamilton–Jacobi–Bellman equation involves the minimization of the Hamiltonian function with respect to a control input. Consequently, results in [8,17,26] can be derived from those presented herein. Since limit sets of partial-state stable dynamical systems are not invariant [15, Chapter 4], the notion of partial-state semistability does not exist, and our differential game framework cannot be specialized to resume results in [27].

The results proven in [8,17,26,27] are not suitable to design optimal feedback controls, which are robust to external disturbances, parameter uncertainties, and unmodeled dynamics. Conversely, one of the key contributions of this paper is the following. Regarding the evader's control policy as an exogenous disturbance, we extend our game-theoretic framework to provide sufficient conditions to solve the *robust optimal control problem* for nonlinear dynamical systems with nonlinear–nonquadratic performance measures. Similar results were achieved in [16] using a dissipativity-based approach and considering classic, but not partial-state, asymptotic stability.

In the second part of this paper, we present an application of differential game theory that has not been considered before. Specifically, we apply our differential game framework to assess the effectiveness of a model reference adaptive control law in the presence of unmodeled nonlinearities. The model reference adaptive control technique requires parametrizing matched nonlinearities in the system state by means of a vector function, known as *regressor*. Since a judicious choice of the regressor vector sensibly improves the controller's performance, the problem of designing regressor vectors has drawn considerable attention, and has been examined for several dynamical systems, such as robotic manipulators [9,39], infinite-impulse response (IIR) systems [3,24], and unmanned vehicles [11,35], to mention a few. Finding adequate regressor vectors is so critical, that several regressor-free adaptive control schemes have been developed as well [18,40].

The regressor vector is a finite-dimensional basis for the infinite-dimensional space of nonlinearities and hence it follows from Baire's category theorem [25, Th. 4–7.2] that it can only *approximate* the system nonlinearities [11]. In this paper, we consider a nonlinear dynamical system controlled by a model reference adaptive control law, and model matched and unmatched nonlinearities, which are *not* captured by the regressor vector, as the pursuer's and the evader's control inputs in a differential game. Then, we apply our differential game approach to *explicitly* parametrize matched and unmatched nonlinearities that do not disrupt the adaptive controller's ability to guarantee satisfactory trajectory tracking.

A relevant application of partial stability and partial stabilization theory is the unification it provides between time-invariant stability theory and stability theory for time-varying systems [10,15]. Results presented in this paper can be easily extended to differential games over the infinite time horizon, which involve time-varying dynamical systems with nonlinear–nonquadratic, time-varying performance measures. Moreover, our partial-state stabilization framework provides clear connections with the $\mathcal{H}_\infty$ control theory for time-varying dynamical systems [19]. Since the differential game problem over the infinite time horizon for time-varying systems is a straightforward specialization of the problem discussed in this paper, these results are omitted.

2. Notation, definitions, and mathematical preliminaries

In this section, we establish notation, definitions, and review some basic results. Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}^n$ denote the set of $n \times 1$ real column vectors, and $\mathbb{R}^{n\times m}$ denote the set of $n \times m$ real matrices. We write $\|\cdot\|$ for the Euclidean vector norm, $V'(x) \triangleq \frac{\partial V(x)}{\partial x}$ for the Fréchet derivative of V at x, I_n or I for the $n \times n$ identity matrix, $0_{n\times m}$ or 0 for the zero $n \times m$ matrix, and A^{T} for the transpose of the matrix A.

Given $f : X \times Y \to \mathbb{R}$, where $X \subseteq \mathbb{R}^{m_1}$ and $Y \subseteq \mathbb{R}^{m_2}$, we define

$$\arg \operatorname*{minmax}_{(x,y)\in(X,Y)} f(x,y) \triangleq \Big\{(x^*,y^*)\in(X,Y) : f(x^*,y^*) \le f(x,y^*),\ \forall x\in X, \text{ and } \\ f(x^*,y^*) \ge f(x^*,y),\ \forall y \in Y\Big\}$$

and

$$\operatorname*{minmax}_{(x,y)\in(X,Y)} f(x,y) \triangleq f(x^*,y^*), \quad (x^*,y^*) \in \arg \operatorname*{minmax}_{(x,y)\in(X,Y)} f(x,y). \tag{1}$$

If $(x^*,y^*) \in \arg\operatorname{minmax}_{(x,y)\in(X,Y)} f(x,y)$, then we say that (x^*,y^*) is a *saddle point* for $f(\cdot,\cdot)$ on $X \times Y$. The next result states a key property of saddle points.

Lemma 2.1 ([5]). *Consider $f : X \times Y \to \mathbb{R}$, where $X \subseteq \mathbb{R}^{m_1}$ and $Y \subseteq \mathbb{R}^{m_2}$, and let $(x^*,y^*) \in \arg\operatorname{minmax}_{(x,y)\in(X,Y)} f(x,y)$. Then,*

$$\min_{x\in X}\max_{y\in Y} f(x,y) = \max_{y\in Y}\min_{x\in X} f(x,y), \quad (x,y)\in X\times Y. \tag{2}$$

In this paper, we consider nonlinear autonomous dynamical systems of the form

$$\dot{x}_1(t) = f_1(x_1(t),x_2(t)), \quad x_1(0) = x_{10}, \quad t \ge 0, \tag{3}$$

$$\dot{x}_2(t) = f_2(x_1(t),x_2(t)), \quad x_2(0) = x_{20}, \tag{4}$$

where, for every $t \ge 0$, $x_1(t) \in \mathcal{D} \subseteq \mathbb{R}^{n_1}$ and $x_2(t) \in \mathbb{R}^{n_2}$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $f_1 : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1}$ is such that, for every $x_2 \in \mathbb{R}^{n_2}$, $f_1(0,x_2) = 0$ and $f_1(\cdot,x_2)$ is locally Lipschitz continuous in x_1, and $f_2 : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$ is such that, for every $x_1 \in \mathcal{D}$, $f_2(x_1,\cdot)$ is locally Lipschitz continuous in x_2.

Definition 2.1 ([15, Def. 4.1]). (i) The nonlinear dynamical system $\mathcal{G}$ given by Eqs. (3) and (4) is *Lyapunov stable with respect to x_1 uniformly in x_{20}* if, for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $\|x_{10}\| \le \delta$ implies that $\|x_1(t)\| < \varepsilon$ for all $t \ge 0$ and for all $x_{20} \in \mathbb{R}^{n_2}$.

(ii) $\mathcal{G}$ is *asymptotically stable with respect to x_1 uniformly in x_{20}* if $\mathcal{G}$ is Lyapunov stable with respect to x_1 uniformly in x_{20} and there exists $\delta > 0$ such that $\|x_{10}\| < \delta$ implies that $\lim_{t\to\infty} x_1(t) = 0$ uniformly in x_{10} and x_{20} for all $x_{20} \in \mathbb{R}^{n_2}$.

(iii) $\mathcal{G}$ is *globally asymptotically stable with respect to x_1 uniformly in x_{20}* if $\mathcal{G}$ is Lyapunov stable with respect to x_1 uniformly in x_{20} and $\lim_{t\to\infty} x_1(t) = 0$ uniformly in x_{10} and x_{20} for all $x_{10} \in \mathbb{R}^{n_1}$ and $x_{20} \in \mathbb{R}^{n_2}$.

Next, we provide sufficient conditions for partial stability of the nonlinear dynamical system given by Eqs. (3) and (4). For the statement of the following result, let $f(x_1,x_2) \triangleq [f_1^{\mathrm{T}}(x_1,x_2), f_2^{\mathrm{T}}(x_1,x_2)]^{\mathrm{T}}$.

Theorem 2.1 (*[15, Th. 4.1]*). *Consider the nonlinear dynamical system (3) and (4). Then the following statements hold*:

(*i*) *If there exist a continuously differentiable function $V : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$ and class $\mathcal{K}$ functions $\alpha(\cdot)$, $\beta(\cdot)$, and $\theta(\cdot)$ such that*

$$\alpha(\|x_1\|) \le V(x_1,x_2) \le \beta(\|x_1\|), \quad (x_1,x_2) \in \mathcal{D} \times \mathbb{R}^{n_2}, \tag{5}$$

$$V'(x_1,x_2)f(x_1,x_2) \le -\theta(\|x_1\|), \quad (x_1,x_2) \in \mathcal{D} \times \mathbb{R}^{n_2}, \tag{6}$$

then the nonlinear dynamical system given by Eqs. (3) and (4) is asymptotically stable with respect to x_1 uniformly in x_{20}.

(*ii*) *If there exist a continuously differentiable function $V : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, a class $\mathcal{K}$ function $\theta(\cdot)$, and class $\mathcal{K}_\infty$ functions $\alpha(\cdot)$ and $\beta(\cdot)$ satisfying Eqs. (5) and (6), then the nonlinear dynamical system given by Eqs. (3) and (4) is globally asymptotically stable with respect to x_1 uniformly in x_{20}.*

In the following, we recall a result that provides connections between Lyapunov functions and nonquadratic cost evaluation. Specifically, we consider the problem of evaluating a nonlinear–nonquadratic performance measure that depends on the solution of the nonlinear dynamical system given by Eqs. (3) and (4). In particular, consider the nonlinear–nonquadratic performance measure

$$J(x_{10},x_{20}) \triangleq \int_0^\infty L(x_1(t),x_2(t))\mathrm{d}t, \tag{7}$$

where $L : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$ is jointly continuous in x_1 and x_2, and $x_1(t)$ and $x_2(t)$, $t \ge 0$, satisfy Eqs. (3) and (4). If there exists a Lyapunov function that is positive definite and decrescent with respect to x_1 and proves asymptotic stability of Eqs. (3) and (4) with respect to x_1 uniformly in x_{20}, then the next theorem shows that (7) can be evaluated so long as Eqs. (3) and (4) are related to this Lyapunov function.

Theorem 2.2 (*[26]*). *Consider the nonlinear dynamical system $\mathcal{G}$ given by Eqs. (3) and (4) with performance measure (7). Assume that there exists a continuously differentiable function $V : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$ and class $\mathcal{K}$ functions $\alpha(\cdot)$, $\beta(\cdot)$, and $\theta(\cdot)$ such that*

$$\alpha(\|x_1\|) \le V(x_1,x_2) \le \beta(\|x_1\|), \quad (x_1,x_2) \in \mathcal{D} \times \mathbb{R}^{n_2}, \tag{8}$$

$$V'(x_1,x_2)f(x_1,x_2) \le -\theta(\|x_1\|), \quad (x_1,x_2) \in \mathcal{D} \times \mathbb{R}^{n_2}, \tag{9}$$

$$L(x_1,x_2) + V'(x_1,x_2)f(x_1,x_2) = 0, \quad (x_1,x_2) \in \mathcal{D} \times \mathbb{R}^{n_2}. \tag{10}$$

Then the nonlinear dynamical system $\mathcal{G}$ is asymptotically stable with respect to x_1 uniformly in x_{20} and there exists a neighborhood $\mathcal{D}_0 \subseteq \mathcal{D}$ of $x_1 = 0$ such that, for all $(x_{10},x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}$,

$$J(x_{10},x_{20}) = V(x_{10},x_{20}). \tag{11}$$

Finally, if $\mathcal{D} = \mathbb{R}^{n_1}$ and the functions $\alpha(\cdot)$ and $\beta(\cdot)$ satisfying Eq. (8) are class $\mathcal{K}_\infty$, then $\mathcal{G}$ is globally asymptotically stable with respect to x_1 uniformly in x_{20}.

