Optimal Control for Linear and Nonlinear Semistabilization

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***Abstract*— The state feedback linear-quadratic optimal control problem for asymptotic stabilization has been extensively studied in the literature. In this paper, the optimal linear and nonlinear control problem is extended to address a weaker version of closed-loop stability, namely, semistability, which involves convergent trajectories and Lyapunov stable equilibria and which is of paramount importance for consensus control of network dynamical systems. Specifically, we show that the optimal semistable state-feedback controller can be solved using a form of the Hamilton-Jacobi-Bellman conditions that does not require the cost-to-go function to be sign-definite. This result is then used to solve the optimal linear-quadratic regulator problem using a Riccati equation approach.**

I. INTRODUCTION

A form of stability that lies between Lyapunov stability and asymptotic stability is *semistability* [1], [2], that is, the property whereby every trajectory that starts in a neighborhood of a Lyapunov stable equilibrium converges to a (possibly different) Lyapunov stable equilibrium. Semistability implies Lyapunov stability, and is implied by asymptotic stability [1], [2], [3]. This notion of stability arises naturally in systems having a continuum of equilibria and includes such systems as mechanical systems having rigid body modes, chemical reaction systems [4], compartmental systems [5], [6], and isospectral matrix dynamical systems. Semistability also arises naturally in dynamical network systems [7], [8], [9], which cover a broad spectrum of applications including cooperative control of unmanned air vehicles, autonomous underwater vehicles, distributed sensor networks, air and ground transportation systems, swarms of air and space vehicle formations, and congestion control in communication networks, to cite but a few examples.

A unique feature of the closed-loop dynamics under any control algorithm that achieves consensus in dynamic networks is the existence of a continuum of equilibria representing a desired state of consensus [8], [9]. Under such dynamics, the desired limiting state is not determined completely by the system dynamics, but depends on the initial system state as well [8], [9], [10], [11]. From a practical viewpoint, it is not sufficient to only guarantee that a network converges to a state of consensus since steady-state convergence is not sufficient to guarantee that small perturbations from the limiting state will lead to only small transient excursions from the state of consensus. It is also necessary to guarantee that the equilibrium states representing consensus are Lyapunov stable, and consequentially, semistable.

In [10], [11], the authors develop $\mathcal{H}_2$ optimal semistable control theory for linear dynamical systems. Specifically, unlike the standard $\mathcal{H}_2$ optimal control problem, it is shown in [10], [11] that a complicating factor of the $\mathcal{H}_2$ optimal semistable stabilization problem is that the closed-loop Lyapunov equation guaranteeing semistability can admit multiple solutions. In addition, the authors show that the $\mathcal{H}_2$ optimal solution is given by a least squares solution to the closed-loop Lyapunov equation over all possible semistabilizing solutions. Moreover, it is shown that this least squares solution can be characterized by a linear matrix inequality minimization problem.

In this paper, we address the problem of finding a state-feedback nonlinear control law $u = \phi(x)$ that minimizes the performance measure

$$J(x_0, u(\cdot)) \triangleq \int_0^\infty L(x(t), u(t))\mathrm{d}t \tag{1}$$

and guarantees semistability of the nonlinear dynamical system

$$\dot{x}(t) = F(x(t), u(t)), \qquad x(0) = x_0, \qquad t \ge 0, \tag{2}$$

$$y(t) = H(x(t), u(t)), \tag{3}$$

where, for every $t \ge 0$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $\mathcal{D}$ is an open set, $u(t) \in U \subseteq \mathbb{R}^m$, $y(t) \in Y \subseteq \mathbb{R}^l$, $L : \mathcal{D} \times U \to \mathbb{R}$, $F : \mathcal{D} \times U \to \mathbb{R}^n$ is Lipschitz continuous in x and u on $\mathcal{D} \times U$, and $H : \mathcal{D} \times U \to Y$. Specifically, our approach focusses on the role of the Lyapunov function guaranteeing semistability of (2) with a feedback control law $u = \phi(x)$, and we provide sufficient conditions for optimality in a form that corresponds to a steady-state version of the Hamilton-Jacobi-Bellman equation.

In addition, we provide sufficient conditions for the existence of a feedback gain $K \in \mathbb{R}^{m\times n}$ such that the state feedback control law $u = Kx$ minimizes the quadratic performance measure

$$J(x_0, u(\cdot)) = \int_0^\infty [(x(t) - x_\mathrm{e})^\mathrm{T} C^\mathrm{T} C(x(t) - x_\mathrm{e}) + (u(t) - u_\mathrm{e})^\mathrm{T} R_2(u(t) - u_\mathrm{e})]\mathrm{d}t \tag{4}$$

and guarantees semistability of the linear dynamical system

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0, \qquad t \ge 0, \tag{5}$$

$$y(t) = Cx(t), \tag{6}$$

where $u_\mathrm{e} \triangleq Kx_\mathrm{e}$, $x_\mathrm{e} \triangleq \lim_{t\to\infty} x(t)$, R_2 is positive definite, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{l\times n}$. The proposed Riccati equation-based framework for optimal linear semistable stabilization presented in this paper is different

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from the framework presented in [10], [11] using linear matrix inequalities.

