Finite-Time Stabilization and Optimal Feedback Control

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***Abstract*—Finite-time stability involves dynamical systems whose trajectories converge to an equilibrium state in finite time. Since finite-time convergence implies nonuniqueness of system solutions in reverse time, such systems possess non-Lipschitzian dynamics. Sufficient conditions for finite-time stability have been developed in the literature using continuous Lyapunov functions. In this technical note, we develop a framework for addressing the problem of optimal nonlinear analysis and feedback control for finite-time stability and finite-time stabilization. Finite-time stability of the closed-loop nonlinear system is guaranteed by means of a Lyapunov function that satisfies a differential inequality involving fractional powers. This Lyapunov function can clearly be seen to be the solution to a partial differential equation that corresponds to a steady-state form of the Hamilton-Jacobi-Bellman equation, and hence, guaranteeing both finite-time stability and optimality.**

***Index Terms*—Differential inequalities, finite-time stability, finite-time stabilization, Hamilton-Jacobi-Bellman theory, optimal control.**

I. Introduction

The notions of asymptotic and exponential stability in dynamical systems theory imply convergence of the system trajectories to an equilibrium state over the infinite horizon. In many applications, however, it is desirable that a dynamical system possesses the property that trajectories that converge to a Lyapunov stable equilibrium state must do so in finite time rather than merely asymptotically. Most of the existing control techniques in the literature ensure that the closed-loop system dynamics of a controlled system are Lipschitz continuous, which implies uniqueness of system solutions in forward and backward times. Hence, convergence to an equilibrium state is achieved over an infinite time interval.

In order to achieve convergence in finite time, the closed-loop system dynamics need to be non-Lipschitzian giving rise to nonuniqueness of solutions in backward time. Uniqueness of solutions in forward time, however, can be preserved in the case of finite-time convergence. Sufficient conditions that ensure uniqueness of solutions in forward time in the absence of Lipschitz continuity are given in [1]–[4]. In addition, it is shown in [5, Theorem 4.3, p. 59] that uniqueness of solutions in forward time along with continuity of the system dynamics ensure that the system solutions are continuous functions of the system initial conditions even when the dynamics are not Lipschitz continuous.

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Finite-time convergence to a Lyapunov stable equilibrium, that is, finite-time stability, was first addressed by Roxin [6] and rigorously studied in [7], [8] for time-invariant systems using continuous Lyapunov functions. Extensions of finite-time stability to time-varying nonlinear dynamical systems are presented in [9], [10]. Finite-time stabilization of second-order systems was considered in [11], [12]. More recently, researchers have considered finite-time stabilization of higher-order systems [13] as well as finite-time stabilization using output feedback [14]. Design of globally strongly stabilizing continuous controllers for linear and nonlinear systems using the theory of homogeneous systems was studied in [8], [15]. In addition, the universal controller given by Sontag [16] is extended in [17] to design a feedback controller for finite-time stabilization. Alternatively, discontinuous finite-time stabilizing feedback controllers have also been developed in the literature [18]–[20]. However, for practical implementations, discontinuous feedback controllers can lead to chattering due to system uncertainty or measurement noise, and hence, may excite unmodeled high-frequency system dynamics. In addition, sampled-data implementations of continuous finite-time controllers can also display a chattering behavior due to infinite local gains at the equilibrium [21].

In [22], the current status of continuous-time, nonlinear nonquadratic optimal control problems was presented in a simplified and tutorial manner. The basic underlying ideas of the results in [22] are based on the fact that the steady-state solution of the Hamilton-Jacobi-Bellman equation is a Lyapunov function for the nonlinear system and thus guaranteeing both *asymptotic* stability and optimality [22], [23]. Specifically, a feedback control problem over an infinite horizon involving a nonlinear-nonquadratic performance functional is considered. The performance functional is then evaluated in closed form as long as the nonlinear nonquadratic cost functional considered is related in a specific way to an underlying Lyapunov function that guarantees asymptotic stability of the nonlinear closed-loop system. This Lyapunov function is shown to be the solution of the steady-state Hamilton-Jacobi-Bellman equation. The overall framework provides the foundation for extending linear-quadratic control to nonlinear-nonquadratic problems.

Currently, optimal finite-time controllers are only obtainable using the maximum principle which generally does not yield feedback controllers. In this technical note, we extend the framework developed in [22] and [23] to address the problem of *optimal finite-time stabilization*, that is, the problem of finding state-feedback control laws that minimize a given performance measure and guarantee finite-time stability of the closed-loop system. Specifically, an optimal finite-time control problem is stated and sufficient Hamilton-Jacobi-Bellman conditions are used to characterize an optimal feedback controller. The steady-state solution of the Hamilton-Jacobi-Bellman equation is clearly shown to be a Lyapunov function for the closed-loop system that additionally satisfies a differential inequality involving a fractional power, and hence, guaranteeing both finite-time stability and optimality. Finally, we explore connections of our approach with inverse optimal control [24]–[28], wherein we parametrize a family of finite-time stabilizing sublinear controllers that minimize a derived cost functional involving subquadratic terms. Subquadratic performance criteria have been studied in [11], [29], [30] and have been shown to permit a unified treatment of a broad range of design goals.

