53rd IEEE Conference on Decision and Control
December 15-17, 2014. Los Angeles, California, USA

Partial-State Stabilization and Optimal Feedback Control

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Abstract— **In this paper, we develop a unified framework to address the problem of optimal nonlinear analysis and feedback control for partial stability and partial-state stabilization. Partial asymptotic stability of the closed-loop nonlinear system is guaranteed by means of a Lyapunov function that is positive definite and decrescent with respect to part of the system state which can clearly be seen to be the solution to the steady-state form of the Hamilton-Jacobi-Bellman equation, and hence, guaranteeing both partial stability and optimality. The overall framework provides the foundation for extending optimal linear-quadratic controller synthesis to nonlinear-nonquadratic optimal partial-state stabilization. Connections to optimal linear and nonlinear regulation for linear and nonlinear time-varying systems with quadratic and nonlinear nonquadratic cost functionals are also provided. An illustrative numerical example is presented to demonstrate the efficacy of the proposed linear and nonlinear partial stabilization framework.**

I. INTRODUCTION

In [1] the current status of continuous-time, nonlinear nonquadratic optimal control problems was presented in a simplified and tutorial manner. The basic underlying ideas of the results in [1] are based on the fact that the steady-state solution of the Hamilton-Jacobi-Bellman equation is a Lyapunov function for the nonlinear system and thus guaranteeing both stability and optimality [2], [1]. Specifically, a feedback control problem over an infinite horizon involving a nonlinear-nonquadratic performance functional is considered. The performance functional is then evaluated in closed form as long as the nonlinear nonquadratic cost functional considered is related in a specific way to an underlying Lyapunov function that guarantees asymptotic stability of the nonlinear closed-loop system. This Lyapunov function is shown to be the solution of the steady-state Hamilton-Jacobi-Bellman equation. The overall framework provides the foundation for extending linear-quadratic control to nonlinear-nonquadratic problems.

In this paper, we extend the framework developed in [1] and [2] to address the problem of optimal *partial-state stabilization*, wherein stabilization with respect to a subset of the system state variables is desired. Partial-state stabilization arises in many engineering applications [3] [4]. Specifically, in spacecraft stabilization via gimballed gyroscopes asymptotic stability of an equilibrium position of the spacecraft is sought while requiring Lyapunov stability of the axis of the gyroscope relative to the spacecraft [4]. Alternatively, in the control of rotating machinery with mass imbalance, spin stabilization about a nonprincipal axis of inertia requires motion stabilization with respect to a subspace instead of the origin [3]. Perhaps the most common application where partial stabilization is necessary is adaptive control, wherein asymptotic stability of the closed-loop plant states is guaranteed without necessarily achieving parameter error convergence. The need to consider partial stability of the closed-loop system in the aforementioned systems arises from the fact that stability notions involve equilibrium coordinates as well as a manifold of coordinates that is closed but *not* compact. Hence, partial stability involves motion lying in a subspace instead of an equilibrium point.

Even though partial-state stabilization has been considered in the literature [3], [4], the problem of optimal partial-state stabilization has received very little attention. In this paper, we consider a notion of optimality that is directly related to a given Lyapunov function that is positive definite and decrescent with respect to part of the system state. Specifically, an optimal partial-state stabilization control problem is stated and sufficient Hamilton-Jacobi-Bellman conditions are used to characterize an optimal feedback controller. Another important application of partial stability and partial stabilization theory is the unification it provides between time-invariant stability theory and stability theory for time-varying systems [2], [5]. We exploit this unification and specialize our results to address optimal linear and nonlinear regulation for linear and nonlinear time-varying systems with quadratic and nonlinear nonquadratic cost functionals.

The contents of this paper are as follows. In Section II, we establish notation, definitions, and recall some basic results on partial stability of nonlinear dynamical systems. In Section III, we consider a nonlinear system with a performance functional evaluated over the infinite horizon. The performance functional is then evaluated in terms of a Lyapunov function that guarantees partial asymptotic stability. We then state an optimal control problem and provide sufficient conditions for characterizing an optimal nonlinear feedback controller guaranteeing partial asymptotic stability of the closed-loop system. These results are then used to address an optimal control problem for uniform asymptotic stabilization of nonlinear time-varying dynamical systems. In Section IV, we specialize the results developed in Section III to affine in the control dynamical systems as well as provide connections to the time-varying, linear-quadratic regulator problem [6]. In Section V, we provide an illustrative numerical example that highlights the optimal partial-state stabilization framework. Finally, in Section VI, we present conclusions and highlight some future research directions. Due to space limitations, we

This work was supported in part by the Air Force Office of Scientific Research under Grant FA9550-12-1-0192 and the Domenica Rea D'Onofrio Fellowship.

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omit all proofs in this paper. Detailed proofs of our results are provided in [7].

II. Notation, Definitions, and Mathematical Preliminaries

In this section, we establish notation, definitions, and review some basic results on partial stability [2]. Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}^n$ denote the set of $n \times 1$ real column vectors, and $\mathbb{R}^{n\times m}$ denote the set of $n \times m$ real matrices. We write $V'(x) \triangleq \frac{\partial V(x)}{\partial x}$ for the Fréchet derivative of V at x, $\|\cdot\|$ for the Euclidean vector norm, A^{T} for the transpose of the matrix A, I_n or I for the $n \times n$ identity matrix, and $0_{n\times m}$ or 0 for the zero $n \times m$ matrix.