The contents of the paper are as follows. In Section II, we establish notation, definitions, and develop some key results on semistability, semicontrollability, semiobservability, and semistabilization. In Section III, we consider a nonlinear system with a performance functional evaluated over the infinite horizon. The performance functional is then evaluated in terms of a Lyapunov function that guarantees semistability. This result is then specialized to the linear-quadratic case. We then state an optimal control problem and provide sufficient conditions for characterizing an optimal nonlinear feedback controller guaranteeing semistable stabilization. Finally, in Section IV we draw conclusions and highlight recommendations for future research. Due to space limitations, we omit all the proofs in this paper. Detailed proofs of our results are provided in [12].

II. Notation, Definitions, and Mathematical Preliminaries

The notation used in this paper is fairly standard. Specifically, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex numbers, $\mathbb{R}_+$ denotes the set of positive real numbers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, and $\mathbb{R}^{n\times m}$ denotes the set of $n \times m$ real matrices. We write $V'(x) \triangleq \frac{\partial V(x)}{\partial x}$ for the Fréchet derivative of V at x, $\|\cdot\|$ for the Euclidean vector norm, $\|\cdot\|_{\mathrm{F}}$ for the Frobenius matrix norm, $\mathcal{S}^\perp$ for the orthogonal complement of a set $\mathcal{S}$, $\operatorname{span}\mathcal{S}$ for the span of the set $\mathcal{S}$, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ for the range space and the null space of a matrix A, respectively, $\operatorname{spec}(A)$ for the spectrum of the square matrix A including multiplicity, $\operatorname{rank}(A)$ for the rank of the matrix A, and $(\cdot)^\#$ for the group generalized inverse.

Consider the nonlinear dynamical system given by

$$\dot{x}(t) = f(x(t)), \qquad x(0) = x_0, \qquad t \geq 0, \tag{7}$$

where, for every $t \geq 0$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$ and $f : \mathcal{D} \to \mathbb{R}^n$ is locally Lipschitz continuous on $\mathcal{D}$. The solution of (7) with initial condition $x(0) = x$ defined on $[0,\infty)$ is denoted by $s(\cdot, x)$. Given $t \geq 0$ and $x \in \mathcal{D}$, we denote the map $s(t,\cdot) : \mathcal{D} \to \mathcal{D}$ by s_t and the map $s(\cdot, x) : [0,\infty) \to \mathcal{D}$ by s^x. The *orbit* $\mathcal{O}_x$ of a point $x \in \mathcal{D}$ is the set $s^x([0,\infty))$. A set $\mathcal{D}_{\mathrm{p}} \subseteq \mathcal{D}$ is *positively invariant* relative to (7) if $s_t(\mathcal{D}_{\mathrm{p}}) \subseteq \mathcal{D}_{\mathrm{p}}$ for all $t \geq 0$ or, equivalently, $\mathcal{D}_{\mathrm{p}}$ contains the orbits of all its points. The set $\mathcal{D}_{\mathrm{p}}$ is *invariant* relative to (7) if $s_t(\mathcal{D}_{\mathrm{p}}) = \mathcal{D}_{\mathrm{p}}$ for all $t \geq 0$. Finally, the set of equilibrium points of (7) is denoted by $f^{-1}(0) \triangleq \{x \in \mathcal{D} : f(x) = 0\}$.

The following definition is needed.

Definition 2.1 ([3]): Let $\mathcal{D} \subseteq \mathbb{R}^n$ be an open positively invariant set with respect to (7). An equilibrium point $x_{\mathrm{e}} \in \mathcal{D}$ of (7) is *semistable* with respect to $\mathcal{D}$ if x_{e} is Lyapunov stable and there exists an open subset $\mathcal{D}_0$ of $\mathcal{D}$ containing x_{e} such that, for all initial conditions in $\mathcal{D}_0$, the solutions of (7) converge to a Lyapunov stable equilibrium point. The system (7) is *semistable* with respect to $\mathcal{D}$ if every solution with initial condition in $\mathcal{D}$ converges to a Lyapunov stable equilibrium. Finally, (7) is said to be *globally semistable* if (7) is semistable with respect to $\mathbb{R}^n$.

Note that if for $\varepsilon > 0$, $\mathcal{B}_\varepsilon(x_{\mathrm{e}}) \cap f^{-1}(0) = \{x_{\mathrm{e}}\}$ is a singleton, where $\mathcal{B}_\varepsilon(x_{\mathrm{e}})$ denotes the *ball centered* at x_{e} with *radius* ε, then Definition 2.1 reduces to the definitions of local and global asymptotic stability. Recall that for $B = 0$, (5) is semistable if and only if $\operatorname{spec}(A) \subset \{s \in \mathbb{C} : \operatorname{Re} s < 0\} \cup \{0\}$ and, if $0 \in \operatorname{spec}(A)$, then 0 is semisimple [13, Def. 11.8.1]. In this case, we say that A is *semistable*. Furthermore, if A is semistable, then the index of A is zero or one, and hence, A is group invertible. The group inverse $A^\#$ of A is a special case of the Drazin inverse A^{D} in the case where A has index zero or one [13, p. 369]. In this case, for every $x_0 \in \mathbb{R}^n$, $x_{\mathrm{e}} = \lim_{t\to\infty} x(t) = (I_n - AA^\#)x_0$ or, equivalently, $\lim_{t\to\infty} e^{At} = I_n - AA^\#$ [13, Prop. 11.8.1]. Next, we introduce the definitions of semicontrollability and semiobservability for linear systems.