In this paper, we consider controlled nonlinear dynamical systems of the form

$$\dot{x}_1(t) = F_1(x_1(t),x_2(t),u(t),w(t)), \quad x_1(0) = x_{10}, \quad t \ge 0, \tag{12}$$

$$\dot{x}_2(t) = F_2(x_1(t), x_2(t), u(t), w(t)), \quad x_2(0) = x_{20}, \tag{13}$$

where, for every $t \geq 0$, $x_1(t) \in \mathcal{D} \subseteq \mathbb{R}^{n_1}$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $x_2(t) \in \mathbb{R}^{n_2}$, $u(t) \in U \subseteq \mathbb{R}^{m_1}$ with $0 \in U$, $w(t) \in W \subseteq \mathbb{R}^{m_2}$ with $0 \in W$, $F_1 : \mathcal{D} \times \mathbb{R}^{n_2} \times U \times W \to \mathbb{R}^{n_1}$ and $F_2 : \mathcal{D} \times \mathbb{R}^{n_2} \times U \times W \to \mathbb{R}^{n_2}$ are locally Lipschitz continuous in x_1, x_2, u, and w, and $F_1(0, x_2, 0, 0) = 0$ for every $x_2 \in \mathbb{R}^{n_2}$. The controls $u(\cdot)$ and $w(\cdot)$ in Eqs. (12) and (13) are restricted to the class of *admissible* controls consisting of continuous functions $u(\cdot)$ and $w(\cdot)$ such that $u(t) \in U$, $t \geq 0$, and $w(t) \in W$, respectively.

Continuous functions $\phi : \mathcal{D} \times \mathbb{R}^{n_2} \to U$ and $\psi : \mathcal{D} \times \mathbb{R}^{n_2} \to W$ satisfying $\phi(0, x_2) = 0$, $x_2 \in \mathbb{R}^{n_2}$, and $\psi(0, x_2) = 0$ are called *control laws*. If $u(t) = \phi(x_1(t), x_2(t))$, $t \geq 0$, and $w(t) = \psi(x_1(t), x_2(t))$, where $\phi(\cdot, \cdot)$ and $\psi(\cdot, \cdot)$ are control laws and $x_1(t)$ and $x_2(t)$ satisfy Eqs. (12) and (13), respectively, then we call $u(\cdot)$ and $w(\cdot)$ *feedback control laws*. Given control laws $\phi(\cdot, \cdot)$ and $\psi(\cdot, \cdot)$, and feedback control laws $u(t) = \phi(x_1(t), x_2(t))$, $t \geq 0$, and $w(t) = \psi(x_1(t), x_2(t))$, the *closed-loop system* (12) and (13) is given by

$$\dot{x}_1(t) = F_1(x_1(t), x_2(t), \phi(x_1(t), x_2(t)), \psi(x_1(t), x_2(t))), \quad x_1(0) = x_{10}, \quad t \geq 0, \tag{14}$$

$$\dot{x}_2(t) = F_2(x_1(t), x_2(t), \phi(x_1(t), x_2(t)), \psi(x_1(t), x_2(t))), \quad x_2(0) = x_{20}. \tag{15}$$

Next, we introduce the notion of partial-state asymptotically stabilizing feedback control laws. To this goal, consider the controlled nonlinear dynamical system (12) and (13) and define the set of regulation controllers

$$\mathcal{S}(x_{10}, x_{20}) \triangleq \{(u(\cdot), w(\cdot)) : u(\cdot) \text{ and } w(\cdot) \text{ are admissible and } x_1(\cdot) \text{ given by Eq. (12)} \\ \text{satisfies } x_1(t) \to 0 \text{ as } t \to \infty\}.$$

In addition, given the control law $\psi(\cdot, \cdot)$, let

$$\mathcal{S}_\psi(x_{10}, x_{20}) \triangleq \{u(\cdot) : (u(\cdot), \psi(x_1(\cdot), x_2(\cdot))) \in \mathcal{S}(x_{10}, x_{20})\}$$

and given the control law $\phi(\cdot, \cdot)$, let

$$\mathcal{S}_\phi(x_{10}, x_{20}) \triangleq \{w(\cdot) : (\phi(x_1(\cdot), x_2(\cdot)), w(\cdot)) \in \mathcal{S}(x_{10}, x_{20})\}.$$

Definition 2.2. Consider the controlled dynamical system given by Eqs. (12) and (13). The feedback control law $u(\cdot) = \phi(x_1(\cdot), x_2(\cdot))$ is *asymptotically stabilizing with respect to* x_1 *uniformly in* x_{20} if the closed-loop system (14) and (15) is asymptotically stable with respect to x_1 uniformly in x_{20} for all admissible controls $w(\cdot) \in \mathcal{S}_\phi(x_{10}, x_{20})$. Furthermore, the feedback control law $u(\cdot) = \phi(x_1(\cdot), x_2(\cdot))$ is *globally asymptotically stabilizing with respect to* x_1 *uniformly in* x_{20} if the closed-loop system (14) and (15) is globally asymptotically stable with respect to x_1 uniformly in x_{20} for all admissible controls $w(\cdot) \in \mathcal{S}_\phi(x_{10}, x_{20})$.

3. Lyapunov functions and differential games

In this section, we characterize partial-state asymptotically stabilizing feedback control laws that solve differential games involving nonlinear dynamical systems over the infinite time horizon. Specifically, we characterize the pursuer's and evader's feedback control laws, which guarantee both partial-state asymptotic stability of Eqs. (12) and (13) and the existence of a saddle point for a nonlinear–nonquadratic performance measure. For the statement of the next theorem, which is one of the main results of this paper, define $F(x_1, x_2, u, w) \triangleq [F_1^{\mathrm{T}}(x_1, x_2, u, w), F_2^{\mathrm{T}}(x_1, x_2, u, w)]^{\mathrm{T}}$ and let $L : \mathcal{D} \times \mathbb{R}^{n_2} \times U \times W \to \mathbb{R}$ be jointly continuous in x_1, x_2, u, and w.

Theorem 3.1. *Consider the controlled nonlinear dynamical system* (12) and (13) *with*

$$J(x_{10}, x_{20}, u(\cdot), w(\cdot)) \triangleq \int_0^\infty L(x_1(t), x_2(t), u(t), w(t))\mathrm{d}t, \tag{16}$$

where $u(\cdot)$ and $w(\cdot)$ are admissible controls. Assume that there exist a continuously differentiable function $V : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$, class $\mathcal{K}$ functions $\alpha(\cdot)$, $\beta(\cdot)$, and $\theta(\cdot)$, and control laws $\phi : \mathcal{D} \times \mathbb{R}^{n_2} \to U$ and $\psi : \mathcal{D} \times \mathbb{R}^{n_2} \to W$, such that

$$\alpha(\|x_1\|) \le V(x_1, x_2) \le \beta(\|x_1\|), \quad (x_1, x_2) \in \mathcal{D} \times \mathbb{R}^{n_2}, \tag{17}$$

$$V'(x_1, x_2)F(x_1, x_2, \phi(x_1, x_2), \psi(x_1, x_2)) \le -\theta(\|x_1\|), \quad (x_1, x_2) \in \mathcal{D} \times \mathbb{R}^{n_2}, \tag{18}$$

$$\phi(0, x_2) = 0, \quad x_2 \in \mathbb{R}^{n_2}, \tag{19}$$

$$\psi(0, x_2) = 0, \quad x_2 \in \mathbb{R}^{n_2}, \tag{20}$$

$$\begin{aligned} &L(x_1, x_2, \phi(x_1, x_2), \psi(x_1, x_2)) + V'(x_1, x_2)F(x_1, x_2, \phi(x_1, x_2), \psi(x_1, x_2)) = 0, \\ &\quad (x_1, x_2) \in \mathcal{D} \times \mathbb{R}^{n_2}, \end{aligned} \tag{21}$$

$$L(x_1, x_2, u, \psi(x_1, x_2)) + V'(x_1, x_2)F(x_1, x_2, u, \psi(x_1, x_2)) \ge 0, \quad (x_1, x_2, u) \in \mathcal{D} \times \mathbb{R}^{n_2} \times U, \tag{22}$$

$$L(x_1, x_2, \phi(x_1, x_2), w) + V'(x_1, x_2)F(x_1, x_2, \phi(x_1, x_2), w) \le 0, \quad (x_1, x_2, w) \in \mathcal{D} \times \mathbb{R}^{n_2} \times W. \tag{23}$$

Then with the feedback controls $u = \phi(x_1, x_2)$ and $w = \psi(x_1, x_2)$, the closed-loop system given by Eqs. (14) *and* (15) *is asymptotically stable with respect to x_1 uniformly in x_{20} and there exists a neighborhood $\mathcal{D}_0 \subseteq \mathcal{D}$* of $x_1 = 0$ *such that*

$$J(x_{10}, x_{20}, \phi(x_1(\cdot), x_2(\cdot)), \psi(x_1(\cdot), x_2(\cdot))) = V(x_{10}, x_{20}), \quad (x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}. \tag{24}$$

In addition, if $(x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}$, then

$$\begin{aligned} &J(x_{10}, x_{20}, \phi(x_1(\cdot), x_2(\cdot)), \psi(x_1(\cdot), x_2(\cdot))) \\ &\quad = \min_{(u(\cdot), w(\cdot)) \in \mathcal{S}_\psi(x_{10}, x_{20}) \times \mathcal{S}_\phi(x_{10}, x_{20})} \max J(x_{10}, x_{20}, u(\cdot), w(\cdot)) \end{aligned} \tag{25}$$

and

$$J(x_{10}, x_{20}, \phi(x_1(\cdot), x_2(\cdot)), w(\cdot)) \le V(x_{10}, x_{20}), \quad w(\cdot) \in \mathcal{S}_\phi(x_{10}, x_{20}). \tag{26}$$

Finally, if $\mathcal{D} = \mathbb{R}^{n_1}$, $U = \mathbb{R}^{m_1}$, $W = \mathbb{R}^{m_2}$, and the functions $\alpha(\cdot)$ and $\beta(\cdot)$ satisfying Eq. (17) *are class $\mathcal{K}_\infty$, then the closed-loop system* (14) *and* (15) *is globally asymptotically stable with respect to x_1 uniformly in x_{20}.*

Theorem 3.1, whose proof is presented in the Appendix, provides sufficient conditions to solve differential games that involve nonlinear controlled dynamical systems of the form (12) and (13) with performance measure (16), and are concluded if $x_1(t) \to 0$ as $t \to \infty$. Specifically, Eq. (21) is the steady-state, Hamilton–Jacobi–Isaacs equation and Eqs. (21)–(23) guarantee that the saddle point condition (25) is verified. Given the control laws $\phi(\cdot, \cdot)$ and $\psi(\cdot, \cdot)$, it holds that $\mathcal{S}_\psi(x_{10}, x_{20}) \times \mathcal{S}_\phi(x_{10}, x_{20}) \subseteq \mathcal{S}(x_{10}, x_{20})$, and restricting our minimization problem to $(u(\cdot), w(\cdot)) \in \mathcal{S}(x_{10}, x_{20})$, that is, inputs corresponding to partial-state null convergent solutions, can be interpreted as incorporating a system detectability condition through the cost. However, it is important to note that an explicit characterization of $\mathcal{S}(x_{10}, x_{20})$, $\mathcal{S}_\psi(x_{10}, x_{20})$, and $\mathcal{S}_\phi(x_{10}, x_{20})$ is not required.

The feedback control laws $u = \phi(x_1, x_2)$ and $w = \psi(x_1, x_2)$ are independent of the initial conditions x_{10} and x_{20} and, using Eqs. (21)–(23), are such that

$$\left[\phi^{\mathrm{T}}(x_1, x_2), \psi^{\mathrm{T}}(x_1, x_2)\right]^{\mathrm{T}} \in \underset{(u(\cdot),w(\cdot)) \in \mathcal{S}_\psi(x_{10},x_{20}) \times \mathcal{S}_\phi(x_{10},x_{20})}{\arg\,\mathrm{minmax}} [L(x_1, x_2, u, w) + V'(x_1, x_2)F(x_1, x_2, u, w)]. \tag{27}$$

It follows from Theorem 3.1 that the pair of control laws $(\phi(\cdot,\cdot), \psi(\cdot,\cdot))$ guarantees asymptotic stability with respect to x_1 uniformly in x_{20} of the closed-loop system. However, the state feedback control law $\psi(\cdot,\cdot)$ may be destabilizing in the sense that, given an admissible control $u(\cdot) \notin \mathcal{S}_\psi(x_{10}, x_{20})$, the solution $x_1(t) = 0$, $t \geq 0$, of

$$\begin{aligned} \dot{x}_1(t) &= F_1(x_1(t), x_2(t), u(t), \psi(x_1(t), x_2(t))), \quad x_1(0) = x_{10}, \quad t \geq 0, \\ \dot{x}_2(t) &= F_2(x_1(t), x_2(t), u(t), \psi(x_1(t), x_2(t))), \quad x_2(0) = x_{20}, \end{aligned} \tag{28}$$

is *not* asymptotically stable with respect to x_1 uniformly in x_{20} or may even be unstable.