In this paper, we consider nonlinear autonomous dynamical systems of the form

$$\dot{x}_1(t) = f_1(x_1(t), x_2(t)), \qquad x_1(0) = x_{10}, \qquad t \geq 0, \quad (1)$$

$$\dot{x}_2(t) = f_2(x_1(t), x_2(t)), \qquad x_2(0) = x_{20}, \quad (2)$$

where, for every $t \geq 0$, $x_1(t) \in \mathcal{D} \subseteq \mathbb{R}^{n_1}$ and $x_2(t) \in \mathbb{R}^{n_2}$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $f_1 : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1}$ is such that, for every $x_2 \in \mathbb{R}^{n_2}$, $f_1(0, x_2) = 0$ and $f_1(\cdot, x_2)$ is locally Lipschitz continuous in x_1, and $f_2 : \mathcal{D}\times\mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$ is such that, for every $x_1 \in \mathcal{D}$, $f_2(x_1, \cdot)$ is locally Lipschitz continuous in x_2.

Definition 2.1 ([2, Def. 4.1]): i) The nonlinear dynamical system $\mathcal{G}$ given by (1) and (2) is *Lyapunov stable with respect to x_1 uniformly in x_{20}* if, for every $\varepsilon > 0$ and $x_{20} \in \mathbb{R}^{n_2}$, there exists $\delta = \delta(\varepsilon) > 0$ such that $\|x_{10}\| \leq \delta$ implies that $\|x_1(t)\| < \varepsilon$ for all $t \geq 0$.

ii) $\mathcal{G}$ is *asymptotically stable with respect to x_1 uniformly in x_{20}* if $\mathcal{G}$ is Lyapunov stable with respect to x_1 uniformly in x_{20} and there exists $\delta > 0$ such that $\|x_{10}\| < \delta$ implies that $\lim_{t\to\infty} x_1(t) = 0$ uniformly in x_{10} and x_{20} for all $x_{20} \in \mathbb{R}^{n_2}$.

iii) $\mathcal{G}$ is *globally asymptotically stable with respect to x_1 uniformly in x_{20}* if $\mathcal{G}$ is Lyapunov stable with respect to x_1 uniformly in x_{20} and $\lim_{t\to\infty} x_1(t) = 0$ uniformly in x_{10} and x_{20} for all $x_{10} \in \mathbb{R}^{n_1}$ and $x_{20} \in \mathbb{R}^{n_2}$.

As shown in [2] and [5], an important application of partial stability theory is the unification it provides between time-invariant stability theory and stability theory for time-varying systems. Specifically, consider the nonlinear time-varying dynamical system given by

$$\dot{x}(t) = f(t, x(t)), \qquad x(t_0) = x_0, \qquad t \geq t_0, \quad (3)$$

where, for every $t \geq t_0$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $f(t, 0) = 0$, $f : [t_0, \infty) \times \mathcal{D} \to \mathbb{R}^n$ is jointly continuous in t and x, and $f(t, \cdot)$ is locally Lipschitz continuous in x uniformly in t for all t in compact subsets of $[t_0, \infty)$. Now, defining $x_1(\tau) \triangleq x(t)$ and $x_2(\tau) \triangleq t$, where $\tau \triangleq t - t_0$, it follows that the solution $x(t)$, $t \geq t_0$, to the nonlinear time-varying dynamical system (3) can be equivalently characterized by the solution $x_1(\tau)$, $\tau \geq 0$, to the nonlinear autonomous dynamical system

$$\dot{x}_1(\tau) = f(x_2(\tau), x_1(\tau)), \qquad x_1(0) = x_0, \qquad \tau \geq 0, \quad (4)$$

$$\dot{x}_2(\tau) = 1, \qquad x_2(0) = t_0. \quad (5)$$

Note that (4) and (5) are in the same form as the system given by (1) and (2), and Definition 2.1 applied to (4) and (5) specializes to the definitions of uniform Lyapunov stability, uniform asymptotic stability, and global uniform asymptotic stability of (3); for details see [2, Def. 4.2].

III. Optimal Partial-State Stabilization

In the first part of this section, we provide connections between Lyapunov functions and nonquatratic cost evaluation. Specifically, we consider the problem of evaluating a nonlinear-nonquadratic performance measure that depends on the solution of the nonlinear dynamical system given by (1) and (2). In particular, we show that the nonlinear-nonquadratic performance measure

$$J(x_{10}, x_{20}) \triangleq \int_0^\infty L(x_1(t), x_2(t))\mathrm{d}t, \quad (6)$$

where $L : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$ is jointly continuous in x_1 and x_2, and $x_1(t)$ and $x_2(t)$, $t \geq 0$, satisfy (1) and (2), can be evaluated in a convenient form so long as (1) and (2) are related to an underlying Lyapunov function that is positive definite and decrescent with respect to x_1 and proves asymptotic stability of (1) and (2) with respect to x_1 uniformly in x_{20}. For the statement of the following results, define $\dot{V}(x_1, x_2) = V'(x_1, x_2)f(x_1, x_2)$, where $f(x_1, x_2) = [f_1^{\mathrm{T}}(x_1, x_2), f_2^{\mathrm{T}}(x_1, x_2)]^{\mathrm{T}}$, for a given continuously differentiable function $V : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$.