Definition 2.2 ([10]): Consider the system given by (5). The pair (A, B) is *semicontrollable* if

$$\left[\bigcap_{k=1}^{n} \mathcal{N}(B^{\mathrm{T}}(A^{k-1})^{\mathrm{T}})\right]^\perp = [\mathcal{N}(A^{\mathrm{T}})]^\perp, \tag{8}$$

where $A^0 \triangleq I_n$.

In [14], the pair (A, B) is called semicontrollable if and only if

$$\sum_{k=1}^{n} \mathcal{R}(A^{k-1}B) = \mathcal{R}(A), \tag{9}$$

where for the given sets $\mathcal{S}_1$ and $\mathcal{S}_2$, $\mathcal{S}_1 + \mathcal{S}_2 \triangleq \{x + y : x \in \mathcal{S}_1, y \in \mathcal{S}_2\}$ denotes the Minkowski sum.

Proposition 2.1: Consider the dynamical system given by (5). Then (9) holds if and only if (8) holds. Furthermore, (8) is equivalent to

$$\operatorname{span}\left\{\bigcup_{k=1}^{n} \mathcal{R}(A^{k-1}B)\right\} = \mathcal{R}(A). \tag{10}$$

Definition 2.3 ([10]): Consider the system given by (5) and (6) with $B = 0$. The pair (A, C) is *semiobservable* if

$$\bigcap_{k=1}^{n} \mathcal{N}(CA^{k-1}) = \mathcal{N}(A). \tag{11}$$

Semicontrollability and semiobservability are extensions of controllability and observability. In particular, semicontrollability is an extension of null controllability to *nonisolated equilibrium controllability*, whereas semiobservability is an extension of zero-state observability to *nonisolated equilibrium observability*. Next, we introduce the notions of *semistabilizability* and *semidetectability* [15] as generalizations of stabilizability and detectability.

Definition 2.4 ([15]): Consider the dynamical system given by (5) and (6). The pair (A, B) is *semistabilizable* if

$$\operatorname{rank}\begin{bmatrix} B & \jmath\omega I_n - A \end{bmatrix} = n \tag{12}$$

for every nonzero $\omega \in \mathbb{R}$. The pair (A, C) is *semidetectable* if

$$\operatorname{rank}\begin{bmatrix} C \\ \jmath\omega I_n - A \end{bmatrix} = n \tag{13}$$

for every nonzero $\omega \in \mathbb{R}$.

Note that (A, C) is semidetectable if and only if $(A^{\mathrm{T}}, C^{\mathrm{T}})$ is semistabilizable. Furthermore, it is important to note that semistabilizability and semidetectability are *different* notions from the standard notions of stabilizability and detectability used in linear system theory. Recall that (A, B) is stabilizable if and only if $\operatorname{rank}\begin{bmatrix}B & \lambda I_n - A\end{bmatrix} = n$ for every $\lambda \in \mathbb{C}$ in the closed-right-half complex plane, and (A, C) is detectable if and only if $\operatorname{rank}\begin{bmatrix}C \\ \lambda I_n - A\end{bmatrix} = n$ for every $\lambda \in \mathbb{C}$ in the closed-right-half complex plane. Hence, if (A, C) is detectable, then (A, C) is semidetectable; however, the converse is not true. A similar remark holds for the notions of controllability and observability. Namely, if (A, C) is observable, then (A, C) is semidetectable; however, the converse is not true. Hence, semidetectability (resp., semistabilizability) is a weaker notion than both observability and detectability (resp., controllability and stabilizability). Since (13) only concerns detectability of (A, C) on the imaginary axis, we refer to this notion as semidetectability.

Remark 2.1: It follows from Facts 2.11.1-2.11.3 of [13, pp. 130-131] that (12) and (13) are equivalent to

$$\dim[\mathcal{R}(\jmath\omega I_n - A) + \mathcal{R}(B)] = n \tag{14}$$

and

$$\mathcal{N}(\jmath\omega I_n - A) \cap \mathcal{N}(C) = \{0\}, \tag{15}$$

respectively, where $\dim$ denotes the dimension of a set. ♦

As in the case of stabilizability, state feedback control does not destroy semistabilizability and semicontrollability. This is shown in the next lemma.

Lemma 2.1: Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $K \in \mathbb{R}^{m\times n}$. If (A, B) is semistabilizable (resp., semicontrollable), then $(A + BK, B)$ is semistabilizable (resp., semicontrollable).

The following result gives necessary and sufficient conditions for semistability of $\mathcal{G}$.