If we consider the input $w(\cdot)$ in Eqs. (12) and (13) as a disturbance, then the framework developed in Theorem 3.1 provides an analytical expression for the least upper bound attained with respect to $u(\cdot)$ of the performance measure (16) over the class of disturbances $\mathcal{S}_\phi(x_{10}, x_{20})$. Specifically, it follows from Eqs. (24)–(26) that

$$\begin{aligned} J(x_{10}, x_{20}, \phi(x_1(\cdot), x_2(\cdot)), w(\cdot)) &\leq J(x_{10}, x_{20}, \phi(x_1(\cdot), x_2(\cdot)), \psi(x_1(\cdot), x_2(\cdot))) \\ &= V(x_{10}, x_{20}), \quad (x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}, \end{aligned} \tag{29}$$

for all admissible inputs $w(\cdot)$ such that $\lim_{t\to\infty} x_1(t) = 0$, where $x_1(\cdot)$ is the solution of Eq. (12) with $u = \phi(x_1, x_2)$.

Possible applications of Theorem 3.1 involve pursuer–evader differential games, such as the *game of two cars* [21, pp. 237–244]. Specifically, if the game of two cars evolves on a plane, then the system dynamics is characterized by a state vector comprised of three components. Two components of the state vector identify the position of the pursuer with respect to the evader, whereas the third component corresponds to the angle between the pursuer velocity vector and the evader velocity vector. If we impose that the game terminates when the pursuer reaches the evader, then this end-of-game condition involves only two of the three state vector components and is irrespective of the angle between the pursuer's velocity vector and the evader's velocity vector. Therefore, Theorem 3.1 is adequate to solve this game of degree [21, p. 12]. The following remark introduces additional applications of Theorem 3.1.

Remark 3.1. If the conditions of Theorem 3.1 are satisfied, then the closed-loop dynamical system (14) and (15) is asymptotically stable with respect to x_1 uniformly in x_{20}. Hence, for every $l > 0$, there exists $\hat{t}_{\mathrm{f}} \geq 0$ such that if $t > \hat{t}_{\mathrm{f}}$, then $\|x_1(t)\| < l$, where $x_1(\cdot)$ denotes the solution of Eq. (14). Now, consider a game involving the nonlinear dynamical system (12) and (13) and the performance measure (16), whose terminal condition is given by $\|x_1(t_{\mathrm{f}})\| = l$ for some $t_{\mathrm{f}} \geq 0$ that is finite and not specified *a priori*. Then Theorem 3.1 provides sufficient conditions to find state-feedback control laws that solve this differential game.

The implications of this remark on the range of possible applications of Theorem 3.1 can be better appreciated considering again the game of two cars. If we impose that this game terminates when the pursuer is at a distance $l>0$ from the evader at some finite time $t_f \geq 0$ that is not predetermined, then also this end-of-game condition involves only two of the three components of the state vector. Hence, it follows from Remark 3.1 that Theorem 3.1 is adequate to solve this version of the game of two cars.

Remark 3.2. Setting $m_1 = m$ and $m_2 = 0$, Eqs. (12) and (13) reduce to

$$\dot{x}_1(t) = F_1(x_1(t), x_2(t), u(t)), \quad x_1(0) = x_{10}, \quad t \geq 0,$$
$$\dot{x}_2(t) = F_2(x_1(t), x_2(t), u(t)), \quad x_2(0) = x_{20}, \tag{30}$$

and the conditions of Theorem 3.1 reduce to the conditions of Theorem 3.2 of [26] characterizing the optimal control problem for partial-state stabilization. Furthermore, setting $n_2 = 0$, $m_1 = m$, and $m_2 = 0$, the nonlinear controlled dynamical system given by Eqs. (12) and (13) reduces to

$$\dot{x}(t) = F(x(t), u(t)), \quad x(0) = x_0, \quad t \geq 0. \tag{31}$$

In this case, the conditions of Theorem 3.1 reduce to the conditions of Theorem 3.1 of [8] characterizing the classical optimal control problem for time-invariant systems on the infinite time interval.

4. Spin stabilization of a spacecraft with input disturbance

In this section, we provide a numerical example to highlight the theoretical framework developed in Section 3. Consider the equations of motion of a spacecraft given by [38]

$$\dot{\omega}_1(t) = I_{23}\omega_2(t)\omega_3(t) + u(t), \quad \omega_1(0) = \omega_{10}, \quad t \geq 0, \tag{32}$$

$$\dot{\omega}_2(t) = I_{31}\omega_3(t)\omega_1(t) + w(t), \quad \omega_2(0) = \omega_{20}, \tag{33}$$

$$\dot{\omega}_3(t) = I_{12}\omega_1(t)\omega_2(t), \quad \omega_3(0) = \omega_{30}, \tag{34}$$

where $I_{23} \triangleq (I_2 - I_3)/I_1$, $I_{31} \triangleq (I_3 - I_1)/I_2$, $I_{12} \triangleq (I_1 - I_2)/I_3$, I_1, I_2, and I_3 are the principal moments of inertia of the spacecraft such that $0 < I_1 < I_2 < I_3$, $[\omega_1, \omega_2, \omega_3]^{\mathrm{T}} : [0, \infty) \to \mathbb{R}^3$ denote the angular velocity vector with respect to a given inertial reference frame expressed in a central body reference frame, and $u : [0, \infty) \to \mathbb{R}$ and $w : [0, \infty) \to \mathbb{R}$ are the spacecraft control moments. For this example, we seek state feedback controllers $u = \phi(x_1, x_2)$ and $w = \psi(x_1, x_2)$, where $x_1 = [\omega_1, \ \omega_2]^{\mathrm{T}}$ and $x_2 = \omega_3$, such that the performance measure

$$J(x_{10}, x_{20}, u(\cdot), w(\cdot)) = \int_0^\infty \left[I_{31}^2\omega_1^2(t) - I_{23}^2\omega_2^2(t) + 4I_{23}\omega_2(t)w(t) + u^2(t) - w^2(t)\right]\mathrm{d}t, \tag{35}$$

where $x_{10} = [\omega_{10}, \ \omega_{20}]^{\mathrm{T}}$ and $x_{20} = \omega_{30}$, satisfies Eq. (25) and the affine dynamical system given by Eqs. (32)–(34) is globally asymptotically stable with respect to x_1 uniformly in x_{20}. Since thrusters are usually aligned to the spacecraft principal axes of inertia, $\int_0^\infty u^2(t)\mathrm{d}t$ and $\int_0^\infty w^2(t)\mathrm{d}t$ capture the control effort, and hence the fuel consumption, for the thrusters aligned with the first and the second principal axis of inertia, respectively. Thus, minimizing with respect to u and maximizing with respect to w the term $\int_0^\infty \left[u^2(t) - w^2(t)\right]\mathrm{d}t$ in Eq. (35) implies minimizing the difference in the fuel consumption between the thrusters aligned with the first and the second inertia axes. The term $\int_0^\infty \left[I_{31}^2\omega_1^2(t) - I_{23}^2\omega_2^2(t)\right]\mathrm{d}t$ in Eq. (35) captures the difference in kinetic energy due to the angular velocities $\omega_1(\cdot)$ and $\omega_2(\cdot)$.

Note that Eqs. (32)–(34) with performance measure (35) is in the form of Eqs. (12) and (13) with performance measure (16). In this case, Theorem 3.1 can be applied with $n_1 = 2$, $n_2 = 1$, $m_1 = 1$, $m_2 = 1$, $F(x_1, x_2, u, w) = f(x_1, x_2) + G_u(x_1, x_2)u + G_w(x_1, x_2)$, $f(x_1, x_2) = [I_{23}\omega_3\omega_2, I_{31}\omega_3\omega_1, I_{12}\omega_3\omega_1]^{\mathrm{T}}$, $G_u(x_1, x_2) = [1,\ 0,\ 0]^{\mathrm{T}}$, and $G_w(x_1, x_2) = [0,\ 1,\ 0]^{\mathrm{T}}$ to characterize the partial-state asymptotically stabilizing controllers.

It follows from Eq. (27) and Lemma 2.1 that

$$\left[\phi^{\mathrm{T}}(x_1,x_2),\ \psi^{\mathrm{T}}(x_1,x_2)\right]^{\mathrm{T}} \in \underset{u(\cdot)\in\mathcal{S}_\psi(x_{10},x_{20})}{\arg\min}\ \underset{w(\cdot)\in\mathcal{S}_\phi(x_{10},x_{20})}{\arg\max}\ H(x_1,x_2,V'(x_1,x_2),u,w)$$
$$= \underset{w(\cdot)\in\mathcal{S}_\phi(x_{10},x_{20})}{\arg\max}\ \underset{u(\cdot)\in\mathcal{S}_\psi(x_{10},x_{20})}{\arg\min}\ H(x_1,x_2,V'(x_1,x_2),u,w), \tag{36}$$

where

$$H(x_1,x_2,\lambda^{\mathrm{T}},u,w) \triangleq L(x_1,x_2,u,w) + \lambda^{\mathrm{T}}[f(x_1,x_2) + G_u(x_1,x_2)u + G_w(x_1,x_2)w],$$

and $L(x_1,x_2,u,w) = I_{31}^2\omega_1^2 - I_{23}^2\omega_2^2 + 4I_{23}\omega_2 w + u^2 - w^2$. Thus, the feedback control laws

$$\phi(x_1,x_2) = -\tfrac{1}{2}\left[G_u^{\mathrm{T}}(x_1,x_2)V'^{\mathrm{T}}(x_1,x_2)\right], \tag{37}$$

$$\psi(x_1,x_2) = \tfrac{1}{2}\left[G_w^{\mathrm{T}}(x_1,x_2)V'^{\mathrm{T}}(x_1,x_2) + 4I_{23}\omega_2\right], \tag{38}$$

follow from Eq. (36) by setting

$$\frac{\partial}{\partial[u^{\mathrm{T}}w^{\mathrm{T}}]^{\mathrm{T}}}[L(x_1,x_2,u,w) + V'(x_1,x_2)f(x_1,x_2)$$
$$+V'(x_1,x_2)G_u(x_1,x_2)u + V'(x_1,x_2)G_w(x_1,x_2)w] = 0. \tag{39}$$

Next, we verify that the conditions of Theorem 3.1 are satisfied. Specifically, Eq. (21) with $\phi(x_1,x_2)$ and $\psi(x_1,x_2)$ given by Eqs. (37) and (38), respectively, reduces to

$$0 = x_1^{\mathrm{T}}\begin{bmatrix} I_{31}^2 & 0 \\ 0 & -I_{23}^2 \end{bmatrix}x_1 + V'(x_1,x_2)f(x_1,x_2) - \frac{1}{4}[V'(x_1,x_2)G_u(x_1,x_2)]^2$$
$$+\frac{1}{4}[V'(x_1,x_2)G_w(x_1,x_2) + 4I_{23}\omega_2]^2, \quad (x_1,x_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}, \tag{40}$$

which is verified by

$$V(x_1,x_2) = x_1^{\mathrm{T}}\begin{bmatrix} I_{31} & 0 \\ 0 & -I_{23} \end{bmatrix}x_1, \quad (x_1,x_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}. \tag{41}$$