II. Notation, Definitions, and Mathematical Preliminaries

In this section, we establish notation, definitions, and review some basic results on finite-time stability [7], [23]. Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}_+$ denote the set of positive real numbers, $\overline{\mathbb{R}}_+$ denote the set of nonnegative numbers, $\mathbb{R}^n$ denote the set of $n \times 1$ real column vectors, $\mathbb{R}^{n\times m}$ denote the set of $n \times m$ real matrices, and $\mathcal{B}_\varepsilon(x)$ denote the *open ball centered* at x with *radius* ε. We write $V'(x) \triangleq \partial V(x)/\partial x$ for the Fréchet derivative of V at x, $\|\cdot\|$ for the Euclidean vector norm, A^{T} for the transpose of the matrix A, and I_n or I for the $n \times n$ identity matrix.

Consider the nonlinear dynamical system given by

$$\dot{x}(t) = f(x(t)), \qquad x(0) = x_0, \qquad t \in \mathcal{I}_{x_0} \tag{1}$$

where, for every $t \in \mathcal{I}_{x_0}$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $\mathcal{I}_{x_0} \subseteq \overline{\mathbb{R}}_+$ is the maximal interval of existence of a solution $x(t)$ of (1), $0 \in \mathcal{I}_{x_0}$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $f(0) = 0$, and $f(\cdot)$ is continuous on $\mathcal{D}$. A continuously differentiable function $x : \mathcal{I}_{x_0} \to \mathcal{D}$ is said to be the *solution* of (1) on the interval $\mathcal{I}_{x_0} \subset \mathbb{R}$ if $x(\cdot)$ satisfies (1) for all $t \in \mathcal{I}_{x_0}$. The continuity of $f(\cdot)$ implies that, for every $x \in \mathcal{D}$, there exists $\tau_0 < 0 < \tau_1$ and a solution $x(\cdot)$ of (1) defined on the open interval (τ_0, τ_1) such that $x(0) = x$ [23, Th. 2.24]. A solution $t \mapsto x(t)$ is said to be *right maximally* defined if x cannot be extended (either uniquely or nonuniquely) forward in time. We assume that all right maximal solutions to (1) exist on $[0, \infty)$, and hence, we assume that (1) is *forward complete*. Recall that every bounded solution to (1) can be extended on a semi-infinite interval $[0, \infty)$ [23]. That is, if $x : [0, \tau) \to \mathcal{D}$ is the right maximally defined solution of (1) such that $x(t) \in \mathcal{D}_{\mathrm{c}}$ for all $t \in [0, \tau)$, where $\mathcal{D}_{\mathrm{c}} \subset \mathcal{D}$ is compact, then $\tau = \infty$ [23, Cor. 2.5].

We assume that (1) possesses unique solutions in forward time for all initial conditions except possibly the origin in the following sense. For every $x \in \mathcal{D} \setminus \{0\}$ there exists $\tau_x > 0$ such that, if $y_1 : [0, \tau_1) \to \mathcal{D}$ and $y_2 : [0, \tau_2) \to \mathcal{D}$ are two solutions of (1) with $y_1(0) = y_2(0) = x$, then $\tau_x \le \min\{\tau_1, \tau_2\}$ and $y_1(t) = y_2(t)$ for all $t \in [0, \tau_x)$. Without loss of generality, we assume that for each x, τ_x is chosen to be the largest such number in $\overline{\mathbb{R}}_+$. In this case, given $x \in \mathcal{D}$, we denote by the continuously differentiable map $s^x(\cdot) \triangleq s(\cdot, x)$ the *trajectory* or the unique *solution curve* of (1) on $[0, \tau_x)$ satisfying $s(0, x) = x$. Sufficient conditions for forward uniqueness in the absence of Lipschitz continuity can be found in [1], [2, Section 10], [3], [4 Section 1].

The following definition introduces the notion of finite-time stability.

Definition 2.1 ([7]): Consider the nonlinear dynamical system (1). The zero solution $x(t) \equiv 0$ to (1) is *finite-time stable* if there exist an open neighborhood $\mathcal{N} \subseteq \mathcal{D}$ of the origin and a function $T : \mathcal{N} \setminus \{0\} \to (0, \infty)$, called the *settling-time function*, such that the following statements hold:

i) *Finite-time convergence*. For every $x \in \mathcal{N} \setminus \{0\}$, $s^x(t)$ is defined on $[0, T(x))$, $s^x(t) \in \mathcal{N} \setminus \{0\}$ for all $t \in [0, T(x))$, and $\lim_{t \to T(x)} s^x(t) = 0$.

ii) *Lyapunov stability*. For every $\varepsilon > 0$ there exists $\delta > 0$ such that $\mathcal{B}_\delta(0) \subset \mathcal{N}$ and for every $x \in \mathcal{B}_\delta(0) \setminus \{0\}$, $s^x(t) \in \mathcal{B}_\varepsilon(0)$ for all $t \in [0, T(x))$.

The zero solution $x(t) \equiv 0$ of (1) is *globally finite-time stable* if it is finite-time stable with $\mathcal{N} = \mathcal{D} = \mathbb{R}^n$.

