Optimal Singular Control for Nonlinear Semistabilization

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Abstract— **The singular optimal control problem for asymptotic stabilization has been extensively studied in the literature. In this paper, the optimal singular control problem is extended to address a weaker version of closed-loop stability, namely, semistability, which is of paramount importance for consensus control of network dynamical systems. Two approaches are presented to address the nonlinear semistable singular control problem. Namely, we solve the nonlinear semistable singular control problem by using the cost-to-go function to cancel the singularities in the corresponding Hamilton-Jacobi-Bellman equation. For this case, we show that the minimum value of the singular performance measure is zero. In the second approach, we provide a framework based on the concepts of state-feedback linearization and feedback equivalence to solve the singular control problem for semistabilization of nonlinear dynamical systems. For this approach, we also show that the minimum value of the singular performance measure is zero. A numerical example is presented to demonstrate the efficacy of the proposed singular semistabilization frameworks.**

I. Introduction

Semistability [1], [2] is the property whereby every trajectory that starts in a neighborhood of a Lyapunov stable equilibrium converges to a (possibly different) Lyapunov stable equilibrium. This notion of stability, which implies Lyapunov stability and is implied by asymptotic stability [1]–[3], arises naturally in dynamical network systems [4]–[6], which cover a broad spectrum of applications including cooperative control of unmanned air vehicles, autonomous underwater vehicles, distributed sensor networks, air and ground transportation systems, swarms of air and space vehicle formations, and congestion control in communication networks, to cite but a few examples.

A unique feature of the closed-loop dynamics under any control algorithm that achieves consensus in dynamic networks is the existence of a continuum of equilibria representing a desired state of consensus [5], [6]. Under such dynamics, the desired limiting state is not determined completely by the system dynamics, but depends on the initial system state as well [5]–[8]. From a practical viewpoint, it is not sufficient to only guarantee that a network converges to a state of consensus since steady-state convergence is not sufficient to guarantee that small perturbations from the limiting state will lead to only small transient excursions from the state of consensus. It is also necessary to guarantee that the equilibrium states representing consensus are Lyapunov stable, and consequentially, semistable.

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The optimal singular control problem for asymptotic stabilization has received considerable attention in the literature as a limiting case of the linear-quadratic regulator problem [9]. This problem provides an interesting perspective in system characterization, such as the invertibility problem [10], and it is used in the design of high-gain feedback systems [11], [12]. The singular control problem has been extended to non-square systems [13], affine nonlinear systems [14], and discrete-time systems [15]. A complicating factor in the solution of the optimal singular control problem for affine nonlinear dynamical systems is the fact that the Hamilton-Jacobi-Bellman equation involves singularities that cannot be canceled since the cost-to-go function is required to be positive definite.

In [16], [17], the authors address an optimal control problem for semistabilization of linear and nonlinear dynamical systems. Specifically, given a nonlinear dynamical system with a nonlinear-nonquadratic performance measure, it is shown that the optimal semistable state-feedback controller can be solved using Hamilton-Jacobi-Bellman-type conditions that do not require the cost-to-go function to be sign definite. This result is then used to solve the $\mathcal{H}_2$ optimal semistable stabilization problem using a Riccati equation approach.

In this paper, we provide two approaches to address the nonlinear semistable optimal singular control problem. Specifically, we solve the nonlinear semistable optimal singular control problem using the results of [16], [17]. Specifically, since the cost-to-go function that solves the Hamilton-Jacobi-Bellman-like equation for semistabilization is not required to be sign definite, we use this extra flexibility in the semistable singular control problem to cancel the singularities in the corresponding Hamilton-Jacobi-Bellman-like equation. In this case, we show that the minimum value of the singular performance measure is zero. Finally, a solution to the singular semistabilization problem using differential geometric methods [18] and the concepts of output-feedback linearization and feedback equivalence is also presented. Specifically, we construct an output-feedback linearizing controller and find the control parameters that solve the optimal singular control problem for semistabilization of the linearized system. Due to space limitations, we omit all proofs in this paper. Detailed proofs of our results are provided in [19].