Hence, Eq. (17) holds with $\alpha(\|x_1\|) = \alpha\|x_1\|$ and $\beta(\|x_1\|) = \beta\|x_1\|$, where $\alpha = \min\{I_{31},\ -I_{23}\}$ and $\beta = \max\{I_{31},\ -I_{23}\}$. In addition, Eqs. (19) and (20) are trivially satisfied, and since

$$V'(x_1,x_2)f(x_1,x_2) - \tfrac{1}{2}V'(x_1,x_2)G_w(x_1,x_2)R_{2w}^{-1}(x_1,x_2)L_{2w}^{\mathrm{T}}(x_1,x_2)$$
$$-\tfrac{1}{2}V'(x_1,x_2)\left[G_u(x_1,x_2)G_u^{\mathrm{T}}(x_1,x_2) - G_w(x_1,x_2)G_w^{\mathrm{T}}(x_1,x_2)\right]V'^{\mathrm{T}}(x_1,x_2)$$
$$= -2I_{31}^2\omega_1^2 - 2I_{23}^2\omega_2^2, \quad (x_1,x_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}, \tag{42}$$

condition (18) is satisfied with $\theta(\|x_1\|) = 2\alpha^2\|x_1\|$. Lastly, it holds that

$$L(x_1,x_2,u,\psi(x_1,x_2)) + V'(x_1,x_2)[f(x_1,x_2) + G_u(x_1,x_2)u + G_w(x_1,x_2)\psi(x_1,x_2)]$$
$$= L(x_1,x_2,u,\psi(x_1,x_2)) + V'(x_1,x_2)[f(x_1,x_2) + G_u(x_1,x_2)u$$
$$+G_w(x_1,x_2)\psi(x_1,x_2)] - L(x_1,x_2,\phi(x_1,x_2),\psi(x_1,x_2))$$
$$-V'(x_1,x_2)[f(x_1,x_2) + G_u(x_1,x_2)\phi(x_1,x_2) + G_w(x_1,x_2)\psi(x_1,x_2)]$$

$$= [u-\phi(x_1,x_2)]^{\mathrm{T}}R_{2u}(x_1,x_2)[u-\phi(x_1,x_2)]$$
$$\geq 0, \quad (x_1,x_2,u) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{m_1}, \tag{43}$$

and similarly

$$L(x_1,x_2,\phi(x_1,x_2),w) + V'(x_1,x_2)[f(x_1,x_2) + G_u(x_1,x_2)\phi(x_1,x_2) + G_w(x_1,x_2)w]$$
$$= [w-\psi(x_1,x_2)]^{\mathrm{T}}R_{2w}(x_1,x_2)[w-\psi(x_1,x_2)]$$
$$\leq 0, \quad (x_1,x_2,w) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{m_2}. \tag{44}$$

Hence Eqs. (22) and (23) are verified.

Since all of the conditions of Theorem 3.1 hold, the feedback control laws

$$\phi(x_1,x_2) = -I_{31}\omega_1, \quad \psi(x_1,x_2) = I_{23}\omega_2, \quad (x_1,x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2},$$

guarantee that the dynamical system (32)–(34) is globally asymptotically stable with respect to x_1 uniformly in x_{20}, and

$$J(x_{10},x_{20},\phi(\cdot,\cdot),\psi(\cdot,\cdot)) = \min_{(u(\cdot),w(\cdot)) \in \mathcal{S}_\psi(x_{10},x_{20}) \times \mathcal{S}_\phi(x_{10},x_{20})} \max J(x_{10},x_{20},u(\cdot),w(\cdot))$$
$$= I_{13}\omega_{10}^2 - I_{23}\omega_{20}^2 \tag{45}$$

for all $(x_{10},x_{20}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$.

Now, suppose the thruster delivering the control moment w is defective and

$$w = \left[\varepsilon - \frac{1}{I_{23}}\delta(t)\right]\psi(x_1,x_2), \quad (t,x_1,x_2) \in [0,\infty) \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \tag{46}$$

where $\varepsilon > 0$ and $\delta : [0,\infty) \to [0,\infty)$ is continuous on the set of nonnegative real numbers. Then the closed-loop dynamical system is given by

$$\dot\omega_1(t) = I_{23}\omega_2(t)\omega_3(t) + \phi(x_1,x_2), \quad \omega_1(0) = \omega_{10}, \quad t \geq 0, \tag{47}$$

$$\dot\omega_2(t) = I_{31}\omega_3(t)\omega_1(t) + [\varepsilon\psi(x_1(t),x_2(t)) - \omega_2(t)\delta(t)], \quad \omega_2(0) = \omega_{20}, \tag{48}$$

$$\dot\omega_3(t) = I_{12}\omega_1(t)\omega_2(t), \quad \omega_3(0) = \omega_{30}, \tag{49}$$

which is equivalent to the time-invariant nonlinear dynamical system

$$\dot\omega_1(t) = I_{23}\omega_2(t)\omega_3(t) + \phi(x_1,\hat x_2), \quad \omega_1(0) = \omega_{10}, \quad t \geq 0, \tag{50}$$

$$\dot\omega_2(t) = I_{31}\omega_3(t)\omega_1(t) + [\varepsilon\psi(x_1(t),\hat x_2(t)) - \omega_2(t)\delta(\hat x_2(t))], \quad \omega_2(0) = \omega_{20}, \tag{51}$$

$$\dot\omega_3(t) = I_{12}\omega_1(t)\omega_2(t), \quad \omega_3(0) = \omega_{30}, \tag{52}$$

$$\dot\omega_4(t) = 1, \quad \omega_4(0) = 0, \tag{53}$$

where $\hat x_2 \triangleq [x_2^{\mathrm{T}},\omega_4]^{\mathrm{T}}$.

In this case, the Lyapunov function (41) is such that

$$\dot V(x_1,\hat x_2) = V'(x_1,\hat x_2)[f(x_1,\hat x_2) + G_u(x_1,\hat x_2)\phi(x_1,\hat x_2) + \varepsilon G_w(x_1,\hat x_2)\psi(x_1,\hat x_2)]$$
$$-\omega_2 V'(x_1,\hat x_2)G_w(x_1,\hat x_2)\delta(\hat x_2)$$
$$= -2I_{23}^2\omega_1^2 - 2\varepsilon I_{23}^2\omega_2^2 + 2I_{23}\omega_2^2\delta(\hat x_2)$$
$$\leq -2I_{31}^2\omega_1^2 - 2\varepsilon I_{23}^2\omega_2^2, \quad (x_1,\hat x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2+1}. \tag{54}$$

Since $\theta(\|x_1\|) = 2I_{31}^2\omega_1^2 + 2\varepsilon I_{23}^2\omega_2^2$, $x_1 \in \mathbb{R}^{n_1}$ is a class $\mathcal{K}$ function, it follows from Theorem 2.1 that the closed-loop dynamical system (47)–(49) is globally asymptotically stable with respect to x_1 uniformly in $\hat x_2(0)$, which implies that $\lim_{t\to\infty}x_1(t) = 0$. Hence, the input function (46) is such

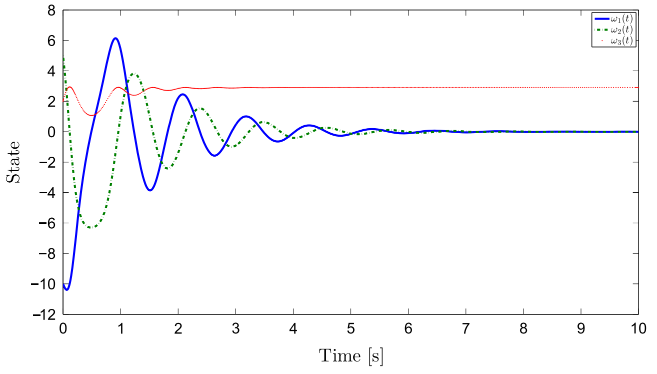

Fig. 1. Closed-loop system trajectories versus time.

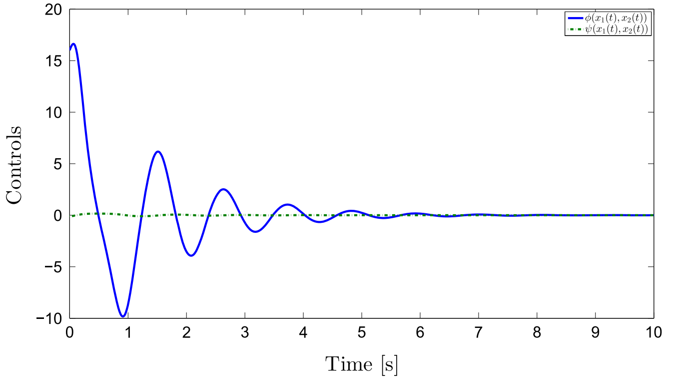

Fig. 2. Control signal versus time.

that $w(\cdot) \in \mathcal{S}_\phi(x_{10}, x_{20})$ and it follows from Theorem 3.1 and Eq. (45) that

$$J(x_{10}, x_{20}, \phi(\cdot,\cdot), w(\cdot)) = I_{31}\omega_{10}^2 - I_{23}\omega_{20}^2, \quad (x_{10}, x_{20}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}. \tag{55}$$

Let $I_1 = 4$ kg m^2, $I_2 = 10$ kg m^2, $I_3 = 20$ kg m^2, $\omega_{10} = -10$ Hz, $\omega_{20} = 5$ Hz, and $\omega_{30} = 2$ Hz. Fig. 1 shows the state trajectories of Eqs. (32) and (33) with $u = \phi(x_1, x_2)$ and $w = \psi(x_1, x_2)$ versus time, and Fig. 2 shows the corresponding control signal versus time. Next, let $\varepsilon = 10^{-2}$ and $\delta(\cdot)$ be a triangle wave, that is,

$$\delta(t) = 200\left[(-1)^{\lfloor t+\frac{1}{2}\rfloor}\left(t - \left\lfloor t + \tfrac{1}{2}\right\rfloor\right) + 1\right], \quad t \geq 0, \tag{56}$$

where $\lfloor\cdot\rfloor$ denotes the floor function. Fig. 3 shows the state trajectories of Eqs. (32) and (33) with $u = \phi(x_1, x_2)$ and $w = \left[\varepsilon - \frac{1}{I_{23}}\delta(t)\right]\psi(x_1, x_2)$ versus time, and Fig. 4 shows the corresponding control signal versus time. Note that $x_1(t) \to 0$ as $t \to \infty$ both in Fig. 1 and in Fig. 3. Finally, $J(x_{10}, x_{20}, \phi(\cdot,\cdot), \psi(\cdot,\cdot)) = J(x_{10}, x_{20}, \phi(\cdot,\cdot), w(\cdot)) = 222.5$ Hz^2, for all $w(\cdot) \in \mathcal{S}_\phi(x_{10}, x_{20})$.

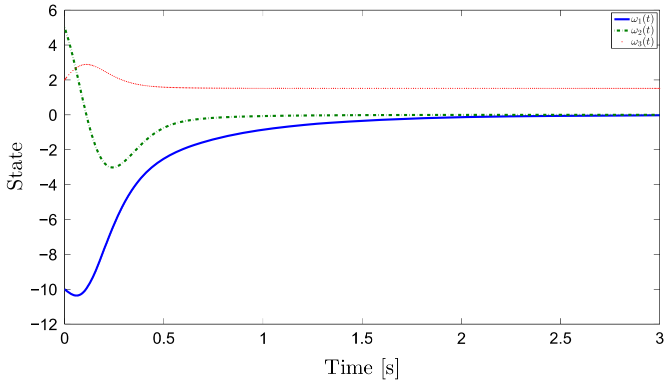

Fig. 3. Closed-loop system trajectories versus time in the presence of disturbances.

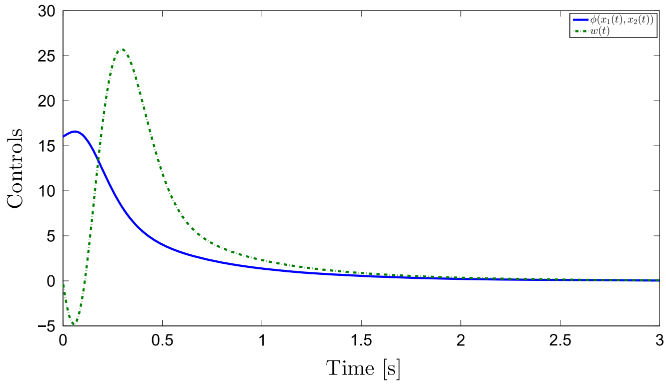

Fig. 4. Control signal versus time.

5. Adaptive control, differential games, and disturbance modeling

In this section, we show how our differential game framework can be applied to assess the ability of robust feedback control laws to satisfactorily regulate uncertain nonlinear dynamical systems. Model reference adaptive control is a technique aimed at stabilizing poorly modeled dynamical systems in spite of unknown plant nonlinearities [20]. This result is achieved parametrizing the matched nonlinearities in the plant state by means of a given vector function, known as regressor vector. However, the regressor vector can only *approximate* the plant nonlinearities. By modeling matched and unmatched nonlinearities, which are not captured by the regressor vector, as pursuer's and evader's control inputs in a differential game, we apply the framework developed in Section 3 to *explicitly* parametrize those matched and unmatched nonlinearities that do not disrupt the ability of an adaptive control law to guarantee satisfactory trajectory tracking.