Note that if the zero solution $x(t) \equiv 0$ to (1) is finite-time stable, then it is asymptotically stable, and hence, finite-time stability is a stronger condition than asymptotic stability. The following result shows that if the zero solution $x(t) \equiv 0$ to (1) is finite-time stable, then (1) has a unique solution $s(\cdot, \cdot)$ defined on $\overline{\mathbb{R}}_+ \times \mathcal{N}$ for every initial condition in an open neighborhood of the origin, including the origin, and $s(t, x) = 0$ for all $t \ge T(x)$, $x \in \mathcal{N}$, where $T(0) \triangleq 0$.

Proposition 2.1 ([7]): Consider the nonlinear dynamical system (1). Assume that the zero solution $x(t) \equiv 0$ to (1) is finite-time stable and let $\mathcal{N} \subseteq \mathcal{D}$ and $T : \mathcal{N} \setminus \{0\} \to (0, \infty)$ be as in Definition 2.1. Then, $s(\cdot, \cdot)$ is a unique solution of (1) and is defined on $\overline{\mathbb{R}}_+ \times \mathcal{N}$, and $s(t, x) = 0$ for all $t \ge T(x)$, $x \in \mathcal{N}$, where $T(0) \triangleq 0$.

It follows from Proposition 2.1 that if the zero solution $x(t) \equiv 0$ to (1) is finite-time stable, then the solutions of (1) define a continuous *global semiflow* on $\mathcal{N}$; that is, $s : \overline{\mathbb{R}}_+ \times \mathcal{N} \to \mathcal{N}$ is jointly continuous and satisfies the consistency property $s(0, x) = x$ and the semigroup property $s(t, s(\tau, x)) = s(t + \tau, x)$ for every $x \in \mathcal{N}$ and $t, \tau \in \overline{\mathbb{R}}_+$. Furthermore, $s(\cdot, \cdot)$ satisfies $s(T(x) + t, x) = 0$ for all $x \in \mathcal{N}$ and $t \in \overline{\mathbb{R}}_+$. Finally, it also follows from Proposition 2.1 that we can extend $T(\cdot)$ to all of $\mathcal{N}$ by defining $T(0) \triangleq 0$. Now, by uniqueness of solutions it follows that $s(T(x) + t, x) = 0$, $t \in \mathbb{R}_+$, and hence, it is easy to see from Definition 2.1 that

$$T(x) = \inf\left\{t \in \overline{\mathbb{R}}_+ : s(t, x) = 0\right\}, \qquad x \in \mathcal{N}. \tag{2}$$

The next proposition shows that the settling time function of a finite-time stable system is continuous on $\mathcal{N}$ if and only if it is continuous at the origin.

Proposition 2.2 ([7]): Consider the nonlinear dynamical system (1). Assume that the zero solution $x(t) \equiv 0$ to (1) is finite-time stable, let $\mathcal{N} \subseteq \mathcal{D}$ be as in Definition 2.1, and let $T : \mathcal{N} \to \mathbb{R}_+$ be the settling-time function. Then $T(\cdot)$ is continuous on $\mathcal{N}$ if and only if $T(\cdot)$ is continuous at $x = 0$.

Next, we provide sufficient conditions for finite-time stability of the nonlinear dynamical system given by (1). For the statement of the following result define $\dot{V}(x) \triangleq V'(x)f(x)$ for a continuously differentiable function $V : \mathcal{D} \to \mathbb{R}$.

Theorem 2.1 ([7], [23, Th. 4.17]): Consider the nonlinear dynamical system (1). Assume there exist a continuously differentiable function $V : \mathcal{D} \to \mathbb{R}$, real numbers $c > 0$ and $\alpha \in (0, 1)$, and a neighborhood $\mathcal{M} \subseteq \mathcal{D}$ of the origin such that

$$V(0) = 0 \tag{3}$$

$$V(x) > 0, \qquad x \in \mathcal{M} \setminus \{0\} \tag{4}$$

$$\dot{V}(x) \le -c\,(V(x))^\alpha, \qquad x \in \mathcal{M} \setminus \{0\}. \tag{5}$$

Then the zero solution $x(t) \equiv 0$, $t \ge 0$, to (1) is finite-time stable. Moreover, there exists an open neighborhood $\mathcal{N} \subset \mathcal{M}$ of the origin and a settling-time function $T : \mathcal{N} \to [0, \infty)$ such that

$$T(x_0) \le \frac{1}{c(1-\alpha)}\,(V(x_0))^{1-\alpha}, \qquad x_0 \in \mathcal{N} \tag{6}$$

and $T(\cdot)$ is continuous on $\mathcal{N}$. If, in addition, $\mathcal{D} = \mathbb{R}^n$, $V(\cdot)$ is radially unbounded, and (5) holds on $\mathbb{R}^n \setminus \{0\}$, then the zero solution $x(t) \equiv 0$ to (1) is globally finite-time stable.

Note that if the conditions of Theorem 2.1 are satisfied, then it follows from Proposition 2.1 that the solution $x(t)$ of (1) is defined for all $t \ge 0$, that is, $\mathcal{I}_{x_0} = [0, \infty)$, and is unique. Furthermore, since the regularity properties of the Lyapunov function and those of the settling-time function are related, and there exist finite-time stable systems that do not admit a continuously differentiable or even a Hölder continuous settling time function, a converse theorem to Theorem 2.1 can only ensure the existence of a continuous Lyapunov function. For details, see [7]. Alternatively, the authors in [17] provide conditions on the system dynamics for the settling-time function to be continuous leading to a stronger converse Lyapunov theorem involving a more regular function $V(\cdot)$ satisfying (5).