Theorem 3.1: Consider the nonlinear dynamical system $\mathcal{G}$ given by (1) and (2) with performance measure (6). Assume that there exists a continuously differentiable function $V : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$ and class $\mathcal{K}$ functions $\alpha(\cdot)$, $\beta(\cdot)$, and $\gamma(\cdot)$ such that

$$\alpha(\|x_1\|) \leq V(x_1, x_2) \leq \beta(\|x_1\|), \quad (x_1, x_2) \in \mathcal{D} \times \mathbb{R}^{n_2}, \quad (7)$$

$$\dot{V}(x_1, x_2) \leq -\gamma(\|x_1\|), \qquad (x_1, x_2) \in \mathcal{D} \times \mathbb{R}^{n_2}, \quad (8)$$

$$L(x_1, x_2) + V'(x_1, x_2)f(x_1, x_2) = 0, \ (x_1, x_2) \in \mathcal{D} \times \mathbb{R}^{n_2}. \quad (9)$$

Then the nonlinear dynamical system $\mathcal{G}$ is asymptotically stable with respect to x_1 uniformly in x_{20} and there exists a neighborhood $\mathcal{D}_0 \subseteq \mathcal{D}$ of $x_1 = 0$ such that, for all $(x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}$,

$$J(x_{10}, x_{20}) = V(x_{10}, x_{20}). \quad (10)$$

Finally, if $\mathcal{D} = \mathbb{R}^{n_1}$ and the functions $\alpha(\cdot)$ and $\beta(\cdot)$ satisfying (7) are class $\mathcal{K}_\infty$, then $\mathcal{G}$ is globally asymptotically stable with respect to x_1 uniformly in x_{20}.

Next, we obtain a characterization of optimal feedback controllers that guarantee closed-loop, partial-state stabilization. Specifically, sufficient conditions for optimality are given in a form that corresponds to a steady-state version of the Hamilton-Jacobi-Bellman equation. To address the problem of characterizing partially stabilizing feedback controllers, consider the controlled nonlinear dynamical system

$$\dot{x}_1(t) = F_1(x_1(t), x_2(t), u(t)), \ \ x_1(0) = x_{10}, \ \ t \geq 0, \quad (11)$$

$$\dot{x}_2(t) = F_2(x_1(t), x_2(t), u(t)), \ \ x_2(0) = x_{20}, \quad (12)$$

where, for every $t \geq 0$, $x_1(t) \in \mathcal{D} \subseteq \mathbb{R}^{n_1}$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $x_2(t) \in \mathbb{R}^{n_2}$, $u(t) \in U \subseteq \mathbb{R}^m$ with $0 \in U$, $F_1 : \mathcal{D} \times \mathbb{R}^{n_2} \times U \to \mathbb{R}^{n_1}$ and $F_2 : \mathcal{D} \times \mathbb{R}^{n_2} \times U \to$

$\mathbb{R}^{n_2}$ are locally Lipschitz continuous in x_1, x_2, and u, and $F_1(0,x_2,0)=0$ for every $x_2\in\mathbb{R}^{n_2}$. The control $u(\cdot)$ in (11) and (12) is restricted to the class of *admissible* controls consisting of measurable functions $u(\cdot)$ such that $u(t)\in U$, $t\geq 0$.

A measurable function $\phi:\mathcal{D}\times\mathbb{R}^{n_2}\to U$ satisfying $\phi(0,x_2)=0$, $x_2\in\mathbb{R}^{n_2}$, is called a *control law*. If $u(t)=\phi(x_1(t),x_2(t))$, $t\geq 0$, where $\phi(\cdot,\cdot)$ is a control law and $x_1(t)$ and $x_2(t)$ satisfy (11) and (12), then we call $u(\cdot)$ a *feedback control law*. Note that the feedback control law is an admissible control since $\phi(\cdot,\cdot)$ has values in U. Given a control law $\phi(\cdot,\cdot)$ and a feedback control law $u(t)=\phi(x_1(t),x_2(t))$, $t\geq 0$, the *closed-loop system* (11) and (12) is given by

$$\dot{x}_1(t)=F_1(x_1(t),x_2(t),\phi(x_1(t),x_2(t))),\quad x_1(0)=x_{10},\quad t\geq 0,\quad (13)$$

$$\dot{x}_2(t)=F_2(x_1(t),x_2(t),\phi(x_1(t),x_2(t))),\quad x_2(0)=x_{20}.\quad (14)$$

We now consider the problem of partial-state stabilization.

Definition 3.1: Consider the controlled dynamical system given by (11) and (12). The feedback control law $u=\phi(x_1,x_2)$ is *asymptotically stabilizing with respect to* x_1 *uniformly in* x_{20} if the closed-loop system (13) and (14) is asymptotically stable with respect to x_1 uniformly in x_{20}. Furthermore, the feedback control law $u=\phi(x_1,x_2)$ is *globally asymptotically stabilizing with respect to* x_1 *uniformly in* x_{20} if the closed-loop system (13) and (14) is globally asymptotically stable with respect to x_1 uniformly in x_{20}.