Consider the nonlinear *plant*

$$\dot{x}(t) = Ax(t) + B\left[\omega(t) + \mu_{\rm m}(t) + \Theta^{\rm T}\Phi(x(t))\right] + \hat{G}_\mu(x(t))\mu_{\rm u}(t), \quad x(0) = x_0, \quad t \ge 0, \tag{57}$$

where $x(t)\in\mathbb{R}^{n_1}$, $t\geq 0$, $\omega(t)\in\mathbb{R}^{m_1}$, $\mu_{\rm m}(t)\in\mathbb{R}^{m_1}$, $\mu_{\rm u}(t)\in\mathbb{R}^{m_2}$, $A\in\mathbb{R}^{n_1\times n_1}$, $B\in\mathbb{R}^{n_1\times m_1}$, $\Theta\in\mathbb{R}^{p\times m_1}$, $\hat{G}_\mu:\mathbb{R}^{n_1}\to\mathbb{R}^{n_1\times m_2}$ is Lipschitz continuous on $\mathbb{R}^{n_1}$, and $\Phi:\mathbb{R}^{n_1}\to\mathbb{R}^p$ is continuously differentiable on $\mathbb{R}^{n_1}$. Although A is unknown, the structure of A is known. The terms $\Theta^{\rm T}\Phi(x)$, $\mu_{\rm m}$, and $\hat{G}_\mu(x)\mu_{\rm u}$ capture the system nonlinearities. Specifically, the matrix Θ is unknown, the *regressor vector* $\Phi(\cdot)$ is assigned *a priori*, and $\Theta^{\rm T}\Phi(x)$ captures the system nonlinearities parametrized by the regressor vector. The vector function $\Phi(\cdot)$ is a finite-dimensional basis for the infinite-dimensional space of nonlinearities, and it follows from Baire's category theorem [25, Th. 4–7.2] that $\Theta^{\rm T}\Phi(x)$ can only approximate the system nonlinearities [11]. Hence, the unknown terms $\mu_{\rm m}$ and $\hat{G}_\mu(x)\mu_{\rm u}$ in (57) are key to capture unmodeled matched and unmatched nonlinearities, respectively.

Consider the *reference model*

$$\dot{x}_{\rm ref}(t)=A_{\rm ref}x_{\rm ref}(t)+B_{\rm ref}r(t),\quad x_{\rm ref}(0)=x_{\rm ref\ 0},\quad t\geq 0, \tag{58}$$

where $x_{\rm ref}(t)\in\mathbb{R}^{n_1}$, $t\geq 0$, $r:[0,\infty)\to\mathbb{R}^{m_1}$ denotes a bounded reference input signal, $A_{\rm ref}\in\mathbb{R}^{n_1\times n_1}$ is Hurwitz, and $B_{\rm ref}\in\mathbb{R}^{n_1\times m_1}$, and assume there exist $K_x\in\mathbb{R}^{n_1\times m_1}$ and $K_r\in\mathbb{R}^{m_1\times m_1}$, such that

$$A_{\rm ref}=A+BK_x^{\rm T}, \tag{59}$$

$$B_{\rm ref}=BK_r^{\rm T}. \tag{60}$$

The objective of the model reference adaptive control is to design $\omega=\eta(x_1,x_2)$, for some $x_2\in\mathbb{R}^{n_2}$, so that the plant state $x(t)$, $t\geq 0$, of Eq. (57) with $\mu_{\rm m}=0$ and $\mu_{\rm u}=0$ eventually mimics the state of the reference model $x_{\rm ref}(t)$, that is, $\lim_{t\to\infty}x_1(t)=0$, where $x_1(t)=x(t)-x_{\rm ref}(t)$. To this goal, the plant control input is set as

$$\eta(x_1,x_2)=\hat{K}_x^{\rm T}x+\hat{K}_r^{\rm T}r(\tau)-\hat{\Theta}^{\rm T}\Phi(x), \tag{61}$$

where $\tau\geq 0$, $x_2=\left[{\rm vec}^{\rm T}\left(\hat{K}_x^{\rm T}(\tau)\right),{\rm vec}^{\rm T}\left(\hat{K}_r^{\rm T}(\tau)\right),{\rm vec}^{\rm T}\left(\hat{\Theta}^{\rm T}(\tau)\right),\tau\right]^{\rm T}$, ${\rm vec}(\cdot)$ denotes the column-stacking operator,

$$\dot{x}_2(t)=f_2(x_1(t),x_2(t)),\quad x_2(0)=x_{20},\quad t\geq 0, \tag{62}$$

$f_2:\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to\mathbb{R}^{n_2}$ is to be determined, $f_2(\cdot,x_2)$ is locally Lipschitz continuous in x_1, and $f_2(x_1,\cdot)$ is locally Lipschitz continuous in x_2. Note that in practical applications, the adaptation gains $\hat{K}_x(\cdot)$, $\hat{K}_r(\cdot)$, and $\hat{\Theta}(\cdot)$ represent variables in a computer and do not have a physical meaning. Therefore, since the adaptation gains do not need to asymptotically converge to zero, the partial-state stability framework presented in this paper is ideal to analyze the adaptive control architecture.

Assuming that the system nonlinearities are exclusively captured by $\Theta^{\rm T}\Phi(x)$, the next result provides a choice of the adaptation gains $\hat{K}_x(\cdot)$, $\hat{K}_r(\cdot)$, and $\hat{\Theta}(\cdot)$, and hence of the control system (62), so that $\lim_{t\to\infty}x_1(t)=0$. For the statement of this result, note that Eqs. (57), (61), and (58) yield

$$\begin{aligned}\dot{x}_1(t)&=A_{\rm ref}x_1(t)+B\left[\tilde{K}_x^{\rm T}(t)x(t)+\tilde{K}_r^{\rm T}(t)r(t)\right]\\&\quad+B\left[\mu_{\rm m}(t)-\tilde{\Theta}^{\rm T}(t)\Phi(x(t))\right]+G_{1\mu}(x_1(t))\mu_{\rm u}(t),\quad x_1(0)=x_{10},\ t\geq 0,\end{aligned} \tag{63}$$

where $\tilde{K}_x(t)\triangleq\hat{K}_x(t)-K_x$, $\tilde{K}_r(t)\triangleq\hat{K}_r(t)-K_r$, and $\tilde{\Theta}(t)\triangleq\hat{\Theta}(t)-\Theta$.

Theorem 5.1 (*[28, Th. 9.2]*). *Consider the nonlinear dynamical system $\mathcal{G}$ given by Eq. (63) with $\mu_{\rm m}=0$, $\mu_{\rm u}=0$, and $\hat{K}_x(\cdot)$, $\hat{K}_r(\cdot)$, and $\hat{\Theta}(\cdot)$ such that*

$$\dot{\hat{K}}_x(t)=-\Gamma_x x(t)x_1^{\rm T}(t)PB,\quad \hat{K}_x(0)=\hat{K}_{x0},\quad t\geq 0, \tag{64}$$

$$\dot{\hat{K}}_r(t)=-\Gamma_r r(t)x_1^{\rm T}(t)PB,\quad \hat{K}_r(0)=\hat{K}_{y0}, \tag{65}$$

$$\dot{\hat{\Theta}}(t)=\Gamma_\Theta \Phi(x(t))x_1^{\rm T}(t)PB,\quad \hat{\Theta}(0)=\hat{\Theta}_0, \tag{66}$$

where $\Gamma_x\in\mathbb{R}^{n_1\times n_1}$, $\Gamma_r\in\mathbb{R}^{l\times l}$, and $\Gamma_\Theta\in\mathbb{R}^{p\times p}$ are positive-definite, $x(t)$, $t\geq 0$, satisfies Eq. (57) with $\mu_{\rm m}(t)=\mu_{\rm u}(t)=0$ and $\omega=\eta(x_1,x_2)$ given by Eq. (61), and $P\in\mathbb{R}^{n_1\times n_1}$ is the positive-definite solution of

$$0=A_{\rm ref}^T P+PA_{\rm ref}+Q, \tag{67}$$

for some positive-definite $Q\in\mathbb{R}^{n_1\times n_1}$. Then $\mathcal{G}$ is uniformly Lyapunov stable, and $x_1(t)\to 0$ as $t\to\infty$ uniformly in x_{20} for all $x_{10}\in\mathbb{R}^{n_1}$ and $x_{20}\in\mathbb{R}^{n_2}$.

Let $x_1(t)$, $t\geq 0$, verify Eq. (63) with $\mu_{\rm m}(t)=\mu_{\rm u}(t)=0$, and let $x_2(t)$ verify Eq. (62). Theorem 5.1 consists in proving uniform Lyapunov stability of closed-loop system by means of a Lyapunov function $V:\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to\mathbb{R}$ such that

$$\dot{V}(x_1(t),x_2(t))=-x_1^{\rm T}(t)Qx_1(t),\quad t\geq 0, \tag{68}$$

along the trajectories of Eqs. (63) and (62) with $\mu_{\rm m}(t)=\mu_{\rm u}(t)=0$, and then applying Barbalat's lemma [15, Lemma 4.1] to prove global uniform convergence of $x_1(\cdot)$.

Considering the matched nonlinearities $\mu_{\rm m}(\cdot)$ and the unmatched nonlinearities $\mu_{\rm u}(\cdot)$ in Eq. (57) as pursuer and evader control inputs in a differential game, respectively, the next theorem provides an *explicit* parametrization for those nonlinearities, which are not captured by the regressor vector and do not disrupt the ability of the adaptive control law (61) to regulate the nonlinear plant (57). For the statement of this theorem, which is the main original result of this section, let $R_{2\mu_{\rm m}}:\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to\mathbb{R}^{m_1\times m_1}$ and $R_{2\mu_{\rm u}}:\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to\mathbb{R}^{m_2\times m_2}$ be continuous on $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ and sign-definite, that is, either $R_{2\mu_{\rm m}}(x_1,x_2)\geq N_{\rm m}(x_1)$, $(x_1,x_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$, or $R_{2\mu_{\rm m}}(x_1,x_2)\leq -N_{\rm m}(x_1)$, for some positive-definite matrix function $N_{\rm m}:\mathbb{R}^{n_1}\to\mathbb{R}^{m_1\times m_1}$, and either $R_{2\mu_{\rm u}}(x_1,x_2)\geq N_{\rm u}(x_1)$ or $R_{2\mu_{\rm u}}(x_1,x_2)\leq -N_{\rm u}(x_1)$ for some positive-definite matrix function $N_{\rm u}:\mathbb{R}^{n_1}\to\mathbb{R}^{m_2\times m_2}$. Let also

$$V(x_1,x_2)=\int_0^\infty s_1^{\rm T}(t,x_1,x_2)Qs_1(t,x_1,x_2){\rm d}t, \tag{69}$$

where $s(t,x_1,x_2)=[s_1^{\rm T}(t,x_1,x_2),s_2^{\rm T}(t,x_1,x_2)]^{\rm T}$, $t\geq 0$, denotes the *trajectory* of Eqs. (63) and (62) with initial condition $s(0,x_1,x_2)=[x_1^{\rm T},x_2^{\rm T}]^{\rm T}$, $s_1(t,x_1,x_2)\in\mathbb{R}^{n_1}$, and $s_2(t,x_1,x_2)\in\mathbb{R}^{n_2}$.