III. Optimal Finite-Time Stabilization

In this section, we obtain a characterization of optimal feedback controllers that guarantee closed-loop, finite-time stabilization.

Specifically, sufficient conditions for optimality are given in a form that corresponds to a steady-state version of the Hamilton-Jacobi-Bellman equation. To address the problem of characterizing finite-time stabilizing feedback controllers, consider the controlled nonlinear dynamical system

$$\dot{x}(t) = F(x(t), u(t)), \qquad x(0) = x_0, \qquad t \geq 0 \tag{7}$$

where, for every $t \geq 0$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $u(t) \in U \subseteq \mathbb{R}^m$ with $0 \in U$, $F : \mathcal{D} \times U \to \mathbb{R}^n$ is jountly continuous in x and u, and $F(0,0) = 0$. The control $u(\cdot)$ in (7) is restricted to the class of *admissible* controls consisting of measurable functions $u(\cdot)$ such that $u(t) \in U$, $t \geq 0$.

A continuous function $\phi : \mathcal{D} \to U$ satisfying $\phi(0) = 0$ is called a *control law*. If $u(t) = \phi(x(t))$, $t \geq 0$, where $\phi(\cdot)$ is a control law and $x(t)$ satisfies (7), then we call $u(\cdot)$ a *feedback control law*. Note that the feedback control law is an admissible control since $\phi(\cdot)$ has values in U. Given a control law $\phi(\cdot)$ and a feedback control law $u(t) = \phi(x(t))$, $t \geq 0$, the *closed-loop system* (7) is given by

$$\dot{x}(t) = F(x(t), \phi(x(t))), \quad x(0) = x_0, \qquad t \geq 0. \tag{8}$$

We now consider the problem of finite-time stabilization.

Definition 3.1: Consider the controlled dynamical system given by (7). The feedback control law $u = \phi(x)$ is *finite-time stabilizing* if the closed-loop system (8) is finite-time stable. Furthermore, the feedback control law $u = \phi(x)$ is *globally finite-time stabilizing* if the closed-loop system (8) is globally finite-time stable.

Next, we present a main theorem for finite-time stabilization characterizing feedback controllers that guarantee finite-time closed-loop stability and minimize a nonlinear-nonquadratic performance functional. For the statement of this result, let $L : \mathcal{D} \times U \to \mathbb{R}$ be jointly continuous in x and u, and define the set of finite-time regulation controllers given by

$$\mathcal{S}(x_0) \triangleq \{u(\cdot) : \ u(\cdot) \text{ is admissible and } x(\cdot) \text{ given by (7) satisfies } x(t) \to 0 \text{ as } t \to T\}$$

where $T > 0$. Note that since finite-time convergence is a stronger condition than asymptotic convergence, $\mathcal{S}(x_0)$ includes the set of all null convergent controllers.

Theorem 3.1: Consider the controlled nonlinear dynamical system (7) with

$$J(x_0, u(\cdot)) \triangleq \int_0^\infty L(x(t), u(t))\,\mathrm{d}t \tag{9}$$

where $u(\cdot)$ is an admissible control. Assume that there exist a continuously differentiable function $V : \mathcal{D} \to \mathbb{R}$, real numbers $c > 0$ and $\alpha \in (0,1)$, a neighborhood $\mathcal{M} \subseteq \mathcal{D}$ of the origin, and a continuous control law $\phi : \mathcal{D} \to U$ such that

$$\phi(0) = 0 \tag{10}$$
$$V(0) = 0 \tag{11}$$
$$V(x) > 0, \quad x \in \mathcal{M} \setminus \{0\} \tag{12}$$
$$V'(x) F(x, \phi(x)) \leq -c\,(V(x))^\alpha, \qquad x \in \mathcal{M} \setminus \{0\} \tag{13}$$
$$L(x, \phi(x)) + V'(x) F(x, \phi(x)) = 0, \qquad x \in \mathcal{D} \tag{14}$$
$$L(x,u) + V'(x)F(x,u) \geq 0, \qquad (x,u) \in \mathcal{D} \times U. \tag{15}$$

Then, with the feedback control $u = \phi(x)$, the zero solution $x(t) \equiv 0$, $t \geq 0$, to (7) is finite-time stable. Moreover, there exist an open neighborhood $\mathcal{D}_0 \subset \mathcal{M}$ of the origin and a settling-time function $T : \mathcal{D}_0 \to [0, \infty)$ such that

$$T(x_0) \leq \frac{1}{c(1-\alpha)} (V(x_0))^{1-\alpha}, \qquad x_0 \in \mathcal{D}_0 \tag{16}$$
$$J(x_0, \phi(x(\cdot))) = V(x_0), \qquad x_0 \in \mathcal{D}_0. \tag{17}$$

In addition, if $x_0 \in \mathcal{D}_0$, then the feedback control $u(\cdot) = \phi(x(\cdot))$ minimizes $J(x_0, u(\cdot))$ in the sense that

$$J(x_0, \phi(\cdot)) = \min_{u(\cdot) \in \mathcal{S}(x_0)} J(x_0, u(\cdot)). \tag{18}$$

Finally, if $\mathcal{D} = \mathbb{R}^n$, $U = \mathbb{R}^m$, $V(\cdot)$ is radially unbounded, and (13) holds on $\mathbb{R}^n \setminus \{0\}$, then the closed-loop system (8) is globally finite-time stable.