Theorem 2.1 ([10]): Consider the dynamical system $\mathcal{G}$ given by (5) with $B = 0$ and output given by (6). Then $\mathcal{G}$ is semistable if and only if for every semiobservable pair (A, C) there exists an $n \times n$ matrix $P = P^{\mathrm{T}} \geq 0$ such that

$$0 = A^{\mathrm{T}}P + PA + C^{\mathrm{T}}C \tag{16}$$

is satisfied. Furthermore, if (A, C) is semiobservable and P satisfies (16), then

$$P = \int_0^\infty e^{A^{\mathrm{T}}t}C^{\mathrm{T}}Ce^{At}\mathrm{d}t + P_0, \tag{17}$$

for some $P_0 = P_0^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ satisfying

$$0 = A^{\mathrm{T}}P_0 + P_0A \tag{18}$$

and

$$P_0 \geq -\int_0^\infty e^{A^{\mathrm{T}}t}C^{\mathrm{T}}Ce^{At}\,\mathrm{d}t. \tag{19}$$

In addition, $\min_{P\in\mathcal{P}} \|P\|_{\mathrm{F}}$ has a unique least squares solution P given by

$$P_{\mathrm{LS}} = \int_0^\infty e^{A^{\mathrm{T}}t}C^{\mathrm{T}}Ce^{At}\mathrm{d}t, \tag{20}$$

where $\mathcal{P}$ denotes the set of all P satisfying (16).

Next, using the notions of semistabilizability and semidetectability, we provide a generalization of Theorem 2.1. First, however, the following lemmas are needed.

Lemma 2.2: Let $A \in \mathbb{R}^{n\times n}$. Then A is semistable if and only if $\mathcal{N}(A) \cap \mathcal{R}(A) = \{0\}$ and $\operatorname{spec}(A) \subseteq \{s \in \mathbb{C} : s + s^* < 0\} \cup \{0\}$, where s^* denotes the complex conjugate of s.

Lemma 2.3: Let $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{l\times n}$. If A is semistable and $\mathcal{N}(A) \subseteq \mathcal{N}(C)$, then $CL = 0$, where L is given by

$$L \triangleq I_n - AA^{\#}. \tag{21}$$

Theorem 2.2: Consider the dynamical system $\mathcal{G}$ given by (5) with $B = 0$ and output given by (6). Then the following statements are equivalent:

$i)$ $\mathcal{G}$ is semistable.

$ii)$ $\operatorname{rank}(\jmath\omega I_n - A) = n$ for every nonzero $\omega \in \mathbb{R}$ and there exist a positive integer p, a $p \times n$ matrix E, and an $n \times n$ matrix $P = P^{\mathrm{T}} \geq 0$ such that

$$0 = A^{\mathrm{T}}P + PA + E^{\mathrm{T}}E. \tag{22}$$

In this case,

$$P = \int_0^\infty e^{A^{\mathrm{T}}t}(E^{\mathrm{T}}E + L^{\mathrm{T}}E^{\mathrm{T}}EL)e^{At}\mathrm{d}t + P_0, \tag{23}$$

where $L = I_n - AA^{\#}$ and P_0 satisfies (18) and (19).

$iii)$ For every matrix $C \in \mathbb{R}^{l\times n}$ such that (A, C) is semiobservable, there exists an $n\times n$ matrix $P = P^{\mathrm{T}} \geq 0$ such that (16) holds.

$iv)$ There exist a positive integer p, a $p \times n$ matrix E, and an $n \times n$ matrix $P = P^{\mathrm{T}} \geq 0$ such that (A, E) is semiobservable and (22) holds.

$v)$ There exist a positive integer p, a $p \times n$ matrix E, and an $n \times n$ matrix $P = P^{\mathrm{T}} \geq 0$ such that (A, E) is semidetectable and (22) holds.

The following result follows as a consequence of Theorem 2.2.

Theorem 2.3: Consider the dynamical system $\mathcal{G}$ given by (5) with $B = 0$ and output given by (6). Assume that there exists an $n \times n$ matrix $P = P^{\mathrm{T}} \geq 0$ such that (16) holds. Then $\mathcal{G}$ is semistable if and only if the pair (A, C) is semidetectable. Furthermore, if (A, C) is semidetectable and P satisfies (16), then

$$P = \int_0^\infty e^{A^{\mathrm{T}}t}C^{\mathrm{T}}Ce^{At}\mathrm{d}t + \alpha zz^{\mathrm{T}}, \tag{24}$$

where $\alpha \geq 0$, $z \in \mathcal{N}(A^{\mathrm{T}})$,

$$\alpha zz^{\mathrm{T}} = \int_0^\infty e^{A^{\mathrm{T}}t}L^{\mathrm{T}}C^{\mathrm{T}}CLe^{At}\mathrm{d}t + P_0, \tag{25}$$

$L = I_n - AA^{\#}$, and P_0 satisfies (18) and (19).

Consider the dynamical system given by (5) and (6) with $B = 0$. If the pair (A, C) is semiobservable, then (A, C) is semidetectable and, in this case, it follows from Theorems 2.1 and 2.3 that $\int_0^\infty e^{A^{\mathrm{T}}t}L^{\mathrm{T}}C^{\mathrm{T}}CLe^{At}\mathrm{d}t = 0$.