Theorem 5.2. *Consider the nonlinear dynamical system (63), where $\tilde{K}_x(t)=\hat{K}_x(t)-K_x$, $t\geq 0$, $\tilde{K}_r(t)=\hat{K}_r(t)-K_r$, and $\tilde{\Theta}(t)=\hat{\Theta}(t)-\Theta$, $\hat{K}_x(\cdot)$, $\hat{K}_r(\cdot)$, and $\hat{\Theta}(\cdot)$ satisfy Eqs. (64)–(66), and suppose that $\frac{\partial\Phi(x)}{\partial x_1}$ is globally uniformly bounded and*

$$G_{\mu_{\rm m}}(x_1)R_{2\mu_{\rm m}}^{-1}(x_1,x_2)G_{\mu_{\rm m}}^{\rm T}(x_1)+G_{\mu_{\rm u}}(x_1)R_{2\mu_{\rm u}}^{-1}(x_1,x_2)G_{\mu_{\rm u}}^{\rm T}(x_1)\geq 0,\quad (x_1,x_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}, \tag{70}$$

where $G_{\mu_{\rm m}} = \left[B^{\rm T}, 0_{m_1\times n_2}\right]^{\rm T}$ *and* $G_{\mu_{\rm u}} = \left[G_{1\mu}^{\rm T}(x_1), 0_{m_2\times n_2}\right]^{\rm T}$. *If*

$$\mu_{\rm m} = \phi(x_1,x_2) = -\tfrac{1}{4}R_{2\mu_{\rm m}}^{-1}(x_1,x_2)\left[B^{\rm T},0_{m_1\times n_2}\right]V'^{\rm T}(x_1,x_2), \quad (x_1,x_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}, \tag{71}$$

$$\mu_{\rm u} = \psi(x_1,x_2) = -\tfrac{1}{4}R_{2\mu_{\rm u}}^{-1}(x_1,x_2)\left[G_{1\mu}^{\rm T}(x_1),0_{m_2\times n_2}\right]V'^{\rm T}(x_1,x_2), \tag{72}$$

then $x_1(t)\to 0$ *as* $t\to\infty$, *uniformly in* $\hat{K}_{x0}$, $\hat{K}_{y0}$, *and* $\hat{\Theta}_0$.

This result, whose proof is presented in the Appendix, proves that the adaptive control law (61) guarantees satisfactory trajectory tracking in spite of the matched nonlinearity $\mu_{\rm m}=\phi(x_1,x_2)$ and the unmatched nonlinearity $\mu_{\rm u}=\psi(x_1,x_2)$, which are not captured by the regressor vector $\Phi(x)$. Remarkably, both $\phi(x_1,x_2)$ and $\psi(x_1,x_2)$ are parametrized by the sign-definite matrix functions $R_{2\mu_{\rm m}}(x_1,x_2)$ and $R_{2\mu_{\rm u}}(x_1,x_2)$ and the positive-definite matrix Q. Therefore, Theorem 5.2 provides an explicit parametrization of the nonlinearities $\mu_{\rm m}(\cdot)$ and $\mu_{\rm u}(\cdot)$ by means of $\frac{m_1(m_1+1)+m_2(m_2+1)}{2}$ scalar functions of x_1 and x_2, and $\frac{n_1(n_1+1)}{2}$ real numbers.

Note that if $R_{2\mu_{\rm m}}(x_1,x_2)\geq N_{\rm m}(x_1)>0$ and $R_{2\mu_{\rm u}}(x_1,x_2)\geq N_{\rm u}(x_1)>0$, then Eq. (70) is always satisfied. Alternatively, if $R_{2\mu_{\rm m}}(x_1,x_2)\leq -N_{\rm m}(x_1)<0$ and $R_{2\mu_{\rm u}}(x_1,x_2)\leq -N_{\rm u}(x_1)<0$, then Eq. (70) is never satisfied. Let

$$\begin{aligned}J(x_{10},x_{20},\mu_{\rm m}(\cdot),\mu_{\rm u}(\cdot)) = \int_0^\infty \Big[&\big(\phi(x_1(t),x_2(t))+\mu_{\rm m}(t)\big)^{\rm T}R_{2\mu_{\rm m}}(x_1(t),x_2(t))\cdot\\ &\big(\phi(x_1(t),x_2(t))+\mu_{\rm m}(t)\big)\\ &+\big(\psi(x_1(t),x_2(t))+\mu_{\rm u}(t)\big)^{\rm T}R_{2\mu_{\rm u}}(x_1(t),x_2(t))\cdot\\ &\big(\psi(x_1(t),x_2(t))+\mu_{\rm u}(t)\big)\\ &-V'(x_1(t),x_2(t))f(x_1(t),x_2(t))\Big]{\rm d}t,\end{aligned} \tag{73}$$

where $x_1(t)$, $t\geq 0$, and $x_2(t)$ satisfy Eqs. (63) and (62), $\phi(\cdot,\cdot)$ and $\psi(\cdot,\cdot)$ are given by Eqs. (71) and (72), respectively, $V(\cdot,\cdot)$ is given by Eq. (69), $f(x_1,x_2)=[f_1^{\rm T}(x_1,x_2),f_2^{\rm T}(x_1,x_2)]^{\rm T}$, and

$$f_1(x_1,x_2) = A_{\rm ref}x_1 + B\left[\tilde{K}_x^{\rm T}x + \tilde{K}_r^{\rm T}r(\tau) - \tilde{\Theta}^{\rm T}\Phi(x)\right]. \tag{74}$$

The performance measure (73) captures the energy of the nonlinearities $\mu_{\rm m}(\cdot)$ and $\mu_{\rm u}(\cdot)$ by means of the weighted $\mathcal{L}_2$ norms of $\big(\phi(x_1,x_2)+\mu_{\rm m}\big)$ and $\big(\psi(x_1,x_2)+\mu_{\rm u}\big)$. It follows from the proof of Theorem 5.2 that if $R_{2\mu_{\rm m}}(x_1,x_2)$ is positive-definite and $R_{2\mu_{\rm u}}(x_1,x_2)$ is negative-definite, or if $R_{2\mu_{\rm m}}(x_1,x_2)$ is negative-definite and $R_{2\mu_{\rm u}}(x_1,x_2)$ is positive-definite, then Eqs. (71) and (72) guarantee the existence of a saddle point for Eq. (73) along the trajectories of Eqs. (63) and (62). Lastly, if both $R_{2\mu_{\rm m}}(x_1,x_2)$ and $R_{2\mu_{\rm u}}(x_1,x_2)$ are positive-definite, then it follows from the proof of Theorem 5.2 that Eqs. (71) and (72) guarantee the existence of a minimum for the cost function (73).

Example 5.1. Consider the nonlinear plant

$$\dot{x}(t) = ax(t) + b\left[\omega(t)+\mu_{\rm m}(t)+\theta\Phi(x(t))\right] + \hat{g}_\mu\mu_{\rm u}(t), \quad x(0)=x_0, \quad t\geq 0, \tag{75}$$

where $x(t)\in\mathbb{R}$ denotes the plant state, $\omega(t)\in\mathbb{R}$ denotes the plant control input, $a\in\mathbb{R}$ is unknown, $b>0$, $\theta\in\mathbb{R}$ is unknown, $\Phi:\mathbb{R}\to\mathbb{R}$ is given and denotes the regressor function, $\hat{g}_\mu\in\mathbb{R}$ is given, $\mu_{\rm m}:[0,\infty)\to\mathbb{R}$ captures the matched nonlinearities, and $\mu_{\rm u}:[0,\infty)\to\mathbb{R}$ captures the

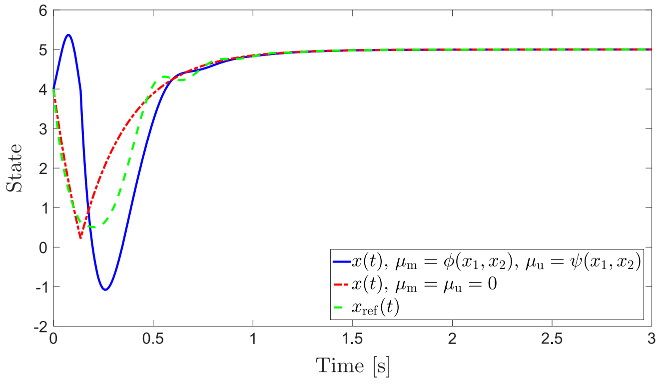

Fig. 5. Plant and reference trajectories versus time.

unmatched nonlinearities. Consider also the reference model

$$\dot{x}_{\rm ref}(t) = a_{\rm ref}x_{\rm ref}(t) + b_{\rm ref}r(t), \quad x_{\rm ref}(0) = x_{\rm ref\ 0}, \quad t \geq 0, \tag{76}$$

where $x_{\rm ref}(t) \in \mathbb{R}$ denotes the reference model state, $r(t) \in \mathbb{R}$ denotes a bounded input signal, $a_{\rm ref} < 0$, and $b_{\rm ref} \in \mathbb{R}$. The dynamical systems (75) and (76) are in the same form as Eqs. (57) and (58), respectively, with $n_1 = 1$, $m_1 = 1$, $m_2 = 1$, and $p = 1$.

Assuming that $\mu_{\rm m}(t) = \mu_{\rm u}(t) = 0$, $t \geq 0$, Theorem 5.1 provides an adaptive control law $\omega = \eta(x_1, x_2)$, such that the plant state $x(\cdot)$ asymptotically mimics the reference model state $x_{\rm ref}(\cdot)$. Our goal is to apply Theorem 5.2 and parametrize the nonlinear terms $\mu_{\rm m}(\cdot)$ and $\mu_{\rm u}(\cdot)$ that do not disrupt the adaptive control law's ability to regulate the plant dynamics.

It follows from Theorem 5.1 that if $\mu_{\rm m}(t) = \mu_{\rm u}(t) = 0$, $t \geq 0$, and

$$\omega = \eta(x_1, x_2) = \hat{k}_x x + \hat{k}_r r(\tau) - \hat{\theta}\Phi(x), \tag{77}$$

where $x_1 = x - x_{\rm ref}$, $\tau \geq 0$, $x_2 = [\hat{k}_x, \hat{k}_r, \hat{\theta}, \tau]^{\rm T}$,

$$\dot{\hat{k}}_x(t) = -\gamma_x x(t)x_1(t)pb, \quad \hat{k}_x(0) = \hat{k}_{x0}, \quad t \geq 0, \tag{78}$$

$$\dot{\hat{k}}_r(t) = -\Gamma_r r(t)x_1(t)pb, \quad \hat{k}_r(0) = \hat{k}_{r0}, \tag{79}$$

$$\dot{\hat{\theta}}(t) = \gamma_\theta \Phi(x(t))x_1(t)pb, \quad \hat{\theta}(0) = \hat{\theta}_0, \tag{80}$$

$\gamma_x > 0$, $\gamma_r > 0$, $\gamma_\theta > 0$, $p = -\frac{q}{2a_{\rm ref}}$, and $q > 0$, then $x_1(t) \to 0$ as $t \to 0$ uniformly in $x_2(0)$ for all $x_1(0) \in \mathbb{R}^{n_1}$ and $x_2(0) \in \mathbb{R}^{n_2}$. It follows from Theorem 9.1 of [28] that

$$V(x_1, x_2) = x_1^2 + b\left(\gamma_x^{-1}\hat{k}_x^2 + \gamma_r^{-1}\hat{k}_r^2 + \gamma_\theta^{-1}\hat{\theta}^2\right) \tag{81}$$

is a Lyapunov function, such that Eq. (68) with $Q = q$ is satisfied along the trajectory of Eq. (75) with adaptive control input (77). In this case, Eq. (70) specializes to

$$\frac{b^2}{r_{2\mu_{\rm m}}} + \frac{\hat{g}_\mu^2}{r_{2\mu_{\rm u}}} \geq 0, \tag{82}$$

where $r_{2\mu_{\rm m}} \in \mathbb{R}\backslash\{0\}$ and $r_{2\mu_{\rm u}} \in \mathbb{R}\backslash\{0\}$ are design parameters, and Eqs. (71) and (72) specialize to

$$\mu_{\rm m} = \phi(x_1, x_2) = -\tfrac{1}{2}r_{2\mu_{\rm m}}^{-1}bx_1, \quad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \tag{83}$$

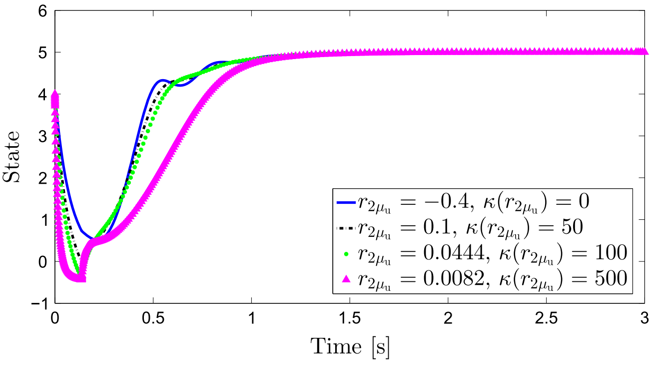

Fig. 6. Plant trajectories versus time for several values of $r_{2\mu_u}$.

$$\mu_u = \psi(x_1, x_2) = -\tfrac{1}{2} r_{2\mu_u}^{-1} \hat{g}_\mu x_1, \tag{84}$$

respectively.