Proof: Local and global finite-time stability along with the existence of a settling-time function $T : \mathcal{D}_0 \to [0, \infty)$ such that (16) holds are a direct consequence of (11)–(13) by applying Theorem 2.1 to the closed-loop system given by (8).

Next, let $u(\cdot) = \phi(x(\cdot))$ and $x(t)$, $t \geq 0$, satisfy (8). Then, since

$$0 = -\dot{V}(x(t)) + V'(x(t)) F(x(t), \phi(x(t))), \quad t \geq 0 \tag{19}$$

it follows from (14) that

$$\begin{aligned} L(x(t), \phi(x(t))) &= -\dot{V}(x(t)) + L(x(t), \phi(x(t))) \\ &\quad + V'(x(t)) F(x(t), \phi(x(t))) \\ &= -\dot{V}(x(t)), \qquad t \geq 0. \end{aligned} \tag{20}$$

Now, integrating (20) over $[0,t]$ yields

$$\int_0^t L(x(s), \phi(x(s)))\,\mathrm{d}s = -V(x(t)) + V(x_0), \quad t \geq 0. \tag{21}$$

Using (11) and letting $t \to \infty$ it follows from (21) that

$$\int_0^\infty L(x(s), \phi(x(s)))\,\mathrm{d}s = -V\left(\lim_{t\to\infty} x(t)\right) + V(x_0) \tag{22}$$

and hence, (17) is a direct consequence of (22) using the fact that $\lim_{t \to T(x_0)} x(t) = \lim_{t\to\infty} x(t) = 0$.

Next, let $x_0 \in \mathcal{D}_0$, let $u(\cdot) \in \mathcal{S}(x_0)$, and let $x(t)$, $t \geq 0$, be the solution of (7). Then, it follows that

$$0 = -\dot{V}(x(t)) + V'(x(t)) F(x(t), u(t)), \quad t \geq 0. \tag{23}$$

Hence

$$\begin{aligned} L(x(t), u(t)) = -\dot{V}(x(t)) &+ L(x(t), u(t)) \\ &+ V'(x(t)) F(x(t), u(t)), \qquad t \geq 0. \end{aligned} \tag{24}$$

Thus, it follows from (24), (15), (17), (11), and the fact that $u(\cdot) \in \mathcal{S}(x_0)$, that

$$\begin{aligned} \int_0^\infty L(x(t), u(t))\,\mathrm{d}t &= \int_0^\infty -\dot{V}(x(t))\,\mathrm{d}t + \int_0^\infty L(x(t), u(t))\,\mathrm{d}t \\ &\quad + \int_0^\infty V'(x) F(x(t), u(t))\,\mathrm{d}t \\ &\geq \int_0^\infty -\dot{V}(x(t))\,\mathrm{d}t \\ &= -\lim_{t\to\infty} V(x(t)) + V(x_0) \\ &= -V\left(\lim_{t\to\infty} x(t)\right) + V(x_0) \\ &= -V\left(\lim_{t\to T} x(t)\right) + V(x_0) \\ &= J(x_0, \phi(x(\cdot))) \end{aligned} \tag{25}$$

which yields (18). ■

Note that (14) is the steady-state, Hamilton-Jacobi-Bellman equation for the controlled nonlinear dynamical system (7) with performance criterion (9). Furthermore, conditions (14) and (15) guarantee optimality with respect to the set of admissible finite-time stabilizing controllers $\mathcal{S}(x_0)$. However, it is important to note that an explicit characterization of $\mathcal{S}(x_0)$ is not required. In addition, the optimal finite-time stabilizing *feedback* control law $u = \phi(x)$ is independent of the initial condition x_0 and is given by

$$\phi(x) = \arg\min_{u\in\mathcal{S}(x_0)} \left[L(x,u) + \frac{\partial V(x)}{\partial x}F(x,u)\right]. \tag{26}$$

Finally, setting $\mathcal{M} = \mathcal{D}$ in Theorem 3.1 and replacing (13) with

$$V'(x)F(x,\phi(x)) < 0, \qquad x \in \mathcal{D} \tag{27}$$

Theorem 3.1 reduces to Theorem 8.2 of [23] characterizing the classical asymptotically stabilizing optimal control problem for time-invariant systems on an infinite interval.