II. Notation, Definitions, and Mathematical Preliminaries

The notation used in this paper is fairly standard. Specifically, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex numbers, $\mathbb{R}_+$ denotes the set of positive real numbers, $\mathbb{R}^n$ denotes the set of $n\times 1$ real column vectors, and $\mathbb{R}^{n\times m}$ denotes the set of $n \times m$ real matrices. Furthermore, $\mathbb{R}_{\rm prop}(s)$ denotes the set of proper rational transfer functions with coefficients in $\mathbb{R}$ and $\mathbb{R}_{\mathrm{prop}}^{l\times m}(s)$ denotes the set of $l\times m$ matrices with entries in $\mathbb{R}_{\mathrm{prop}}(s)$. The $n\times m$ zero matrix is denoted by $0_{n\times m}$ or 0 and the $n\times n$ identity matrix is denoted by I_n or I. We write $V'(x)\triangleq\frac{\partial V(x)}{\partial x}$ for the Fréchet derivative of V at x, $\|\cdot\|$ for the Euclidean vector norm, $\|\cdot\|_{\mathrm{F}}$ for the Frobenius matrix norm, $\mathcal{S}^{\perp}$ for the orthogonal complement of a set $\mathcal{S}$, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ for the range space and the null space of a matrix A, respectively, $\mathrm{spec}(A)$ for the spectrum of the square matrix A including multiplicity, $\det(A)$ for the determinant of the square matrix A, and $\mathrm{rank}\,A$ for the rank of the matrix A. Finally, given the vector $v\in\mathbb{R}^n$, $v_i\in\mathbb{R}$, $i=1,\ldots,n$, denotes the ith component of v, and given the matrix $G\in\mathbb{R}^{n\times m}$, $G_i\in\mathbb{R}^n$, $i=1,\ldots,m$, denotes the ith column of G.

Consider the nonlinear dynamical system given by

$$\dot{x}(t)=f(x(t)),\qquad x(0)=x_0,\qquad t\geq 0,\tag{1}$$

where $f:\mathcal{D}\to\mathbb{R}^n$ is locally Lipschitz continuous on the open set $\mathcal{D}\subseteq\mathbb{R}^n$. In this paper, we assume that for every $t\geq 0$, there exists a unique solution $x(t)\in\mathcal{D}\subseteq\mathbb{R}^n$ of (1). The solution of (1) with initial condition $x(0)=x$ defined on $[0,\infty)$ is denoted by $s(\cdot,x)$. The above assumptions imply that the map $s:[0,\infty)\times\mathcal{D}\to\mathcal{D}$ is continuous [20, Th. 2.1], satisfies the *consistency* property $s(0,x)=x$, and possesses the *semigroup* property $s(t,s(\tau,x))=s(t+\tau,x)$ for all $t,\tau\geq 0$ and $x\in\mathcal{D}$. Given $t\geq 0$, we denote the map $s(t,\cdot):\mathcal{D}\to\mathcal{D}$ by s_t. A set $\mathcal{D}_{\mathrm{p}}\subseteq\mathcal{D}$ is *positively invariant* with respect to (1) if $s_t(\mathcal{D}_{\mathrm{p}})\subseteq\mathcal{D}_{\mathrm{p}}$ for all $t\geq 0$ and the set $\mathcal{D}_{\mathrm{p}}$ is *invariant* with respect to (1) if $s_t(\mathcal{D}_{\mathrm{p}})=\mathcal{D}_{\mathrm{p}}$ for all $t\geq 0$, where $s_t(\mathcal{D}_{\mathrm{p}})\triangleq\{s_t(x),x\in\mathcal{D}_{\mathrm{p}}\}$. Finally, the set of equilibrium points of (1) is denoted by $f^{-1}(0)\triangleq\{x\in\mathcal{D}:f(x)=0\}$.

The following definition is needed.