The following theorem is a direct consequence of Theorem 2.1. For the statement of this theorem, define the set of semistabilizing controllers $\mathcal{S}(x_0)$ for every initial condition $x_0 \in \mathcal{D}$, that is,

$$\mathcal{S}(x_0) \triangleq \{u(\cdot) : u(\cdot) \text{ is measurable and } x(\cdot) \text{ given by (2)} \\ \text{is bounded and satisfies } x(t) \to x_{\mathrm{e}} \text{ as } t \to \infty\},$$

where $x_{\mathrm{e}} \in \mathcal{D}$ is an equilibrium point of (2) for some $u_{\mathrm{e}} \in U$.

Theorem 2.4 ([10]): Consider the closed-loop system $\mathcal{G}$ given by (5) and (6) with feedback controller $u(t) = Kx(t)$, where $K \in \mathbb{R}^{m\times n}$. Then $\mathcal{G}$ is semistable if and only if for every semicontrollable pair (A, B) and semiobservable pair (A, C) there exists an $n \times n$ matrix $P = P^{\mathrm{T}} \geq 0$ such that

$$0 = \tilde{A}^{\mathrm{T}} P + P\tilde{A} + C^{\mathrm{T}}C + K^{\mathrm{T}}R_2K, \qquad (26)$$

where $\tilde{A} \triangleq A + BK$. Furthermore, the least squares solution of (26) is given by

$$P_{\mathrm{LS}} \triangleq \int_0^\infty e^{\tilde{A}^{\mathrm{T}}t}(C^{\mathrm{T}}C + K^{\mathrm{T}}R_2K)e^{\tilde{A}t}\,\mathrm{d}t. \qquad (27)$$

Finally, in this case (4) is given by

$$J(x_0, K) = x_0^{\mathrm{T}} P_{\mathrm{LS}} x_0. \qquad (28)$$

Next, we give an alternative form of Theorem 2.4 using semidetectability.

Theorem 2.5: Consider the closed-loop system $\mathcal{G}$ given by (5) and (6) with feedback controller $u(t) = Kx(t)$, where $K \in \mathbb{R}^{m\times n}$. Assume that there exists an $n \times n$ matrix $P = P^{\mathrm{T}} \geq 0$ such that (26) holds. Then $\mathcal{G}$ is semistable if and only if (A, C) is semidetectable. Furthermore, (4) is given by (28).

The following classical result from optimal control theory is needed.

Theorem 2.6 ([16, p. 24]): Consider the dynamical system given by (5) and (6) with performance measure

$$J_{t_f}(x_0, u(\cdot)) \triangleq \int_0^{t_f} [x^{\mathrm{T}}(t)C^{\mathrm{T}}Cx(t) + u^{\mathrm{T}}(t)R_2u(t)]\mathrm{d}t. \qquad (29)$$

If $P_{t_f}(t) = P_{t_f}^{\mathrm{T}}(t) \geq 0$, $t \in [0, t_f]$, is a solution to the differential Riccati equation

$$-\dot{P}(t) = A^{\mathrm{T}}P(t) + P(t)A + C^{\mathrm{T}}C - P(t)BR_2^{-1}B^{\mathrm{T}}P(t),\ P(t_f) = 0,\ t \in [0, t_f], \qquad (30)$$

then

$$u(t) = -R_2^{-1}B^{\mathrm{T}}P_{t_f}(t)x(t) \qquad (31)$$

minimizes the performance criterion (29). Furthermore, the minimal cost for (29) is given by

$$J_{t_f}(x_0, K_{t_f}(\cdot)) = x_0^{\mathrm{T}}P_{t_f}(0)x_0, \qquad (32)$$

where $K_{t_f}(t) = -R_2^{-1}B^{\mathrm{T}}P_{t_f}(t)$.

The existence of $P_{t_f}(t) = P_{t_f}^{\mathrm{T}}(t) \geq 0$, $t \in [0, t_f]$, satisfying (30) such that (5) is semistable with $u(t) = -R_2^{-1}B^{\mathrm{T}}P_{t_f}(t)x(t)$ and $t_f \to \infty$ has not been addressed in the literature. Furthermore, in this case, it is not clear whether (30) has a steady-state solution, that is, whether there exists an $n \times n$ matrix $P = P^{\mathrm{T}} \geq 0$ such that $P = \lim_{t\to\infty} P_{t_f}(t)$. These issues are discussed in this paper and have been partially addressed in [10], [11].

III. Optimal Semistable Stabilization

In the first part of this section, we provide connections between Lyapunov functions and nonquatratic cost evaluation. Specifically, we consider the problem of evaluating a nonlinear-nonquadratic cost functional that depends on the solution of the nonlinear dynamical system (7). In particular, we show that the nonlinear-nonquadratic cost functional

$$J(x_0) \triangleq \int_0^\infty L(x(t))\mathrm{d}t, \qquad (33)$$

where $L : \mathcal{D} \to \mathbb{R}$ and $x(t)$, $t \geq 0$, satisfies (7), can be evaluated in a convenient form so long as (7) is related to an underlying Lyapunov-like function that proves semistability of (7).