Next, we present a main theorem for partial-state stabilization characterizing feedback controllers that guarantee partial closed-loop stability and minimize a nonlinear-nonquadratic performance functional. For the statement of this result, define $F(x_1,x_2,u)\triangleq[F_1^{\mathrm{T}}(x_1,x_2,u),\ F_2^{\mathrm{T}}(x_1,x_2,u)]^{\mathrm{T}}$, let $L:\mathcal{D}\times\mathbb{R}^{n_2}\times U\to\mathbb{R}$ be jointly continuous in x_1, x_2, and u, and define the set of partial regulation controllers given by

$$\mathcal{S}(x_{10},x_{20})\triangleq\{u(\cdot):u(\cdot)\text{ is admissible and }x_1(\cdot)\text{ given by (11) satisfies }x_1(t)\to 0\text{ as }t\to\infty\}.$$

Note that restricting our minimization problem to $u(\cdot)\in\mathcal{S}(x_{10},x_{20})$, that is, inputs corresponding to partial-state null convergent solutions, can be interpreted as incorporating a partial-state system detectability condition through the cost.

Theorem 3.2: Consider the controlled nonlinear dynamical system $\mathcal{G}$ given by (11) and (12) with

$$J(x_{10},x_{20},u(\cdot))\triangleq\int_0^\infty L(x_1(t),x_2(t),u(t))\,\mathrm{d}t,\quad (15)$$

where $u(\cdot)$ is an admissible control. Assume that there exist a continuously differentiable function $V:\mathcal{D}\times\mathbb{R}^{n_2}\to\mathbb{R}$, class $\mathcal{K}$ functions $\alpha(\cdot)$, $\beta(\cdot)$, and $\gamma(\cdot)$, and a control law $\phi:\mathcal{D}\times\mathbb{R}^{n_2}\to U$ such that

$$\alpha(\|x_1\|)\leq V(x_1,x_2)\leq\beta(\|x_1\|),\quad (x_1,x_2)\in\mathcal{D}\times\mathbb{R}^{n_2},\quad (16)$$

$$V'(x_1,x_2)F(x_1,x_2,\phi(x_1,x_2))\leq-\gamma(\|x_1\|),\quad (x_1,x_2)\in\mathcal{D}\times\mathbb{R}^{n_2},\quad (17)$$

$$\phi(0,x_2)=0,\qquad x_2\in\mathbb{R}^{n_2},\quad (18)$$

$$0=L(x_1,x_2,\phi(x_1,x_2))+V'(x_1,x_2)F(x_1,x_2,\phi(x_1,x_2)),\quad (x_1,x_2)\in\mathcal{D}\times\mathbb{R}^{n_2},\quad (19)$$

$$L(x_1,x_2,u)+V'(x_1,x_2)F(x_1,x_2,u)\geq 0,\quad (x_1,x_2,u)\in\mathcal{D}\times\mathbb{R}^{n_2}\times U.\quad (20)$$

Then, with the feedback control $u=\phi(x_1,x_2)$, the closed-loop system given by (13) and (14) is asymptotically stable with respect to x_1 uniformly in x_{20} and there exists a neighborhood $\mathcal{D}_0\subseteq\mathcal{D}$ of $x_1=0$ such that

$$J(x_{10},x_{20},\phi(x_1(\cdot),x_2(\cdot)))=V(x_{10},x_{20}),\quad (x_{10},x_{20})\in\mathcal{D}_0\times\mathbb{R}^{n_2}.\quad (21)$$

In addition, if $(x_{10},x_{20})\in\mathcal{D}_0\times\mathbb{R}^{n_2}$, then the feedback control $u(\cdot)=\phi(x_1(\cdot),x_2(\cdot))$ minimizes $J(x_{10},x_{20},u(\cdot))$ in the sense that

$$J(x_{10},x_{20},\phi(\cdot,\cdot))=\min_{u(\cdot)\in\mathcal{S}(x_{10},x_{20})}J(x_{10},x_{20},u(\cdot)).\quad (22)$$

Finally, if $\mathcal{D}=\mathbb{R}^{n_1}$, $U=\mathbb{R}^m$, and the functions $\alpha(\cdot)$ and $\beta(\cdot)$ satisfying (16) are class $\mathcal{K}_\infty$, then the closed-loop system (13) and (14) is globally asymptotically stable with respect to x_1 uniformly in x_{20}.

Note that (19) is the steady-state, Hamilton-Jacobi-Bellman equation for the nonlinear controlled dynamical system (11) and (12) with performance criterion (15). Furthermore, conditions (19) and (20) guarantee optimality with respect to the set of admissible partially asymptotically stabilizing controllers $\mathcal{S}(x_{10},x_{20})$. However, it is important to note that an explicit characterization of $\mathcal{S}(x_{10},x_{20})$ is not required. In addition, the optimal asymptotically stabilizing with respect to x_1 uniformly in x_{20} *feedback* control law $u=\phi(x_1,x_2)$ is independent of the initial condition (x_{10},x_{20}) and is given by

$$\phi(x_1,x_2)=\underset{u\in\mathcal{S}(x_{10},x_{20})}{\arg\min}\big[L(x_1,x_2,u)+V'(x_1,x_2)F(x_1,x_2,u)\big].\quad (23)$$

Remark 3.1: Setting $n_1=n$ and $n_2=0$, the nonlinear controlled dynamical system given by (11) and (12) reduces to

$$\dot{x}(t)=F(x(t),u(t)),\qquad x(0)=x_0,\qquad t\geq 0.\quad (24)$$

In this case, (16) implies that $V(\cdot)$ is positive definite with respect to x and the conditions of Theorem 3.2 reduce to the conditions of Theorem 8.2 of [2] characterizing the classical optimal control problem for time-invariant systems on an infinite interval.