Definition 2.1 ([21]): Let $\mathcal{D}\subseteq\mathbb{R}^n$ be an open positively invariant set with respect to (1). An equilibrium point $x_{\mathrm{e}}\in\mathcal{D}$ of (1) is *semistable* with respect to $\mathcal{D}$ if x_{e} is Lyapunov stable and there exists an open subset $\mathcal{D}_0$ of $\mathcal{D}$ containing x_{e} such that, for all initial conditions in $\mathcal{D}_0$, the solutions of (1) converge to a Lyapunov stable equilibrium point. The system (1) is *semistable* with respect to $\mathcal{D}$ if every solution with initial condition in $\mathcal{D}$ converges to a Lyapunov stable equilibrium. Finally, (1) is said to be *globally semistable* if (1) is semistable with respect to $\mathbb{R}^n$.

Note that if for $\varepsilon>0$, $\mathcal{B}_\varepsilon(x_{\mathrm{e}})\cap f^{-1}(0)=\{x_{\mathrm{e}}\}$ is a singleton, where $\mathcal{B}_\varepsilon(x_{\mathrm{e}})$ denotes the *open ball centered* at x_{e} with *radius* ε, then Definition 2.1 reduces to the definitions of local and global asymptotic stability.

Next, consider the affine in the control nonlinear dynamical system

$$\dot{x}(t)=f(x(t))+G(x(t))u(t),\quad x(0)=x_0,\quad t\geq 0,\tag{2}$$

$$y(t)=h(x(t)),\tag{3}$$

where, for every $t\geq 0$, $x(t)\in\mathcal{D}\subseteq\mathbb{R}^n$, $\mathcal{D}$ is an open set, $u(t)\in U\subseteq\mathbb{R}^m$, $y(t)\in Y\subseteq\mathbb{R}^l$, $0=f(x_{\mathrm{e}})+G(x_{\mathrm{e}})u_{\mathrm{e}}$ for some $(x_{\mathrm{e}},u_{\mathrm{e}})\in\mathcal{D}\times U$, $y_{\mathrm{e}}=h(x_{\mathrm{e}})$, $l=m$, $f:\mathcal{D}\to\mathbb{R}^n$ is Lipschitz continuous on $\mathcal{D}$, and $G:\mathcal{D}\to\mathbb{R}^{n\times m}$ and $h:\mathcal{D}\to\mathbb{R}^l$ are continuous on $\mathcal{D}$. To address the optimal semistabilization problem, we consider the controlled nonlinear dynamical system (2) with $u(\cdot)$ restricted to the class of *admissible* controls consisting of measurable functions $u(\cdot)$ such that $u(t)\in U$, $t\geq 0$. A measurable function $\phi:\mathcal{D}\to U$ satisfying $\phi(x_{\mathrm{e}})=u_{\mathrm{e}}$, for some $(x_{\mathrm{e}},u_{\mathrm{e}})\in\mathcal{D}\times U$ such that $0=f(x_{\mathrm{e}})+G(x_{\mathrm{e}})u_{\mathrm{e}}$, is called a *control law*. If $u(t)=\phi(x(t))$, $t\geq 0$, where $\phi(\cdot)$ is a control law and $x(t)$ satisfies (2), then we call $u(\cdot)$ a *feedback control law*. Note that the feedback control law is an admissible control since $\phi(\cdot)$ has values in U. Given a control law $\phi(\cdot)$ and a feedback control $u(t)=\phi(x(t))$, $t\geq 0$, the closed-loop system (2) and (3) is given by

$$\dot{x}(t)=f(x(t))+G(x(t))\phi(x(t)),\ \ x(0)=x_0,\ \ t\geq 0,\tag{4}$$

$$y(t)=h(x(t)).\tag{5}$$

Given the nonlinear dynamical system (2) and (3) with performance measure

$$J_\varepsilon(x_0,u(\cdot))\triangleq\int_0^\infty[(y(t)-y_{\mathrm{e}})^{\mathrm{T}}(y(t)-y_{\mathrm{e}})+\varepsilon^2(u(t)-u_{\mathrm{e}})^{\mathrm{T}}(u(t)-u_{\mathrm{e}})]\mathrm{d}t,\tag{6}$$

where $\varepsilon>0$, we construct a feedback control law $u(t)=\phi(x(t))$ such that the equilibrium solution $x(t)\equiv x_{\mathrm{e}}$, $t\geq 0$, of (2) and (3) is semistable and