Theorem 3.1: Consider the nonlinear dynamical system $\mathcal{G}$ given by (7) with performance functional (33), and let $\mathcal{Q}$ be an open neighborhood of $f^{-1}(0)$. Suppose the solution $x(t)$, $t \geq 0$, of (7) is bounded for all $x \in \mathcal{Q}$ and assume that there exists a continuously differentiable function $V : \mathcal{D} \to \mathbb{R}$ such that

$$V'(x)f(x) \leq 0, \qquad x \in \mathcal{Q}, \qquad (34)$$

$$L(x) + V'(x)f(x) = 0, \qquad x \in \mathcal{D}. \qquad (35)$$

If every point in the largest invariant set $\mathcal{M}$ of $\{x \in \mathcal{Q} : V'(x)f(x) = 0\}$ is Lyapunov stable, then (7) is semistable and

$$J(x_0) = V(x_0) - V(x_{\mathrm{e}}), \qquad x_0 \in \mathcal{Q}, \qquad (36)$$

where $x_{\mathrm{e}} = \lim_{t\to\infty} x(t)$.

The following corollary specializes Theorem 3.1 for linear dynamical systems with quadratic performance measure

$$J(x_0) = \int_0^\infty [(x(t) - x_{\mathrm{e}})^{\mathrm{T}}C^{\mathrm{T}}C(x(t) - x_{\mathrm{e}})]\mathrm{d}t. \qquad (37)$$

Corollary 3.1: Consider the linear dynamical system $\mathcal{G}$ given by (5) and (6) with $B = 0$ and with quadratic performance measure (37). If (A, C) is semiobservable, then $\mathcal{G}$ is globally semistable and

$$J(x_0) = x_0^{\mathrm{T}}(AA^{\#})^{\mathrm{T}}PAA^{\#}x_0, \qquad (38)$$

where $P = P^{\mathrm{T}} \geq 0$ is a solution of

$$\begin{bmatrix} AA^{\#} \\ I_n \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} P & 0 \\ 0 & -P + P_0 \end{bmatrix} \begin{bmatrix} AA^{\#} \\ I_n \end{bmatrix} = 0 \qquad (39)$$

and P_0 satisfies (18) and (19).

Note that (39) can be written as

$$P = (AA^{\#})^{\mathrm{T}}PAA^{\#} + P_0. \qquad (40)$$

Hence, since $A^{\#}A = AA^{\#}$ and $AA^{\#}A = A$ [13, p. 403], premultiplying and postmultiplying (40) by A^{T} and A, respectively, it follows that $A^{\mathrm{T}}P_0A = 0$, which is implied by (18).

Corollary 3.2: Consider the linear dynamical system $\mathcal{G}$ given by (5) and (6) with $B = 0$ and with quadratic performance measure (37). If (A, C) is semidetectable and there exists $P = P^{\mathrm{T}} \geq 0$ such that (16) holds, then $\mathcal{G}$ is globally semistable and (38) holds. Furthermore, P additionally satisfies

$$\begin{bmatrix} AA^{\#} \\ I_n \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} P & 0 \\ 0 & -P + \alpha zz^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} AA^{\#} \\ I_n \end{bmatrix} = 0, \qquad (41)$$

where $\alpha \geq 0$ and $z \in \mathcal{N}(A^{\mathrm{T}})$ satisfies (25).

Next, we use the approach of Theorem 3.1 to obtain a characterization of optimal feedback controllers that guarantee closed-loop semistability. Specifically, sufficient conditions for optimality are given in a form that corresponds to a steady-state version of the Hamilton-Jacobi-Bellman equation.

Theorem 3.2: Consider the controlled dynamical system (2) with $u(\cdot) \in \mathcal{S}(x_0)$ and performance measure (1), and suppose there exists a continuously differentiable function $V : \mathcal{D} \to \mathbb{R}$ and a control law $\phi : \mathcal{D} \to U$ such that

$$\phi(x_{\rm e}) = u_{\rm e}, \tag{42}$$

$$V'(x)F(x, \phi(x)) \le 0, \qquad x \in \mathcal{Q}, \tag{43}$$

$$L(x, \phi(x)) + V'(x)F(x, \phi(x)) = 0, \qquad x \in \mathcal{D}, \tag{44}$$

$$L(x, u) + V'(x)F(x, u) \ge 0, \qquad (x, u) \in \mathcal{D} \times U, \tag{45}$$

where $\mathcal{Q}$ is an open neighborhood of $F^{-1}(0) \triangleq \{x \in \mathcal{D} : F(x, \phi(x)) = 0\}$, $x_{\rm e} = \lim_{t\to\infty} x(t)$, and $x(t)$, $t \ge 0$, is the solution of

$$\dot{x}(t) = F(x(t), \phi(x)), \qquad x(0) = x_0, \qquad t \ge 0. \tag{46}$$

If every point in the largest invariant set $\mathcal{M}$ of $\{x \in \mathcal{Q} : V'(x)F(x, \phi(x)) = 0\}$ is Lyapunov stable, then, with the feedback control $u(\cdot) = \phi(x(\cdot))$, the solution $x(t) = x_{\rm e}$, $t \ge 0$, of the closed-loop system (46) is semistable and

$$J(x_0, \phi(x(\cdot))) = V(x_0) - V(x_{\rm e}). \tag{47}$$

Furthermore, the feedback control $u(\cdot) = \phi(x(\cdot))$ minimizes $J(x_0, u(\cdot))$ in the sense that

$$J(x_0, \phi(x(\cdot))) = \min_{u(\cdot)\in\mathcal{S}(x_0)} J(x_0, u(\cdot)). \tag{48}$$