Finally, we use Theorem 3.2 to provide a unification between optimal partial-state stabilization and optimal control for nonlinear time-varying systems. Specifically, consider the nonlinear time-varying controlled dynamical system

$$\dot{x}(t)=F(t,x(t),u(t)),\qquad x(t_0)=x_0,\qquad t\geq t_0,\quad (25)$$

with performance measure

$$J(t_0,x_0,u(\cdot))\triangleq\int_{t_0}^\infty L(t,x(t),u(t))\mathrm{d}t,\quad (26)$$

where, for every $t \geq t_0$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $u(t) \in U \subseteq \mathbb{R}^m$ with $0 \in U$, $L : [t_0, \infty) \times \mathcal{D} \times U \to \mathbb{R}$ and $F : [t_0, \infty) \times \mathcal{D} \times U \to \mathbb{R}^n$ are jointly continuous in t, x, and u, $F(t, \cdot, u)$ is Lipschitz continuous in x for every $(t, u) \in [t_0, \infty) \times U$, and $F(t, x, \cdot)$ is Lipschitz continuous in u for every $(t, x) \in [t_0, \infty) \times \mathcal{D}$. For the statement of the next result, define the set of regulation controllers

$$\mathcal{S}(t_0, x_0) \triangleq \{u(\cdot) : u(\cdot) \text{ is admissible and } x(\cdot) \text{ given by (25) satisfies } x(t) \to 0 \text{ as } t \to \infty\}.$$

Corollary 3.1: Consider the controlled nonlinear time-varying dynamical system (25) with performance measure (26) where $u(\cdot)$ is an admissible control. Assume that there exist a continuously differentiable function $V : [t_0, \infty) \times \mathcal{D} \to \mathbb{R}$, class $\mathcal{K}$ functions $\alpha(\cdot)$, $\beta(\cdot)$, and $\gamma(\cdot)$, and a control law $\phi : [t_0, \infty) \times \mathcal{D} \to U$ such that

$$\alpha(\|x\|) \leq V(t, x) \leq \beta(\|x\|), \quad (t, x) \in [t_0, \infty) \times \mathcal{D}, \quad (27)$$

$$\frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x} F(t, x, \phi(t, x)) \leq -\gamma(\|x\|), \quad (t, x) \in [t_0, \infty) \times \mathcal{D}, \quad (28)$$

$$\phi(t, 0) = 0, \qquad t \in [t_0, \infty), \quad (29)$$

$$L(t, x, \phi(t, x)) + \frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x} F(t, x, \phi(t, x)) = 0, \quad (t, x) \in [t_0, \infty) \times \mathcal{D}, \quad (30)$$

$$L(t, x, u) + \frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x} F(t, x, u) \geq 0, \quad (t, x, u) \in [t_0, \infty) \times \mathcal{D} \times U. \quad (31)$$

Then, with the feedback control $u = \phi(t, x)$, the closed-loop system given by (25) is uniformly asymptotically stable and there exists a neighborhood of the origin $\mathcal{D}_0 \subseteq \mathcal{D}$ such that

$$J(t_0, x_0, \phi(\cdot, \cdot)) = V(t_0, x_0), \quad (t_0, x_0) \in [0, \infty) \times \mathcal{D}_0. \quad (32)$$

In addition, if $(t_0, x_0) \in [0, \infty) \times \mathcal{D}_0$, then the feedback control $u(\cdot) = \phi(\cdot, x(\cdot))$ minimizes $J(x_0, u(\cdot))$ in the sense that

$$J(t_0, x_0, \phi(\cdot, \cdot)) = \min_{u(\cdot) \in \mathcal{S}(t_0, x_0)} J(t_0, x_0, u(\cdot)). \quad (33)$$

Finally, if $\mathcal{D} = \mathbb{R}^n$, $U = \mathbb{R}^m$, and the functions $\alpha(\cdot)$ and $\beta(\cdot)$ satisfying (27) are class $\mathcal{K}_\infty$, then the nonlinear dynamical system $\mathcal{G}$ is globally uniformly asymptotically stable.

Note that (30) and (31) give the Hamilton-Jacobi-Bellman equation

$$-\frac{\partial V(t,x)}{\partial t} = \min_{u \in \mathcal{S}(t_0, x_0)} \left[L(t, x, u) + \frac{\partial V(t,x)}{\partial x} F(t, x, u) \right], \quad (t, x) \in [t_0, \infty) \times \mathcal{D}, \quad (34)$$

which characterizes the optimal control

$$\phi(t, x) = \underset{u \in \mathcal{S}(t_0, x_0)}{\arg\min} \left[L(t, x, u) + \frac{\partial V(t,x)}{\partial x} F(t, x, u) \right] \quad (35)$$

for time-varying systems on a finite or infinite interval.