$$J_0(x_0,u(\cdot))\triangleq\lim_{\varepsilon\to 0}\int_0^\infty[(y(t)-y_{\mathrm{e}})^{\mathrm{T}}(y(t)-y_{\mathrm{e}})+\varepsilon^2(u(t)-u_{\mathrm{e}})^{\mathrm{T}}(u(t)-u_{\mathrm{e}})]\mathrm{d}t\tag{7}$$

is minimized in the sense that

$$J_0(x_0,\phi(x(\cdot)))=\min_{u(\cdot)\in\mathcal{S}(x_0)}J_0(x_0,u(\cdot)),\tag{8}$$

where, for every initial condition $x_0\in\mathcal{D}$,

$$\mathcal{S}(x_0)\triangleq\{u(\cdot):u(\cdot)\text{ is admissible and }x(\cdot)\text{ given by (2) is bounded and satisfies }x(t)\to x_{\mathrm{e}}\text{ as }t\to\infty\}$$

denotes the set of convergent controllers.

Theorem 2.1 ([16], [17]): Consider the controlled nonlinear dynamical system (2) and (3) with $u(\cdot)\in\mathcal{S}(x_0)$ and performance measure (6), and assume that there exists a continuously differentiable function $V:\mathcal{D}\to\mathbb{R}$ such that

$$V'(x_{\mathrm{e}})=0,\qquad x_{\mathrm{e}}\in\mathcal{D},\tag{9}$$

$$(y-y_{\mathrm{e}})^{\mathrm{T}}(y-y_{\mathrm{e}})+V'(x)f(x)+V'(x)G(x)u_{\mathrm{e}}-\frac{1}{4\varepsilon^2}V'(x)G(x)G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x)=0,\qquad (x,u_{\mathrm{e}})\in\mathcal{D}\times U.\tag{10}$$

If, with the feedback control

$$u=\phi(x)=-\frac{1}{2\varepsilon^2}G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x)+u_{\mathrm{e}},\tag{11}$$

every equilibrium point $x_{\mathrm{e}}\in F^{-1}(0)=\{x\in\mathcal{D}:f(x)+G(x)\phi(x)=0\}$ of the closed-loop system (4) is Lyapunov stable, then the solution $x(t)=x_{\mathrm{e}}$, $t\geq 0$, of the closed-loop system (4) is semistable and

$$J_\varepsilon(x_0,\phi(x(\cdot)))=V(x_0)-V(x_{\mathrm{e}}).\tag{12}$$

Furthermore, the feedback control (11) minimizes $J_\varepsilon(x_0,u(\cdot))$ in the sense that

$$J_\varepsilon(x_0,\phi(x(\cdot)))=\min_{u(\cdot)\in\mathcal{S}(x_0)}J_\varepsilon(x_0,u(\cdot)).\tag{13}$$

Remark 2.1: Theorem 2.1 requires that $u(\cdot) \in \mathcal{S}(x_0)$ or, equivalently, the solution of the closed-loop system is bounded for all $x \in \mathcal{N}$. One can replace the assumption $u(\cdot) \in \mathcal{S}(x_0)$ in Theorem 2.1 with $u(\cdot)$ being simply admissible and supplementing the conditions of Theorem 2.1 by assuming a nontangency condition of the closed-loop vector field to invariant or negatively invariant subsets of the level sets of $V(\cdot)$ containing the system equilibrium. For details; see [1].

Finally, we address the $\mathcal{H}_2$ optimal semistabilization problem. Consider the linear dynamical system

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0, \qquad t \geq 0, \quad (14)$$

$$y(t) = Cx(t), \quad (15)$$

where, for every $t \geq 0$, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^l$, $0 = Ax_\mathrm{e} + Bu_\mathrm{e}$ for some $u_\mathrm{e} \in \mathbb{R}^m$, $y_\mathrm{e} = Cx_\mathrm{e}$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{l\times n}$. Next, we introduce the definitions of semicontrollability and semiobservability for linear systems.