IV. Finite-Time Stabilization for Affine Dynamical Systems and Connections to Inverse Optimal Control

In this section, we specialize the results of Section III to nonlinear affine dynamical systems of the form

$$\dot{x}(t) = f(x) + G(x)u(t), \quad x(0) = x_0, \qquad t \geq 0 \tag{28}$$

where, for every $t \geq 0$, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $f : \mathbb{R}^n \to \mathbb{R}^n$ and $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are such that $f(\cdot)$ and $G(\cdot)$ are continuous in x and $f(0) = 0$. Furthermore, we consider performance integrands $L(x,u)$ of the form

$$L(x,u) = L_1(x) + L_2(x)u + u^{\mathrm{T}}R_2(x)u, \ (x,u) \in \mathbb{R}^n \times \mathbb{R}^m \tag{29}$$

where $L_1 : \mathbb{R}^n \to \mathbb{R}$, $L_2 : \mathbb{R}^n \to \mathbb{R}^{1\times m}$ is continuous on $\mathbb{R}^n$, and $R_2(x) > 0$, $x \in \mathbb{R}^n$, is continuous on $\mathbb{R}^n$, so that (9) becomes

$$J(x_0, u(\cdot)) = \int_0^\infty \left[L_1(x(t)) + L_2(x(t))u(t) + u^{\mathrm{T}}(t)R_2(x)u(t)\right] \mathrm{d}t. \tag{30}$$

Theorem 4.1: Consider the controlled nonlinear affine dynamical system (28) with performance measure (30). Assume that there exist a continuously differentiable, radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}$ and real numbers $c > 0$ and $\alpha \in (0,1)$ such that

$$V(0) = 0 \tag{31}$$
$$V(x) > 0, \qquad x \in \mathbb{R}^n \setminus \{0\} \tag{32}$$
$$V'(x)\left[f(x) - \frac{1}{2}G(x)R_2^{-1}(x)L_2^{\mathrm{T}}(x) - \frac{1}{2}G(x)R_2^{-1}(x)G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x)\right] \leq -c\,(V(x))^{\alpha}, \quad x \in \mathbb{R}^n \tag{33}$$
$$L_2(0) = 0 \tag{34}$$
$$0 = L_1(x) + V'(x)f(x) - \frac{1}{4}\left[V'(x)G(x) + L_2(x)\right] \cdot R_2^{-1}(x)\left[V'(x)G(x) + L_2(x)\right]^{\mathrm{T}}, \quad x \in \mathbb{R}^n. \tag{35}$$

Then, with the feedback control

$$u = \phi(x) = -\frac{1}{2}R_2^{-1}(x)\left[L_2(x) + V'(x)G(x)\right]^{\mathrm{T}} \tag{36}$$

the zero solution $x(t) \equiv 0$, $t \geq 0$, to

$$\dot{x}(t) = f(x) + G(x(t))\,\phi(x(t)), \quad x(0) = x_0, \quad t \geq 0 \tag{37}$$

is globally finite-time stable. Moreover, there exists a settling-time function $T : \mathbb{R}^n \to [0,\infty)$ such that

$$T(x_0) \leq \frac{1}{c(1-\alpha)}(V(x_0))^{1-\alpha}, \qquad x_0 \in \mathbb{R}^n \tag{38}$$

and the performance measure (30) is minimized in the sense of (18). Finally

$$J(x_0, \phi(x(\cdot))) = V(x_0), \qquad x_0 \in \mathbb{R}^n. \tag{39}$$

Proof: The result is a direct consequence of Theorem 3.1 with $\mathcal{D} = \mathbb{R}^n$, $U = \mathbb{R}^m$, $F(x,u) = f(x) + G(x)u$, and $L(x,u) = L_1(x) + L_2(x)u + u^{\mathrm{T}}R_2(x)u$. Specifically, the feedback control law (36) follows from (26) by setting

$$\frac{\partial}{\partial u}\left[L_1(x) + L_2(x)u + u^{\mathrm{T}}R_2(x)u + V'(x)(f(x) + G(x)u)\right] = 0. \tag{40}$$

Now, with $u = \phi(x)$ given by (36), conditions (31)–(33) and (35) imply (11)–(14), respectively.

Next, since $V(\cdot)$ is continuously differentiable and, by (31) and (32), $V(0)$ is a local minimum of $V(\cdot)$, it follows that $V'(0) = 0$, and hence, it follows from (34) and (36) that $\phi(0) = 0$, which implies (10). Finally, it follows from (14), (29), and (36) that

$$\begin{aligned}
L(x,u) &+ V'(x)\left[f(x) + G(x)u\right] \\
&= L(x,u) + V'(x)\left[f(x) + G(x)u\right] \\
&\quad - L(x,\phi(x)) - V'(x)\left[f(x) + G(x)\phi(x)\right] \\
&= \left[L_2(x) + V'(x)G(x)\right](u - \phi(x)) \\
&\quad + u^{\mathrm{T}}R_2(x)u - \phi^{\mathrm{T}}(x)R_2(x)\phi(x) \\
&= -2\phi^{\mathrm{T}}(x)R_2(x)(u - \phi(x)) \\
&\quad + u^{\mathrm{T}}R_2(x)u - \phi^{\mathrm{T}}(x)R_2(x)\phi(x) \\
&= \left[u - \phi(x)\right]^{\mathrm{T}}R_2(x)\left[u - \phi(x)\right] \\
&\geq 0, \qquad x \in \mathbb{R}^n
\end{aligned} \tag{41}$$

which implies (15). The result now follows as a direct consequence of Theorem 3.1. ■

Next, we construct state feedback controllers for nonlinear affine in the control dynamical systems that are predicated on an *inverse optimal control problem* [24]–[28]. In particular, to avoid the complexity in solving the steady-state, Hamilton-Jacobi-Bellman equation (35) we do not attempt to minimize a given cost functional, but rather, we parameterize a family of stabilizing controllers that minimize some derived cost functional that provides flexibility in specifying the control law. The performance integrand is shown to explicitly depend on the nonlinear system dynamics, the Lyapunov function of the closed-loop system, and the stabilizing feedback control law, wherein the coupling is introduced via the Hamilton-Jacobi-Bellman equation. Hence, by varying the parameters in the Lyapunov function and the performance integrand, the proposed framework can be used to characterize a class of globally finite-time stabilizing controllers that can meet closed-loop system response constraints.