IV. Partial-State Stabilization for Affine Dynamical Systems and Connections to the Time-Varying Linear-Quadratic Regulator Problem

In this section, we specialize the results of Section III to nonlinear affine dynamical systems of the form

$$\dot{x}_1(t) = f_1(x_1(t), x_2(t)) + G_1(x_1(t), x_2(t)) u(t), \quad x_1(0) = x_{10}, \quad t \geq 0, \quad (36)$$

$$\dot{x}_2(t) = f_2(x_1(t), x_2(t)) + G_2(x_1(t), x_2(t)) u(t), \quad x_2(0) = x_{20}, \quad (37)$$

where, for every $t \geq 0$, $x_1(t) \in \mathbb{R}^{n_1}$ and $x_2(t) \in \mathbb{R}^{n_2}$, $u(t) \in \mathbb{R}^m$, and $f_1 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1}$, $f_2 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$, $G_1 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1 \times m}$, and $G_2 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_2 \times m}$ are such that $f_1(0, x_2) = 0$ for all $x_2 \in \mathbb{R}^{n_2}$, $f_1(\cdot, x_2)$, $f_2(\cdot, x_2)$, $G_1(\cdot, x_2)$, and $G_2(\cdot, x_2)$ are locally Lipschitz continuous in x_1, and $f_1(x_1, \cdot)$, $f_2(x_1, \cdot)$, $G_1(x_1, \cdot)$, and $G_2(x_1, \cdot)$ are locally Lipschitz continuous in x_2. Furthermore, we consider performance integrands $L(x_1, x_2, u)$ of the form

$$L(x_1, x_2, u) = L_1(x_1, x_2) + L_2(x_1, x_2) u + u^{\mathrm{T}} R_2(x_1, x_2) u, \quad (x_1, x_2, u) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^m, \quad (38)$$

where $L_1 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, $L_2 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{1 \times m}$, and $R_2(x_1, x_2) \geq N(x_1) > 0$, $(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, so that (15) becomes

$$J(x_{10}, x_{20}, u(\cdot)) = \int_0^\infty \big[L_1(x_1(t), x_2(t)) + L_2(x_1(t), x_2(t)) u(t) + u^{\mathrm{T}}(t) R_2(x_1(t), x_2(t)) u(t) \big] \, \mathrm{d}t. \quad (39)$$

For the statement of the next result, define $f(x_1, x_2) \triangleq [f_1^{\mathrm{T}}(x_1, x_2), f_2^{\mathrm{T}}(x_1, x_2)]^{\mathrm{T}}$ and $G(x_1, x_2) \triangleq [G_1^{\mathrm{T}}(x_1, x_2), G_2^{\mathrm{T}}(x_1, x_2)]^{\mathrm{T}}$.

Theorem 4.1: Consider the controlled nonlinear affine dynamical system (36) and (37) with performance measure (39). Assume that there exist a continuously differentiable function $V : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, class $\mathcal{K}_\infty$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, and a class $\mathcal{K}$ function $\gamma(\cdot)$ such that

$$\alpha(\|x_1\|) \leq V(x_1, x_2) \leq \beta(\|x_1\|), \quad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \quad (40)$$

$$V'(x_1, x_2) \Big[f(x_1, x_2) - \frac{1}{2} G(x_1, x_2) R_2^{-1}(x_1, x_2) L_2^{\mathrm{T}}(x_1, x_2) - \frac{1}{2} G(x_1, x_2) R_2^{-1}(x_1, x_2) G^{\mathrm{T}}(x_1, x_2) V'^{\mathrm{T}}(x_1, x_2) \Big] \leq -\gamma(\|x_1\|), \quad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \quad (41)$$

$$L_2(0, x_2) = 0, \quad x_2 \in \mathbb{R}^{n_2}, \quad (42)$$

$$0 = L_1(x_1, x_2) + V'(x_1, x_2) f(x_1, x_2) - \frac{1}{4} \Big[V'(x_1, x_2) G(x_1, x_2) + L_2(x_1, x_2) \Big] R_2^{-1}(x_1, x_2) \cdot \Big[V'(x_1, x_2) G(x_1, x_2) + L_2(x_1, x_2) \Big]^{\mathrm{T}}, \quad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \quad (43)$$

Then, with the feedback control

$$u = \phi(x_1, x_2) = -\frac{1}{2} R_2^{-1}(x_1, x_2) \big[L_2(x_1, x_2) + V'(x_1, x_2) G(x_1, x_2) \big]^{\mathrm{T}}, \quad (44)$$

the closed-loop system

$$\dot{x}_1(t) = f_1(x_1(t), x_2(t)) + G_1(x_1(t), x_2(t)) \phi(x_1(t), x_2(t)), \quad x_1(0) = x_{10}, \quad t \geq 0, \quad (45)$$

$$\dot{x}_2(t) = f_2(x_1(t), x_2(t)) + G_2(x_1(t), x_2(t)) \phi(x_1(t), x_2(t)), \quad x_2(0) = x_{20}, \quad (46)$$

is globally asymptotically stable with respect to x_1 uniformly in x_{20} and the performance measure (39) is minimized in the sense of (22). Finally,

$$J(x_{10}, x_{20}, \phi(x_1(\cdot), x_2(\cdot)) = V(x_{10}, x_{20}), \quad (x_{10}, x_{20}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}. \quad (47)$$

Next, Theorem 4.1 can be specialized to address the classical time-varying, linear-quadratic optimal control problem. Specifically, consider the linear time-varying dynamical system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(t_0) = x_0, \quad t \geq t_0, \quad (48)$$

with performance measure

$$J(t_0, x_0, u(\cdot)) = \int_{t_0}^{\infty} \big[x^{\mathrm{T}}(t) R_1(t) x(t) + u^{\mathrm{T}}(t) R_2(t) u(t) \big] \mathrm{d}t, \quad (49)$$

where, for all $t \geq t_0$, $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$, $A : [t_0, \infty) \to \mathbb{R}^{n \times n}$ and $B : [t_0, \infty) \to \mathbb{R}^{n \times m}$ are continuous and uniformly bounded, and $R_1 : [t_0, \infty) \to \mathbb{R}^{n \times n}$ and $R_2 : [t_0, \infty) \to \mathbb{R}^{m \times m}$ are continuous, uniformly bounded, and positive definite, and hence, there exist γ, $\sigma > 0$ such that $R_1(t) \geq \gamma I_n > 0$ and $R_2(t) \geq \sigma I_m > 0$ for all $t \geq t_0$.