Definition 2.2 ([7]): Consider the linear system given by (14). The pair (A, B) is *semicontrollable* if

$$\left[\bigcap_{k=1}^{n} \mathcal{N}(B^\mathrm{T}(A^{k-1})^\mathrm{T})\right]^{\perp} = [\mathcal{N}(A^\mathrm{T})]^{\perp}, \quad (16)$$

where $A^0 \triangleq I_n$.

Definition 2.3 ([7]): Consider the linear system given by (14) and (15) with $B = 0$. The pair (A, C) is *semiobservable* if

$$\bigcap_{k=1}^{n} \mathcal{N}(CA^{k-1}) = \mathcal{N}(A). \quad (17)$$

Next, we consider the linear-quadratic regulator problem for semistabilization, that is, we seek controllers $u(\cdot)$ that minimize the performance measure (6) and guarantee semistability of the linear system given by (14) and (15). The feedback gain K that minimizes (6) and guarantees semistability of (14) can be characterized via a solution to a linear matrix inequality [7]. To state this result, recall that the least squares solution to the algebraic Riccati equation

$$0 = A^\mathrm{T}P + PA + C^\mathrm{T}C - \frac{1}{\varepsilon^2}PBB^\mathrm{T}P \quad (18)$$

is defined by

$$P_\mathrm{LS} \triangleq \min_{P\in\mathcal{P}} \|P\|_\mathrm{F}, \quad (19)$$

where $\mathcal{P}$ denotes the set of all P satisfying (18) [7].

Theorem 2.2 ([16], [17]): Consider the linear controlled dynamical system $\mathcal{G}$ given by (14) and (15) with quadratic performance measure (6), assume that the pair (A, B) is semicontrollable and the pair (A, C) is semiobservable, and let $P_\mathrm{LS} = P_\mathrm{LS}^\mathrm{T} \geq 0$ be the least squares solution to the algebraic Riccati equation (18). Then, with $u = Kx = -\frac{1}{\varepsilon^2}B^\mathrm{T}P_\mathrm{LS}x$, the solution $x(t) = x_\mathrm{e}$, $t \geq 0$, to (14) is globally semistable,

$$J_\varepsilon(x_0, K) = x_0^\mathrm{T}\left[\int_0^\infty e^{\tilde{A}^\mathrm{T}t}\left(C^\mathrm{T}C + \varepsilon^2K^\mathrm{T}K\right)e^{\tilde{A}t}\,\mathrm{d}t\right]x_0, \quad (20)$$

where $\tilde{A} = A + BK$, and

$$J_\varepsilon(x_0, K) = \min_{u(\cdot)\in\mathcal{S}(x_0)} J_\varepsilon(x_0, u(\cdot)). \quad (21)$$

III. A Direct Approach to the Optimal Singular Control Problem

In this section, we provide an alternative solution to the semistable optimal singular control problem. Specifically, we apply Theorem 2.1 to the semistable singular control problem and show that this problem can be solved using Hamilton-Jacobi-Bellman-type conditions that do not involve any singularities.

Theorem 3.1: Consider the controlled nonlinear dynamical system (2) and (3) with $u(\cdot) \in \mathcal{S}(x_0)$ and performance measure (7), and assume that there exists a continuously differentiable function $V : \mathcal{D} \to \mathbb{R}$ such that, for all $x \in \mathcal{D}$,

$$(y - y_\mathrm{e})^\mathrm{T}(y - y_\mathrm{e}) - V'(x)G(x)G^\mathrm{T}(x)V'^\mathrm{T}(x) = 0. \quad (22)$$