Note that Theorem 3.2 guarantees optimality with respect to the set $\mathcal{S}(x_0)$ of semistabilizing admissible controllers with the optimal control law given by the state feedback controller

$$\phi(x) = \operatorname*{argmin}_{u\in\mathcal{S}(x_0)} [L(x, u) + V'(x)F(x, u)], \tag{49}$$

which invokes the steady-state Hamilton-Jacobi-Bellman equation. In addition, note that Theorem 3.2 does not require $V(0) = 0$, $V(x) > 0$, $x \in \mathcal{D} \setminus \{0\}$, and $V'(x)F(x, \phi(x)) < 0$, $x \in \mathcal{D}$, as is the case for optimal asymptotic stabilization [3, Th. 8.2].

Next, we consider the linear-quadratic regulator problem for semistabilization, that is, we seek controllers $u(\cdot)$ that minimize (4) and guarantee semistability of the linear system given by (5) and (6). The feedback gain K that minimizes (4) and guarantees semistability of (5) can be characterized via a solution to a linear matrix inequality [10]. The following result provides a useful alternative in finding the optimal gain K via an algebraic Riccati equation.

Corollary 3.3: Consider the linear controlled dynamical system $\mathcal{G}$ given by (5) and (6) with quadratic performance measure (4), assume that the pair (A, B) is semicontrollable and the pair (A, C) is semiobservable, and let $P_{\rm LS} = P_{\rm LS}^{\rm T} \ge 0$ be the least squares solution to the algebraic Riccati equation

$$0 = A^{\rm T}P + PA + C^{\rm T}C - PBR_2^{-1}B^{\rm T}P. \tag{50}$$

Then, with $u = Kx = -R_2^{-1}BP_{\rm LS}x$, the solution $x(t) = x_{\rm e}$, $t \ge 0$, to (5) is globally semistable,

$$J(x_0, K) = x_0^{\rm T}\left[\int_0^\infty (\tilde{A}\tilde{A}^\#)^{\rm T} e^{\tilde{A}^{\rm T}t}(C^{\rm T}C + K^{\rm T}R_2K)e^{\tilde{A}t}\tilde{A}\tilde{A}^\#\,{\rm d}t\right]x_0, \tag{51}$$

where $\tilde{A} = A + BK$, and

$$J(x_0, K) = \min_{u(\cdot)\in\mathcal{S}(x_0)} J(x_0, u(\cdot)). \tag{52}$$

Corollary 3.4: Consider the linear controlled dynamical system $\mathcal{G}$ given by (5) and (6) with quadratic performance measure (4), assume the pair (A, B) is semistabilizable and the pair (A, C) is semidetectable, and assume that there exists $P = P^{\rm T} \ge 0$ such that (50) holds. Then, with $u = Kx = -R_2^{-1}B^{\rm T}Px$, the equilibrium solution $x(t) \equiv x_{\rm e}$ to (5) is globally semistable and

$$J(x_0, K) = x_0^{\rm T}\left[\int_0^\infty e^{\tilde{A}^{\rm T}t}(C^{\rm T}C + K^{\rm T}R_2K)e^{\tilde{A}t}\,{\rm d}t\right]x_0, \tag{53}$$

where $\tilde{A} = A + BK$, and

$$\begin{aligned}\min_{u(\cdot)\in\mathcal{S}(x_0)} J(x_0, u(\cdot)) &= J(x_0, K_*)\\ &\le J(x_0, K)\\ &= 2x_0^{\rm T}P\tilde{A}\tilde{A}^\# x_0,\end{aligned} \tag{54}$$

where $K_* = -R_2^{-1}B^{\rm T}P_{\rm LS}$ and $P_{\rm LS} = P_{\rm LS}^{\rm T} \ge 0$ is the least squares solution to (50).

Corollary 3.5: Consider the linear controlled dynamical system $\mathcal{G}$ given by (5) and (6) with quadratic performance measure (4), assume that the pair (A, B) is semicontrollable and the pair (A, C) is semiobservable, and let $P_{\rm LS} = P_{\rm LS}^{\rm T} \ge 0$ be the least squares solution to (50). Then, with $u = Kx = -R_2^{-1}B^{\rm T}P_{\rm LS}x$, the equilibrium solution $x(t) \equiv x_{\rm e}$ to (5) is globally semistable and (53) holds. Furthermore, (52) holds.

Given the linear dynamical system given by (5) and (6), if the pair (A, B) is semicontrollable and the pair (A, C) is semiobservable, then it follows from Lemma 2.1 that, for every $K \in \mathbb{R}^{m\times n}$, the pair $(A+BK, B)$ is semicontrollable, and by Proposition 2.1 in [10], it follows that, for every $R_2 \in \mathbb{R}^{n\times n}$ such that $R_2 = R_2^{\rm T} > 0$, the pair $(A + BK, C^{\rm T}C+K^{\rm T}R_2K)$ is semiobservable. Furthermore, if the pair (A, C) is semiobservable, then (A, C) is semidetectable and it follows from Theorems 2.1 and 2.3 that every solution $P = P^{\rm T} \ge 0$ of (26) is given by