Theorem 4.2: Consider the controlled nonlinear affine dynamical system (28) with performance measure (30). Assume that there exist a continuously differentiable, radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}$ and real numbers $c > 0$ and $\alpha \in (0, 1)$ such that (31)–(34) hold. Then, with the feedback control

$$u = \phi(x) = -\frac{1}{2}R_2^{-1}(x)\left[L_2(x) + V'(x)G(x)\right]^{\mathrm{T}} \quad (42)$$

the zero solution $x(t) \equiv 0$, $t \geq 0$, to

$$\dot{x}(t) = f(x) + G\left(x(t)\right)\phi\left(x(t)\right), \; x(0) = x_0, \quad t \geq 0 \quad (43)$$

is globally finite-time stable. Moreover, there exists a settling-time function $T : \mathbb{R}^n \to [0, \infty)$ such that

$$T(x_0) \leq \frac{1}{c(1-\alpha)}\left(V(x_0)\right)^{1-\alpha}, \qquad x_0 \in \mathbb{R}^n \quad (44)$$

and the performance functional (30), with

$$L_1(x) = \phi^{\mathrm{T}}(x)R_2(x)\phi(x) - V'(x)f(x) \quad (45)$$

is minimized in the sense of (18). Finally

$$J(x_0, \phi\left(x(\cdot)\right) = V(x_0), \qquad x_0 \in \mathbb{R}^n. \quad (46)$$

Proof: The proof is similar to the proof of Theorem 4.1 and, hence, is omitted. ■

Remark 4.1: As noted in the Introduction, the universal controller given by Sontag's formula [16] has been extended in [17] to design finite-time feedback controllers. Even though this result can be used to construct inverse optimal value functions and inverse optimal finite-time feedback control laws using the ideas presented in [28], such connections are not explored in [17].

Example 4.1: Consider a spacecraft with one axis of symmetry given by [31, p. 753]

$$\dot{\omega}_1(t) = I_{23}\omega_3\omega_2(t) + u_1(t), \; \omega_1(0) = \omega_{10}, \quad t \geq 0 \quad (47)$$
$$\dot{\omega}_2(t) = -I_{23}\omega_3\omega_1(t) + u_2(t), \; \omega_2(0) = \omega_{20} \quad (48)$$

where $I_{23} \triangleq (I_2 - I_3)/I_1$, I_1, I_2, and I_3 are the principal moments of inertia of the spacecraft such that $0 < I_1 = I_2 < I_3$, $\omega_1 : [0, \infty) \to \mathbb{R}$, $\omega_2 : [0, \infty) \to \mathbb{R}$, and $\omega_3 \in \mathbb{R}$ denote the components of the angular velocity vector with respect to a given inertial reference frame expressed in a central body reference frame, and u_1 and u_2 are the spacecraft control moments.

For this example, we apply Theorem 4.2 to find an *inverse optimal* globally finite-time stabilizing control law $u = [u_1, u_2]^{\mathrm{T}} = \phi(x)$, where $x = [\omega_1, \omega_2]^{\mathrm{T}}$, such that the angular velocities $\omega_1(\cdot)$ and $\omega_2(\cdot)$ are regulated to zero in finite time, that is, the dynamical system (47) and (48) is globally finite-time stable, and hence, the spacecraft is spin-stabilized about its third principal inertia axis. Note that (47) and (48) can be cast in the form of (28), with $n = 2$, $m = 2$, $f(x) = [I_{23}\omega_3\omega_2, -I_{23}\omega_3\omega_1]^{\mathrm{T}}$, and $G(x) = I_2$.

To construct an inverse optimal controller for (47) and (48), let

$$V(x_1, x_2) = p^{\frac{2}{3}}(x^{\mathrm{T}}x)^{\frac{2}{3}} \quad (49)$$

where $p > 0$, $L(x, u) = L_1(x) + L_2(x)u + u^{\mathrm{T}}u$, and

$$L_2(x) = 2[-I_{23}\omega_3\omega_2, \quad I_{23}\omega_3\omega_1]. \quad (50)$$

Now, the inverse optimal control law (42) is given by

$$u = \phi(x) = \left[-\frac{2}{3}p^{\frac{2}{3}}\omega_1\|x\|^{-\frac{2}{3}} - I_{23}\omega_3\omega_2, \; -\frac{2}{3}p^{\frac{2}{3}}\omega_2\|x\|^{-\frac{2}{3}} + I_{23}\omega_3\omega_1\right]^{\mathrm{T}} \quad (51)$$

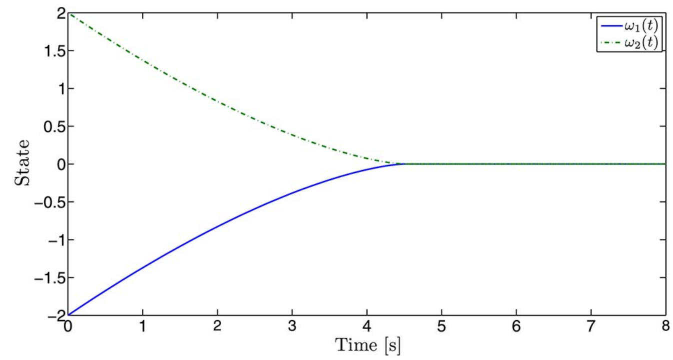

Fig. 1. Closed-loop system trajectories versus time.