Let $a=4$, $b=1$, $\theta=3$, $\hat{g}_\mu=2$, $a_{\rm ref}=-4$, $b_{\rm ref}=4$, $p=1/4$, $r_{2\mu_m}=0.1$, $r_{2\mu_u}=-1$, and

$$r(t) = \mathrm{sgn}(\sin t), \quad t \geq 0, \tag{85}$$

where $\mathrm{sgn}(\cdot)$ denotes the signum function. If Eq. (75) captures the pitch dynamics of an helicopter, then $\Phi(x)=\tanh(x)$, $x\in\mathbb{R}$, is a suitable choice of the regressor function [28, p. 270], $\frac{\partial\Phi(x)}{\partial x}=1-\tanh^2(x)$ is globally uniformly bounded, and Eq. (82) is verified. Hence, the assumptions of Theorem 5.2 are satisfied and the adaptive control law (77) guarantees satisfactory trajectory tracking in spite of the nonlinearities (83) and (84).

Fig. 5 shows the reference trajectory $x_{\rm ref}(t)$, $t\geq 0$, and the plant state vector $x(t)$ both for $\mu_m(t)=\phi(x_1(t),x_2(t))$ and $\mu_u(t)=\psi(x_1(t),x_2(t))$ and for $\mu_m(t)=\mu_u(t)=0$. In both cases, $\lim_{t\to\infty}x(t)=\lim_{t\to\infty}x_{\rm ref}(t)=5$ and hence $\lim_{t\to\infty}|x_1(t)|=\lim_{t\to\infty}|x(t)-x_{\rm ref}(t)|=0$. Lastly, Fig. 6 shows the plant state vector $x(t)$, $t\geq 0$, for several values of the design parameter $r_{2\mu_u}$ and hence, for several values of $\kappa\left(r_{2\mu_u}\right) \triangleq \frac{b^2}{r_{2\mu_m}}+\frac{\hat{g}_\mu^2}{r_{2\mu_u}}$; also in this case, $\lim_{t\to\infty}x(t)$ $=\lim_{t\to\infty}x_{\rm ref}(t)=5$ for any $r_{2\mu_u}$ such that $\kappa(r_{2\mu_u})\geq 0$. □

6. Conclusion

In this paper, we provided a systematic framework to solve the two-player zero-sum differential game problem over the infinite time horizon and guarantee partial-state asymptotic stability of the closed-loop system. Specifically, we proved sufficient conditions for the existence of pursuer's and evader's state feedback control laws that guarantee partial-state asymptotic stability of the closed-loop system and the existence of a saddle point for the system's performance measure. No collaboration between the pursuer and the evader was assumed to achieve closed-loop partial-state asymptotic stability. Indeed, the pursuer's control policy was designed to guarantee closed-loop stability with respect to a class of evader's admissible controls, some of which may lead to system instability if applied in conjunction with other pursuer's admissible controls.

The framework presented in this paper is suitable to address problems, such as the game of two cars, in which the differential game ends when part of the system state trajectory enters a *given* neighborhood of an equilibrium point within some time interval that is finite and not assigned *a priori*. A key contribution of this work is that we extended our approach to the differential game problem and provided a solution of the optimal control problems for partial-state stabilization in the presence of exogenous disturbances. Moreover, results presented in this paper can be easily extended to solve differential games over the infinite time horizon, which involve time-varying dynamical systems with nonlinear–nonquadratic time-varying performance measures, and address the $\mathcal{H}_\infty$ control theory for time-varying dynamical systems [19].

In the second part of this paper, we illustrated how our differential game framework can be applied to analyze the effectiveness of a model reference adaptive control law. Specifically, parametrizing matched nonlinearities by means of the regressor vector, model reference adaptive control is a design technique that guarantees satisfactory trajectory tracking for uncertain nonlinear dynamical systems. We modeled matched and unmatched nonlinerities in the plant state, which are *not* captured by the regressor vector, as pursuer's and evader's controls in a differential game, and provided an *explicit* parametrization of nonlinearities in the plant state that do not disrupt the trajectory tracking capabilities of the adaptive controller. Two numerical examples illustrated the applicability of the results presented.

Appendix A. Proofs of the main results

In this appendix, we present the proofs of the main results of this paper.

Proof of Theorem 3.1. It follows from Eqs. (17) and (18), and Theorem 2.1 that the closed-loop system (14) and (15) is asymptotically stable with respect to x_1 uniformly in x_{20}. Similarly, global asymptotic stability follows if $\mathcal{D}=\mathbb{R}^{n_1}$, $U=\mathbb{R}^{m_1}$, $W=\mathbb{R}^{m_2}$, and the functions $\alpha(\cdot)$ and $\beta(\cdot)$ satisfying Eq. (17) are class $\mathcal{K}_\infty$. Furthermore, it follows from Eqs. (17), (18), and (21), and Theorem 2.2 that Eq. (24) is verified.

Next, let $(x_{10},x_{20})\in\mathcal{D}_0\times\mathbb{R}^{n_2}$, $u(\cdot)$ and $w(\cdot)$ be admissible controls, and $x_1(t)$, $t\geq 0$, and $x_2(t)$ be solutions of Eqs. (12) and (13). Then, it holds that

$$0=-\dot{V}(x_1(t),x_2(t))+V'(x_1(t),x_2(t))F(x_1(t),x_2(t),u(t),w(t)),\quad t\geq 0. \tag{86}$$

Hence,

$$\begin{aligned}L(x_1(t),x_2(t),u(t),w(t))=&-\dot{V}(x_1(t),x_2(t))+L(x_1(t),x_2(t),u(t),w(t))\\&+V'(x_1(t),x_2(t))F(x_1(t),x_2(t),u(t),w(t)),\quad t\geq 0.\end{aligned} \tag{87}$$

Per definition, if $u(\cdot)\in\mathcal{S}_\psi(x_0)$, then $x_1(t)\to 0$ as $t\to\infty$. Thus it follows from Eq. (17) that

$$0=\lim_{t\to\infty}\alpha(\|x_1(t)\|)\leq\lim_{t\to\infty}V(x_1(t),x_2(t))\leq\lim_{t\to\infty}\beta(\|x_1(t)\|)=0, \tag{88}$$

for every $u(\cdot)\in\mathcal{S}_\psi(x_0)$. Consequently, Eqs. (87), (88), and (22) imply that

$$\begin{aligned}J(x_{10},x_{20},u(\cdot),\psi(x_1(\cdot),x_2(\cdot)))&=\int_0^\infty L(x_1(t),x_2(t),u(t),\psi(x_1(t),x_2(t)))\mathrm{d}t\\&=\int_0^\infty -\dot{V}(x_1(t),x_2(t))\mathrm{d}t\\&\quad+\int_0^\infty L(x_1(t),x_2(t),u(t),\psi(x_1(t),x_2(t)))\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&+\int_0^\infty V'(x_1,x_2)F(x_1(t),x_2(t),u(t),\psi(x_1(t),x_2(t)))\mathrm{d}t\\
&\geq \int_0^\infty -\dot{V}(x_1(t),x_2(t))\mathrm{d}t\\
&= -\lim_{t\to\infty} V(x_1(t),x_2(t)) + V(x_{10},x_{20})\\
&= J(x_{10},x_{20},\phi(x_1(\cdot),x_2(\cdot)),\psi(x_1(\cdot),x_2(\cdot))). \qquad (89)
\end{aligned}
$$

Similarly, since Eq. (88) is satisfied for every $w(\cdot)\in\mathcal{S}_\phi(x_0)$, it follows from Eqs. (87), (88), and (23) that

$$
\begin{aligned}
J(x_{10},x_{20},\phi(x_1(\cdot),x_2(\cdot)),w(\cdot)) &= \int_0^\infty L(x_1(t),x_2(t),\phi(x_1(t),x_2(t)),w(t))\mathrm{d}t\\
&\leq \int_0^\infty -\dot{V}(x_1(t),x_2(t))\mathrm{d}t\\
&= -\lim_{t\to\infty} V(x_1(t),x_2(t)) + V(x_{10},x_{20})\\
&= J(x_{10},x_{20},\phi(x_1(\cdot),x_2(\cdot)),\psi(x_1(\cdot),x_2(\cdot))), \qquad (90)
\end{aligned}
$$

and Eqs. (89) and (90) yield Eq. (25).

Finally, it follows from Eq. (25) that $(\phi(x_1(\cdot),x_2(\cdot)),\psi(x_1(\cdot),x_2(\cdot)))$ is a saddle point for the performance measure Eq. (16) on $\mathcal{S}_\psi(x_{10},x_{20})\times\mathcal{S}_\phi(x_{10},x_{20})$, $(x_{10},x_{20})\in\mathcal{D}_0\times\mathbb{R}^{n_2}$. Hence, Eq. (26) directly follows from Eq. (1). □

Proof of Theorem 5.2. The result is a consequence of Theorem 3.1. Consider the nonlinear dynamical system (63)–(66) and recall that Eqs. (64)–(66) are equivalent to Eq. (62) with $x_2=\left[\mathrm{vec}^{\mathrm{T}}\left(\hat{K}_x^{\mathrm{T}}(\tau)\right),\mathrm{vec}^{\mathrm{T}}\left(\hat{K}_r^{\mathrm{T}}(\tau)\right),\mathrm{vec}^{\mathrm{T}}\left(\hat{\Theta}^{\mathrm{T}}(\tau)\right),\tau\right]^{\mathrm{T}}$. In this case, Eqs. (63) and (62) are in the same form as Eqs. (12) and (13), respectively, with $u=\mu_{\mathrm{m}}$, $w=\mu_{\mathrm{u}}$,

$$F_1(x_1,x_2,u,w)=f_1(x_1,x_2)+G_{1u}(x_1,x_2)u+G_{1w}(x_1,x_2)w,$$

$f_1(x_1,x_2)$ given by Eq. (74), $G_{1u}(x_1,x_2)=B$, $G_{1w}(x_1,x_2)=G_{1\mu}(x_1)$, and $F_2(x_1,x_2,u,w)=f_2(x_1,x_2)$. Consider also the performance measure (73), which is in the same form as Eq. (16) with

$$
\begin{aligned}
L(x_1,x_2,u,w) = {} & \left[\phi(x_1,x_2)+\mu_{\mathrm{m}}\right]^{\mathrm{T}}R_{2\mu_{\mathrm{m}}}(x_1,x_2)\left[\phi(x_1,x_2)+\mu_{\mathrm{m}}\right]\\
&+\left[\psi(x_1,x_2)+\mu_{\mathrm{u}}\right]^{\mathrm{T}}R_{2\mu_{\mathrm{u}}}(x_1,x_2)\left[\psi(x_1,x_2)+\mu_{\mathrm{u}}\right]\\
&-V'(x_1,x_2)f(x_1,x_2).
\end{aligned}
$$

It follows from Theorem 5.1 and Definition 2.1 that Eqs. (63) and (62) with $\mu_{\mathrm{m}}(t)=\mu_{\mathrm{u}}(t)=0$ is globally asymptotically stable with respect to x_1 uniformly in $x_2(0)$. Since asymptotic stability implies Lyapunov stability, and Lyapunov stability implies global uniform boundedness, $\hat{K}_x(t)$, $t\geq 0$, $\hat{K}_r(t)$, and $\hat{\Theta}(t)$ are globally uniformly bounded. Thus, since $\frac{\partial\Phi(x)}{\partial x_1}$ is uniformly bounded on $\mathbb{R}^{n_1}$, it follows from Eqs. (63) and (62) that both $\frac{\partial f_1(x_1,x_2)}{\partial x_1}$ and $\frac{\partial f_2(x_1,x_2)}{\partial x_1}$ are globally uniformly bounded. Now, proceeding as in the proof of Theorem 4.3 of [15], one can prove that $V(\cdot,\cdot)$ given by Eq. (69) is a Lyapunov function for Eqs. (63) and (62) such that

$$\alpha(\|x_1\|)\leq V(x_1,x_2)\leq\beta(\|x_1\|),\quad (x_1,x_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}, \qquad (91)$$

$$V'(x_1,x_2)f(x_1,x_2) \le -\theta(\|x_1\|), \quad (x_1,x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \tag{92}$$

for some $\mathcal{K}_\infty$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, some class $\mathcal{K}$ function $\theta(\cdot)$, and $f(x_1,x_2) = \left[f_1^{\mathrm{T}}(x_1,x_2), f_2^{\mathrm{T}}(x_1,x_2)\right]^{\mathrm{T}}$.