$$P = \int_0^\infty e^{\tilde{A}^{\rm T}t}(C^{\rm T}C + K^{\rm T}R_2K)e^{\tilde{A}t}{\rm d}t + zz^{\rm T}, \tag{55}$$

where $\tilde{A} = A + BK$ and $z \in \mathcal{N}(\tilde{A}^{\rm T})$. Now, if $K = -R_2^{-1}B^{\rm T}P$, then (26) is equivalent to (50), where P can be computed using the Schur decomposition of the Hamiltonian matrix [13, pp. 853-859], and the least squares solution $P_{\rm LS} = P_{\rm LS}^{\rm T} \ge 0$ of (50) is given by $P_{\rm LS} = P - zz^{\rm T}$, where z is the solution of the optimization problem

$$\min_{z\in\mathbb{R}^n} \|P - zz^{\rm T}\|_{\rm F} \tag{56}$$

subject to

$$0 \le P - zz^{\rm T}, \tag{57}$$

$$0 = (A^{\rm T} - PBR_2^{-1}B^{\rm T})z. \tag{58}$$

One might surmise that Corollaries 3.3 and 3.5 give different values for $J(x_0, K)$. However, note that

$$J(x_0, u(\cdot)) = \int_0^\infty \left[(x(t) - x_{\rm e})^{\rm T}C^{\rm T}C(x(t) - x_{\rm e}) + (u(t) - u_{\rm e})^{\rm T}R_2(u(t) - u_{\rm e})\right]{\rm d}t$$

$$
\begin{aligned}
&= \int_0^\infty \big[(x_0-x_\mathrm{e})^\mathrm{T} e^{\tilde{A}^\mathrm{T}t}(C^\mathrm{T}C \\
&\qquad + K^\mathrm{T}R_2K)e^{\tilde{A}t}(x_0-x_\mathrm{e})\big]\,\mathrm{d}t \\
&= x_0^\mathrm{T}\bigg[\int_0^\infty (\tilde{A}\tilde{A}^\#)^\mathrm{T} e^{\tilde{A}^\mathrm{T}t}(C^\mathrm{T}C \\
&\qquad + K^\mathrm{T}R_2K)e^{\tilde{A}t}\tilde{A}\tilde{A}^\#\,\mathrm{d}t\bigg]x_0
\end{aligned} \tag{59}
$$

and, since

$$
\begin{aligned}
J(x_0,u(\cdot)) &= \int_0^\infty \big[(x(t)-x_\mathrm{e})^\mathrm{T}C^\mathrm{T}C(x(t)-x_\mathrm{e}) \\
&\qquad + (u(t)-u_\mathrm{e})^\mathrm{T}R_2(u(t)-u_\mathrm{e})\big]\,\mathrm{d}t \\
&= \int_0^\infty x^\mathrm{T}(t)(C^\mathrm{T}C+K^\mathrm{T}R_2K)x(t)\,\mathrm{d}t,
\end{aligned} \tag{60}
$$

it follows that

$$
\begin{aligned}
J(x_0,u(\cdot)) &= \int_0^\infty \big[(x(t)-x_\mathrm{e})^\mathrm{T}C^\mathrm{T}C(x(t)-x_\mathrm{e}) \\
&\qquad + (u(t)-u_\mathrm{e})^\mathrm{T}R_2(u(t)-u_\mathrm{e})\big]\,\mathrm{d}t \\
&= x_0^\mathrm{T}\int_0^\infty e^{\tilde{A}^\mathrm{T}t}(C^\mathrm{T}C+K^\mathrm{T}R_2K)e^{\tilde{A}t}\,\mathrm{d}t\,x_0.
\end{aligned} \tag{61}
$$

Hence, (51) and (53) are equivalent.

Finally, in light of Corollary 3.3 and Lemma 4.3 of [11] the following result is immediate.

Proposition 3.1: Consider the linear controlled dynamical system $\mathcal{G}$ given by (5) and (6). If the pair (A,B) is semicontrollable, the pair (A,C) is semiobservable, and $\mathcal{G}$, with $u=Kx$, is semistable, then

$$
P = \int_0^\infty (\tilde{A}\tilde{A}^\#)^\mathrm{T} e^{\tilde{A}^\mathrm{T}t}(C^\mathrm{T}C+K^\mathrm{T}R_2K)e^{\tilde{A}t}\tilde{A}\tilde{A}^\#\,\mathrm{d}t \tag{62}
$$

satisfies

$$
0 = \tilde{A}^\mathrm{T}(\tilde{A}^\mathrm{T}P+P\tilde{A}+C^\mathrm{T}C+K^\mathrm{T}R_2K)\tilde{A}, \tag{63}
$$

or, equivalently, (26).

IV. Conclusion

In this paper, we presented an optimal control framework for addressing optimal linear and nonlinear semistabilizing controllers with quadratic and nonquadratic cost functionals. Specifically, we considered dynamical systems on the infinite interval and utilized a steady-state Hamilton-Jacobi-Bellman approach to characterize optimal nonlinear feedback controllers that guarantee Lyapunov stability and convergence for closed-loop systems having a continuum of equilibria. Numerical examples highlighting the optimal semistabilization framework developed in this paper are presented in [12].

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