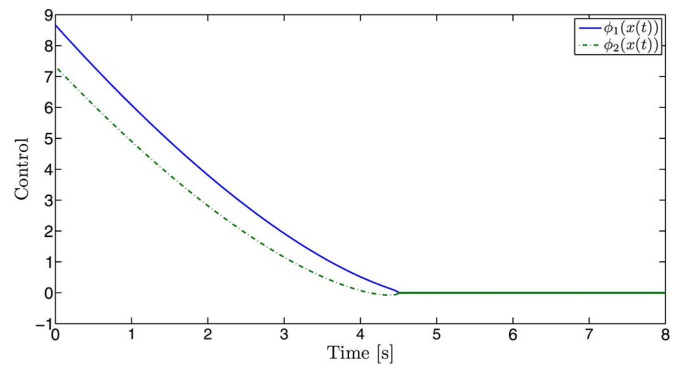

Fig. 2. Control signal versus time.

and the performance functional (30), with

$$L_1(x) = \left(-\frac{2}{3}p^{\frac{2}{3}}\omega_1\|x\|^{-\frac{2}{3}} - I_{23}\omega_3\omega_2\right)^2 + \left(-\frac{2}{3}p^{\frac{2}{3}}\omega_2\|x\|^{-\frac{2}{3}} + I_{23}\omega_3\omega_1\right)^2 \quad (52)$$

is minimized in the sense of (18). Furthermore, since (31) and (32) hold and, since

$$V'(x)\left[f(x) - \frac{1}{2}G(x)L_2^{\mathrm{T}}(x) - \frac{1}{2}G(x)G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x)\right] = -\frac{8}{9}p^{\frac{4}{3}}\left(\omega_1^2 + \omega_2^2\right)^{\frac{1}{3}} = -\frac{8}{9}p\left(V(x)\right)^{\frac{1}{2}}, \qquad x \in \mathbb{R}^2 \quad (53)$$

(33) is verified with $c = (8/9)p$ and $\alpha = 1/2$. Hence, with the feedback control law $\phi(x)$ given by (42), the closed-loop system (47) and (48) is globally finite-time stable. Moreover, there exists a settling-time function $T : \mathbb{R}^2 \to [0, \infty)$ such that

$$T(x_0) \leq \frac{9}{4}p^{-\frac{2}{3}}\left(\omega_{10}^2 + \omega_{20}^2\right)^{\frac{1}{3}}, \qquad x_0 \in \mathbb{R}^2 \quad (54)$$

where $x_0 = [\omega_{10}, \omega_{20}]^{\mathrm{T}}$, and

$$J\left(x_0, \phi\left(x(\cdot)\right)\right) = p^{\frac{2}{3}}\left(\omega_{10}^2 + \omega_{20}^2\right)^{\frac{2}{3}}, \qquad x_0 \in \mathbb{R}^2. \quad (55)$$

Let $I_1 = I_2 = 4\ \mathrm{kg \cdot m^2}$, $I_3 = 20\ \mathrm{kg \cdot m^2}$, $\omega_{10} = -2$ Hz, $\omega_{20} = 2$ Hz, $\omega_3 = 1$ Hz, and $p = 1$, Fig. 1 shows the state trajectories of the controlled system versus time. Note that $x(t) = 0$ for $t = 4.4717$ s $< T(x_0) = 9/2$ s. Fig. 2 shows the control signal versus time. Finally, $J(x(0), \phi(x(\cdot))) = 4\ \mathrm{Hz}^2$. △

V. Conclusion

In this technical note, an optimal control problem for finite-time stabilization is stated and sufficient conditions are derived to characterize an optimal nonlinear feedback controller that stabilizes the closed-loop system in finite-time. Specifically, we utilized a steady-state Hamilton-Jacobi-Bellman framework to characterize optimal nonlinear feedback controllers with a notion of optimality that is directly related to a given Lyapunov function satisfying a differential inequality involving fractional powers. A numerical example was presented to show the utility of the developed framework.

Extensions of this framework for addressing partial finite-time stability and finite-time stabilization, as well as exploring connections between optimal finite-time stabilization and the classical time-optimal control problem are currently under development. Furthermore, since there exist finite-time stable dynamical systems that do not admit a continuously differentiable Lyapunov function that verifies the hypothesis of [7, Theorem 2.1], and hence, Theorem 3.1, a particularly important extension is the consideration of continuous Lyapunov functions leading to viscosity solutions [32] or, equivalently, a proximal analysis formalism [33], of the resulting Hamilton-Jacobi-Bellman equations arising in Theorems 3.1 and 4.1. Finally, the proposed framework can be extended to address optimal finite-time controllers for nonlinear stochastic systems using the results developed in [34]–[36].

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