If, with the feedback control

$$u = \phi_\varepsilon(x) = -\frac{1}{2\varepsilon}G^\mathrm{T}(x)V'^\mathrm{T}(x) + u_\mathrm{e}, \quad (23)$$

every equilibrium point $x_{\mathrm{e}\varepsilon} \in F_\varepsilon^{-1}(0) \triangleq \{x \in \mathcal{D} : f(x) + G(x)\phi_\varepsilon(x) = 0\}$ of the closed-loop system

$$\dot{x}(t) = f(x(t)) + G(x(t))\phi_\varepsilon(x(t)), \;\; x(0) = x_0, \;\; t \geq 0, \quad (24)$$

is Lyapunov stable, then the solution $x(t) = x_{\mathrm{e}0}$, $t \geq 0$, of

$$\dot{x}(t) = f(x(t)) + G(x(t))\phi_0(x(t)), \;\; x(0) = x_0, \;\; t \geq 0, \quad (25)$$

where $\phi_0(x) \triangleq \lim_{\varepsilon\to 0}\phi_\varepsilon(x)$, $x \in \mathcal{D}$, is semistable and

$$J_0(x_0, \phi_0(x(\cdot))) = 0. \quad (26)$$

Furthermore, the feedback control $\phi_0(\cdot)$ minimizes $J_0(x_0, u(\cdot))$ in the sense that

$$J_0(x_0, \phi_0(x(\cdot))) = \min_{u(\cdot)\in\mathcal{S}(x_0)} J_0(x_0, u(\cdot)). \quad (27)$$

Remark 3.1: Since the cost-to-go function $V(\cdot)$ is not required to be sign definite, Theorem 3.1 provides a solution of the nonlinear semistable optimal singular control problem. For nonlinear asymptotic singular stabilization, we require $V(0) = 0$ and $V(x) > 0$, $x \in \mathcal{D}\setminus\{0\}$ (see [14]), and hence, the approach used in Theorem 3.1 cannot be applied to address the nonlinear optimal singular control problem for asymptotic stabilization.

IV. A Feedback Linearization Approach to the Optimal Singular Control Problem

In this section, we provide an alternative approach to the optimal singular control problem for semistabilization based on the notions of output-feedback linearization and feedback equivalence.

A. Feedback Linearization of Nonlinear Dynamical Systems

The following definitions are needed for the main results of this section.

Definition 4.1 ([3]): The *Lie derivative* of the continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ along the vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is defined as

$$L_fV(x) \triangleq V'(x)f(x). \quad (28)$$

The *zeroth-order* and the *higher-order Lie derivatives* are, respectively, defined as

$$L_f^0 V(x) \triangleq V(x), \;\; L_f^k V(x) \triangleq L_f(L_f^{k-1} V(x)), \;\; k \geq 1. \tag{29}$$

For the statement of the next result, consider the nonlinear dynamical system given by (2) with measured output

$$\hat{y}(t) = \hat{h}(x(t)), \tag{30}$$

where $\hat{y}(t) \in \mathbb{R}^m$, $t \geq 0$, $\hat{y}_\mathrm{e} = \hat{h}(x_\mathrm{e})$, and $\hat{h} : \mathcal{D} \to \mathbb{R}^m$ is smooth (i.e., infinitely differentiable) on $\mathcal{D}$.

Definition 4.2 ([3]): Consider the nonlinear dynamical system $\mathcal{G}$ given by (2) and (30), and let $\bar{x} \in \mathcal{D}_0$, where $\mathcal{D}_0 \subseteq \mathcal{D}$ is a neighborhood of $\bar{x}$. If, for all $x \in \mathcal{D}_0$,

$$L_{G_i} L_f^k \hat{h}_j(x) = 0, \;\; 0 \leq k < r_j - 1, \;\; 1 \leq i, j \leq m, \tag{31}$$

and the matrix

$$\mathcal{L}(x) \triangleq \begin{bmatrix} L_{G_1} L_f^{r_1-1} \hat{h}_1(x) & \ldots & L_{G_m} L_f^{r_1-1} \hat{h}_1(x) \\ \vdots & \ddots & \vdots \\ L_{G_1} L_f^{r_m-1} \hat{h}_m(x) & \ldots & L_{G_m} L_f^{r_m-1} \hat{h}_m(x) \end{bmatrix} \tag{32}$$

is nonsingular, then $\mathcal{G}$ has *vector relative degree* $\{r_1, r_2, \ldots, r_m\}$ at $\bar{x}$. Furthermore, if the system $\mathcal{G}$ has vector relative degree $\{r_1, r_2, \ldots, r_m\}$ at every $x \in \mathcal{D}$, then $\mathcal{G}$ has *uniform vector relative degree* $\{r_1, r_2, \ldots, r_m\}$ on $\mathcal{D}$.