Corollary 4.1: Consider the linear time-varying dynamical system (48) with quadratic performance measure (49) and let $P : [t_0, \infty) \to \mathbb{R}^{n \times n}$ be a continuously differentiable, uniformly bounded, positive definite solution of

$$-\dot{P}(t) = A^{\mathrm{T}}(t) P(t) + P(t) A(t) + R_1(t) - P(t) B(t) R_2^{-1}(t) B^{\mathrm{T}}(t) P(t), \quad \lim_{t_f \to \infty} P(t_f) = 0, \quad t \in [t_0, \infty). \quad (50)$$

Then, with the feedback control

$$u = \phi(t, x) = -R_2^{-1}(t) B^{\mathrm{T}}(t) P(t) x, \ (t, x) \in [t_0, \infty) \times \mathbb{R}^n, \quad (51)$$

the dynamical system (48) is globally uniformly asymptotically stable and

$$J(t_0, x_0, \phi(\cdot, \cdot)) = x_0^{\mathrm{T}} P(t_0) x_0, \ (t_0, x_0) \in [0, \infty) \times \mathbb{R}^n. \quad (52)$$

Furthermore, the feedback control $u(\cdot) = \phi(\cdot, x(\cdot))$ minimizes (49) in the sense of (33).

Corollary 4.1 gives sufficient conditions for global uniform asymptotic stability and optimality of the linear dynamical system (48) with the state feedback control law (51). Since the closed-loop linear dynamical system

$$\dot{x}(t) = \tilde{A}(t) x(t), \quad x(t_0) = x_0, \quad t \geq t_0, \quad (53)$$

where $\tilde{A}(t) \triangleq A(t) + B(t)K(t)$ and $K(t) \triangleq -R_2^{-1}(t) B^{\mathrm{T}}(t) P(t)$, is globally uniformly asymptotically stable, (53) is globally (uniformly) exponentially stable [8]. Corollary 4.1 assumes the existence of a continuously differentiable, uniformly bounded, positive definite $P : [t_0, \infty) \to \mathbb{R}^{n \times n}$ satisfying the differential Riccati equation (50). However, if (48) is completely controllable and completely observable (through the cost), then there exists a unique continuously differentiable, uniformly bounded, nonnegative definite solution $P : [t_0, \infty) \to \mathbb{R}^{n \times n}$ to (50) such that the linear dynamical system (48), with state feedback control law (51), is globally (uniformly) exponentially stable [9, Th. 3.5, 3.6].

V. Illustrative Numerical Example

In this section, we consider a numerical example to highlight the optimal partial-state asymptotic stabilization framework developed in the paper. Specifically, we consider control of thermoacoustic instabilities in the combustion process modeled by [2], [10]

$$\dot{q}_1(t) = -\alpha_1 q_1(t) - \beta q_1(t) q_2(t) \cos q_3(t) + u(t), \quad q_1(0) = q_{10}, \quad t \geq 0, \quad (54)$$

$$\dot{q}_2(t) = -\alpha_2 q_2(t) + \beta q_1^2(t) \cos q_3(t) + u(t), \quad q_2(0) = q_{20}, \quad (55)$$

$$\dot{q}_3(t) = 2\theta_1 - \theta_2 - \beta \left(\frac{q_1^2(t)}{q_2(t)} - 2q_2(t) \right) \sin q_3(t), \quad q_3(0) = q_{30}, \quad (56)$$

representing a time-averaged, two-mode thermoacoustic combustion model where q_1, q_2, and q_3 represent modal shapes, $\alpha_1 > 0$ and $\alpha_2 > 0$ represent decay constants, θ_1 and $\theta_2 \in \mathbb{R}$ represent frequency shift constants, $\beta = \big((\gamma + 1)/8\gamma\big)\omega_1$, where γ denotes the ratio of specific heats and ω_1 is the frequency of the fundamental mode, and u is the control input signal.

For this example, we seek a state feedback controller $u = \phi(x_1, x_2)$, where $x_1 = [q_1, \ q_2]^{\mathrm{T}}$ and $x_2 = q_3$, such that the performance measure

$$J(x_1(0), x_2(0), u(\cdot)) = \int_0^{\infty} \big[x_1^{\mathrm{T}}(t) R_1 x_1(t) + u^2(t) \big] \mathrm{d}t, \quad (57)$$

where

$$R_1 = \rho \begin{bmatrix} 2\alpha_1 + \rho & \rho \\ \rho & 2\alpha_2 + \rho \end{bmatrix}, \quad \rho > 0,$$

is minimized in the sense of (22), and (54)–(56) is globally asymptotically stable with respect to x_1 uniformly in $x_2(0)$.