Next, we prove that the conditions of Theorem 3.1 are verified for $R_{2\mu_{\mathrm{m}}}(x_1,x_2) \ge N_{\mathrm{m}}(x_1) > 0$, $(x_1,x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, and $R_{2\mu_{\mathrm{u}}}(x_1,x_2) \le -N_{\mathrm{u}}(x_1) < 0$; the cases wherein the matrices $R_{2\mu_{\mathrm{m}}}(x_1,x_2)$ and $R_{2\mu_{\mathrm{u}}}(x_1,x_2)$ have different signs are discussed at the end of the proof. It follows from Eq. (27) and Lemma 2.1 that Eq. (36) is satisfied, and hence Eqs. (71) and (72) follow from Eq. (36) by setting

$$\begin{aligned} &\frac{\partial}{\partial[u^{\mathrm{T}} w^{\mathrm{T}}]^{\mathrm{T}}}[L(x_1,x_2,u,w) + V'(x_1,x_2)f(x_1,x_2) \\ &\qquad + V'(x_1,x_2)G_u(x_1,x_2)u + V'(x_1,x_2)G_w(x_1,x_2)w] = 0. \end{aligned} \tag{93}$$

Eq. (17) is immediately satisfied by Eq. (91), and since $V(\cdot,\cdot)$ is continuously differentiable, it follows from Eq. (91) that $V(0,x_2)$ is a local minimum of $V(\cdot,\cdot)$. Thus, $V'(0,x_2) = 0$, $x_2 \in \mathbb{R}^{n_2}$, and it follows from Eqs. (71) and (72) that Eqs. (19) and (20) are satisfied. Eq. (21) is immediately verified and it follows from Eq. (70) that

$$\begin{aligned} -\theta(\|x_1\|) &\ge V'(x_1,x_2)f(x_1,x_2) \\ &\ge V'(x_1,x_2)f(x_1,x_2) - \tfrac{1}{2}V'(x_1,x_2)\Big[G_{\mu_{\mathrm{m}}}(x_1)R_{2\mu_{\mathrm{m}}}^{-1}(x_1,x_2)G_{\mu_{\mathrm{m}}}^{\mathrm{T}}(x_1) \\ &\quad + G_{\mu_{\mathrm{u}}}(x_1)R_{2\mu_{\mathrm{u}}}^{-1}(x_1,x_2)G_{\mu_{\mathrm{u}}}^{\mathrm{T}}(x_1)\Big] V'^{\mathrm{T}}(x_1,x_2), \end{aligned} \tag{94}$$

for all $(x_1,x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, which implies Eq. (18). Lastly, since Eqs. (43) and (44) are satisfied, Eqs. (22) and (23) yield.

The conditions of Theorem 3.1 are verified and hence the nonlinear dynamical system (63) and (62) with $\mu_{\mathrm{m}} = \phi(x_1,x_2)$ and $\mu_{\mathrm{u}} = \psi(x_1,x_2)$ is globally asymptotically stable with respect to x_1 uniformly in x_{20}. The result now ensues, since asymptotic stability of Eqs. (63) and (62) with respect to x_1 implies that $x_1(t) \to 0$ as $t \to \infty$.

If $R_{2\mu_{\mathrm{m}}}(x_1,x_2) \le -N_{\mathrm{m}}(x_1) < 0$, $(x_1,x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, and $R_{2\mu_{\mathrm{u}}}(x_1,x_2) \ge N_{\mathrm{u}}(x_1) > 0$, then the proof follows in a similar manner by considering $\mu_{\mathrm{m}}(\cdot)$ as the evader and $\mu_{\mathrm{u}}(\cdot)$ as the pursuer. Lastly, if $R_{2\mu_{\mathrm{m}}}(x_1,x_2) \ge N_{\mathrm{m}}(x_1) > 0$, $(x_1,x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, and $R_{2\mu_{\mathrm{u}}}(x_1,x_2) \ge N_{\mathrm{u}}(x_1) > 0$, then the result follows from Theorem 5.1 of [26]. □

References

[1] R.A. Adiatulina, A.M. Taras'yev, A differential game of unlimited duration, J. Appl. Math. Mech. 51 (4) (1987) 415–420.

[2] T. Alpcan, T. Basar, A globally stable adaptive congestion control scheme for internet-style networks with delay, IEEE/ACM Trans. Network. 13 (6) (2005) 1261–1274.

[3] N. Avessta, T. Aboulnasr, Combined regressor methods and adaptive IIR filtering, IEEE Trans. Circuits Syst. I: Reg. Pap. 51 (11) (2004) 2222–2234.

[4] J.A. Ball, J.W. Helton, M.L. Walker, H^∞ control for nonlinear systems with output feedback, IEEE Trans. Autom. Control 38 (4) (1993) 546–559.

[5] J. Ball, J. Helton, H^∞ control for nonlinear plants: connections with differential games, in: IEEE Conference on Decision and Control, vol. 2, 1989, pp. 956–962.

[6] T. Basar, P. Bernhard, H^∞ Optimal Control and Related Minimax Design Problems, Cambridge University Press, Cambridge, UK, 2000.

[7] T. Basar, G. Olsder, Dynamic Noncooperative Game Theory, 2nd ed., Society for Industrial and Applied Mathematics, New York, NY, 1998.
[8] D.S. Bernstein, Nonquadratic cost and nonlinear feedback control, Int. J. Robust Nonlinear Control 3 (3) (1993) 211–229.
[9] C.C. Cheah, C. Liu, J.J.E. Slotine, Adaptive tracking control for robots with unknown kinematic and dynamic properties, Int. J. Robot. Res. 25 (3) (2006) 283–296.
[10] V. Chellaboina, W.M. Haddad, A unification between partial stability and stability theory for time-varying systems, IEEE Control Syst. 22 (6) (2002) 66–75.
[11] G. Chowdhary, H.A. Kingravi, J.P. How, P.A. Vela, Bayesian nonparametric adaptive control using Gaussian processes, IEEE Trans. Neural Netw. Learn. Syst. 26 (3) (2015) 537–550.
[12] E. Dockner, Differential Games in Economics and Management Science, Cambridge University Press, Cambridge, UK, 2000.
[13] J.C. Doyle, K. Glover, P.P. Khargonekar, B.A. Francis, State-space solutions to standard H_2 and H_∞ control problems, IEEE Trans. Autom. Control 34 (8) (1989) 831–847.
[14] N. Ekneligoda, W. Weaver, Game-theoretic cold-start transient optimization in DC microgrids, IEEE Trans. Ind. Electron. 61 (12) (2014) 6681–6690.
[15] W.M. Haddad, V. Chellaboina, Nonlinear Dynamical Systems and Control: A Lyapunov-Based Approach, Princeton University Press, Princeton, NJ, 2008.
[16] W.M. Haddad, V. Chellaboina, Optimal nonlinear-nonquadratic feedback control for systems with $\mathcal{L}_2$ and $\mathcal{L}_\infty$ disturbances, Nonlinear Anal.: Theory Meth. Appl. 34 (2) (1998) 229–255.
[17] W.M. Haddad, A. L'Afflitto, Finite-time partial stability and stabilization, and optimal feedback control, J. Frankl. Inst. 352 (6) (2015) 2329–2357.
[18] A. Huang, M. Chien, Adaptive Control of Robot Manipulators: A Unified Regressor-free Approach, World Scientific Publishing, New York, NY, 2010.
[19] A. Ichikawa, H. Katayama, Linear Time Varying Systems and Sampled-Data Systems, Springer, London, UK, 2001.
[20] P. Ioannou, B. Fidan, Adaptive Control Tutorial, Society for Industrial and Applied Mathematics, Philadelphia, PA, 2006.
[21] R. Isaacs, Differential Games: A Mathematical Theory with Applications to Warfare and Pursuit, Control and Optimization, Dover, New York, NY, 1999.
[22] D. Jacobson, On values and strategies for infinite-time linear quadratic games, IEEE Trans. Autom. Control 22 (3) (1977) 490–491.
[23] M.R. James, Asymptotic analysis of nonlinear stochastic risk-sensitive control and differential games, Math. Control Signals Syst. 5 (4) (1992) 401–417.
[24] J.B. Kenney, C.E. Rohrs, The composite regressor algorithm for IIR adaptive systems, IEEE Trans. Signal Process. 41 (2) (1993) 617–628.
[25] E. Kreyszig, Introductory Functional Analysis With Applications, Wiley, Toronto, ON, 1978.
[26] A. L′Afflitto, W.M. Haddad, E. Bakolas, Partial-state stabilization and optimal feedback control, Int. J. Robust Nonlinear Control 26 (5) (2016) 1026–1050.
[27] A. L'Afflitto, W.M. Haddad, Q. Hui, Optimal control for linear and nonlinear semistabilization, J. Frankl. Inst. 352 (3) (2015) 851–881.
[28] E. Lavretsky, K. Wise, Robust and Adaptive Control: With Aerospace Applications, Springer, London, UK, 2012.
[29] D.J.N. Limebeer, B.D.O. Anderson, P.P. Khargonekar, M. Green, A game theoretic approach to $\mathcal{H}^\infty$ control for time-varying systems, SIAM J. Control Optim. 30 (2) (1992) 262–283.
[30] K.-Y. Lum, D. Bernstein, V. Coppola, Global stabilization of the spinning top with mass imbalance, Dyn. Stab. Syst. 10 (4) (1995) 339–365.
[31] E. Mageirou, Values and strategies for infinite time linear quadratic games, IEEE Trans. Autom. Control 21 (4) (1976) 547–550.
[32] W.M. McEneaney, A uniqueness result for the Isaacs equation corresponding to nonlinear H_∞ control, Math. Control Signals Syst. 11 (4) (1998) 303–334.
[33] T. Shima, O. Golan, Bounded differential games guidance law for dual-controlled missiles, IEEE Trans. Control Syst. Technol. 14 (4) (2006) 719–724.
[34] P. Soravia, The concept of value in differential games of survival and viscosity solutions of Hamilton–Jacobi equations, Differ. Integral Equ. 5 (5) (1992) 1049–1068.
[35] Y. Taira, M. Oya, S. Sagara, Adaptive control of underwater vehicle-manipulator systems using radial basis function networks, Artif. Life Robot. 17 (1) (2012) 123–129.
[36] V.I. Vorotnikov, Partial Stability and Control, Birkhäuser, Boston, MA, 1998.

[37] K. Weekly, A. Tinka, L. Anderson, A. Bayen, Autonomous river navigation using the Hamilton–Jacobi framework for underactuated vehicles, IEEE Trans. Robot. 30 (5) (2014) 1250–1255.
[38] B. Wie, Space Vehicle Dynamics and Control, American Institute of Aeronautics and Astronautics, Reston, VA, 1998.
[39] H. Yazarel, C.C. Cheah, Task-space adaptive control of robotic manipulators with uncertainties in gravity regressor matrix and kinematics, IEEE Trans. Autom. Control 47 (9) (2002) 1580–1585.
[40] J. Yuh, J. Nie, Application of non-regressor-based adaptive control to underwater robots: experiment, Comput. Electr. Eng. 26 (2) (2000) 169–179.
[41] V. Zhukovskiy, Lyapunov Functions in Differential Games, Taylor & Francis, New York, NY, 2003.