The scalars r_i denote the number of times that the outputs $\hat{y}_i$ need to be differentiated at $\bar{x}$ until the input u appears explicitly in (30) [18, p. 221]. Note that if $m = 1$, $L_G L_f^k \hat{h}(x) = 0$, $k < r - 1$, $x \in \mathcal{D}_0$, and $L_G L_f^{r-1} \hat{h}(\bar{x}) \neq 0$, then $G(x)$ is a column vector and the system given by (2) and (30) has relative degree r at $\bar{x}$.

Theorem 4.1 ([18, Prop. 5.1.2]): Assume that the nonlinear dynamical system $\mathcal{G}$ given by (2) and (30) has vector relative degree $\{r_1, r_2, \ldots, r_m\}$ at $\bar{x} \in \mathcal{D}$. Then, there exist a neighborhood $\mathcal{N} \subset \mathcal{D}$ of $\bar{x}$, a diffeomorphism $\mathcal{T} : \mathcal{N} \mapsto \mathbb{R}^n$, and functions $q : \mathbb{R}^r \times \mathbb{R}^{n-r} \to \mathbb{R}^{n-r}$ and $p : \mathbb{R}^r \times \mathbb{R}^{n-r} \to \mathbb{R}^{(n-r)\times m}$ such that, in the coordinates

$$z \triangleq \mathcal{T}(x), \qquad x \in \mathcal{N}, \tag{33}$$

$\mathcal{G}$ is equivalent to

$$\dot{z}^j_{k_j}(t) = z^j_{k_j+1}(t), \;\; z^j_{k_j}(0) = L_f^{k_j-1} \hat{h}_j(x_0), \;\; t \geq 0, \tag{34}$$

$$\dot{z}^j_{r_j}(t) = L_f^{r_j} \hat{h}_j(x(t)) + \sum_{l=1}^{m} L_{G_l} L_f^{r_j-1} \hat{h}_j(x(t)) u_l(t), \quad z^j_{r_j}(0) = L_f^{r_j-1} \hat{h}_j(x_0), \tag{35}$$

$$\dot{\eta}(t) = q(\xi(t), \eta(t)) + p(\xi(t), \eta(t)) u(t), \;\; \eta(0) = \eta_0, \tag{36}$$

for all $j = 1, \ldots, m$ and $k_j = 1, \ldots, r_j - 1$, where

$$z \triangleq [z_1^1, \ldots, z_{r_1}^1, \ldots, z_1^m, \ldots, z_{r_m}^m, z_{r+1}, \ldots, z_n]^\mathrm{T},$$

$r \triangleq \sum_j^m r_j \leq n$, $\xi \triangleq [z_1^1, \ldots, z_{r_1}^1, \ldots, z_1^m, \ldots, z_{r_m}^m]^\mathrm{T}$, $\eta \triangleq [z_{r+1}, \ldots, z_n]^\mathrm{T}$, and $\eta_0 \in \mathbb{R}^{n-r}$ is arbitrary.

Theorem 4.1 does not specify any conditions on $q(\cdot,\cdot)$ and $p(\cdot,\cdot)$ other than the existence of the diffeomorphism $\mathcal{T}$ on $\mathcal{N}$. If $r \triangleq \sum_{i=1}^m r_i = n$, then $z = \xi$ and the condition (36) is superfluous. In this paper, we say that the nonlinear dynamical system (2) and (30) is *equivalent* to the dynamical system (34)–(36) if and only if the hypothesis of Theorem 4.1 hold.