Note that (54)–(56) with performance measure (57) can be cast in the form of (36) and (37) with performance measure (39). In this case, Theorem 4.1 can be applied with $n_1 = 2$, $n_2 = 1$, $m = 1$, $f(x_1, x_2) = \big[-\alpha_1 q_1 - \beta q_1 q_2 \cos q_3, \ -\alpha_2 q_2 + \beta q_1^2 \cos q_3, \ 2\theta_1 - \theta_2 - \beta\big(\frac{q_1^2}{q_2} - 2q_2\big) \sin q_3 \big]^{\mathrm{T}}$, $G(x_1, x_2) = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^{\mathrm{T}}$, $L_1(x_1, x_2) = x_1^{\mathrm{T}} R_1 x_1$, $L_2(x_1, x_2) = 0$, and $R_2(x_1, x_2) = 1$ to characterize the optimal partially stabilizing controller. Specifically,

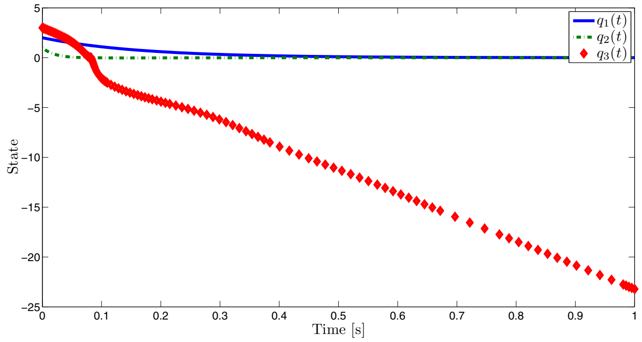

Fig. 1. Closed-loop system trajectories versus time.

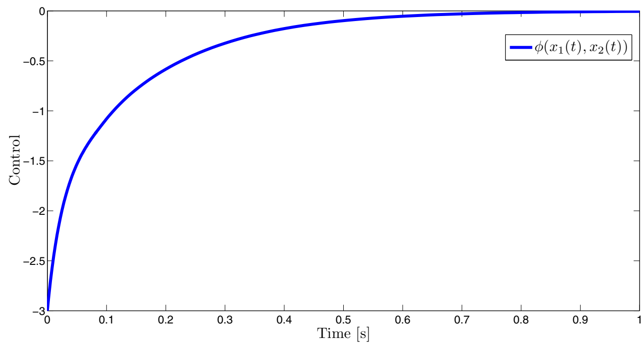

Fig. 2. Control signal versus time.

(43) reduces to

$$0 = x_1^{\mathrm{T}} R_1 x_1 + V'(x_1, x_2) f(x_1, x_2) - \frac{1}{4} V'(x_1, x_2) G(x_1, x_2) G^{\mathrm{T}}(x_1, x_2) V'^{\mathrm{T}}(x_1, x_2), \quad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \tag{58}$$

which implies that $V'(x_1, x_2) = 2\rho\,[q_1,\ q_2,\ 0]$. Furthermore, since $V(0, x_2) = 0$, $x_2 \in \mathbb{R}$,

$$V(x_1, x_2) = \rho\, x_1^{\mathrm{T}} x_1, \tag{59}$$

which is positive definite with respect to x_1, and hence, (40) holds.

Since all of the conditions of Theorem 4.1 hold, it follows that the feedback control (44) given by

$$\phi(x_1, x_2) = -\frac{1}{2} R_2^{-1}(x_1, x_2) G^{\mathrm{T}}(x_1, x_2) V'^{\mathrm{T}}(x_1, x_2) = -\rho \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \tag{60}$$

guarantees that the dynamical system (54)–(56) is globally asymptotically stable with respect to x_1 uniformly in $x_2(0)$ and, for all $(x_1(0), x_2(0)) \in \mathbb{R}^2 \times \mathbb{R}$,

$$J(x_1(0), x_2(0), \phi(x_1(\cdot), x_2(\cdot))) = \rho\, x_1^{\mathrm{T}}(0) x_1(0). \tag{61}$$

Let $\alpha_1 = 5\,\mathrm{Hz}$, $\alpha_2 = 45\,\mathrm{Hz}$, $\gamma = 1.4$, $\omega_1 = 1\,\mathrm{Hz}$, $\theta_1 = 4\,\mathrm{Hz}$, $\theta_2 = 32\,\mathrm{Hz}$, $\rho = 1\,\mathrm{Hz}$, $q_{10} = 2$, $q_{20} = 1$, and $q_{30} = 3$. Figure 1 shows the state trajectories of the controlled system versus time. Note that $x_1(t) = [q_1(t),\ q_2(t)]^{\mathrm{T}} \to 0$ as $t \to \infty$, whereas $x_2(t) = q_3(t)$ is unstable. Figure 2 shows the control signal versus time. Finally, $J(x_1(0), x_2(0), \phi(x_1(\cdot), x_2(\cdot))) = 5\,\mathrm{Hz}$. $\triangle$

VI. Conclusion

In this paper, an optimal control problem for partial-state stabilization is stated and sufficient conditions are derived to characterize an optimal nonlinear feedback controller that guarantees asymptotic stability of part of the closed-loop system state. Specifically, we utilized a steady-state Hamilton-Jacobi-Bellman framework to characterize optimal nonlinear feedback controllers with a notion of optimality that is directly related to a given Lyapunov function that is positive definite and decrescent with respect to part of the system state. This result was then used to address optimal linear and nonlinear regulation for linear and nonlinear time-varying systems with quadratic and nonlinear nonquadratic performance measures. Extensions of this framework for addressing optimal adaptive controllers and inverse optimal feedback controllers for affine nonlinear systems and linear time-varying systems with polynomial and multilinear performance criteria are currently under development [7].

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