For the statement of the next result, the following additional definitions are required. Consider the linear dynamical system $\mathcal{G}$ given by

$$\dot{z}(t) = Az(t) + Bv(t), \qquad z(0) = z_0, \qquad t \geq 0, \tag{37}$$

$$y(t) = Cz(t) + Dv(t), \tag{38}$$

where $z(t) \in \mathbb{R}^n$, $t \geq 0$, $v(t) \in \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, and $D \in \mathbb{R}^{m\times m}$, and let $G(s) \triangleq C(sI_n - A)^{-1}B + D$, $s \in \mathbb{C}$, be the transfer function of $\mathcal{G}$. Then, recall that $G(s) \in \mathbb{R}^{l\times m}_{\mathrm{prop}}(s)$ is *minimum phase* if and only if the zeros of $G(s)$ are nonnpositive, where the *zeros* of $G(s) \in \mathbb{R}^{l\times m}_{\mathrm{prop}}$ are the roots of the numerator polynomials in the nonzero entries of the Smith-McMillan form of $G(s)$ [9], [22, p. 446]. Furthermore, $G(s)$, $s \in \mathbb{C}$, is *right invertible* if and only if $G(s)$ has full row rank for at least one $s \in \mathbb{C}$ [9].

The following result gives sufficient conditions for constructing a feedback controller $u = \mu(x, v)$ such that the nonlinear dynamical system (2) and (30) is locally output-feedback linear, that is, (2) and (30) is equivalent to the linear dynamical system given by (37) and

$$y(t) = Cz(t). \tag{39}$$

Theorem 4.2: Consider the nonlinear dynamical system $\mathcal{G}$ given by (2) and (30). Assume that $\mathcal{G}$ has uniform vector relative degree $\{r_1, r_2, \ldots, r_m\}$ on $f^{-1}(0) \triangleq \{x \in \mathcal{D} : f(x) = 0\}$ and $r = \sum_{i=1}^m r_i = n$. Then, there exists a neighborhood $\mathcal{N} \subset \mathcal{D}$ of the set $f^{-1}(0) = \{x \in \mathcal{D} : f(x) = 0\}$ such that, for all $x \in \mathcal{N}$, the nonlinear dynamical system $\mathcal{G}$ with

$$u = \mu(x, v) = \mathcal{L}^{-1}(x) \left(-b(z) + \psi(z) + v\right) \tag{40}$$

is equivalent to (37), where z is defined as in Theorem 4.1, $z_0 = [\hat{h}_1(x_0), \ldots, L_f^{r_m-1} \hat{h}_m(x_0)]^\mathrm{T}$, $x_0 \in \mathcal{N}$, $\mathcal{L}(x)$ is given by (32),

$$b(z) \triangleq [L_f^{r_1} \hat{h}_1(x), \ldots, L_f^{r_m} \hat{h}_m(x)]^\mathrm{T}, \tag{41}$$

$\psi : \mathbb{R}^n \to \mathbb{R}^m$ is such that $\psi = [\psi_1(z), \ldots, \psi_m(z)]^\mathrm{T}$ with

$$\psi_j(z) = \sum_{i=1}^{r_j} k_{i,j} z_i^j, \tag{42}$$

$k_{i,j} \in \mathbb{R}$, $i = 1, \ldots, r_j$, $j = 1, \ldots, m$, $A \in \mathbb{R}^{n\times n}$ is a block-diagonal matrix with the jth block given by

$$A^j = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ k_{1,j} & k_{2,j} & \ldots & \ldots & k_{r_j,j} \end{bmatrix}, \tag{43}$$

the entry of B in the $\left(\sum_{i=1}^{j} r_i\right)$th row and jth column is equal to one, and the remaining entries of B are equal to zero. Furthermore, the pair (A, B) is controllable and there exists a matrix $C \in \mathbb{R}^{m\times n}$ such that the pair (A, C) is semiobservable and the transfer function $G(s) \triangleq C(sI_n - A)B$, $s \in \mathbb{C}$, of the linear dynamical system (37) and (39) is minimum phase and right invertible.

B. Singular Control for Linear Semistabilization

In Section IV-A, we give sufficient conditions for the existence of a feedback control $u = \mu(x, v)$ such that the nonlinear dynamical system (2) and (30) is feedback equivalent to (37). In this section, we solve the optimal singular control problem for semistabilization of the linear dynamical system (37) with output (39), that is, we find $K \in \mathbb{R}^{m\times n}$ such that, with $v = Kz$, (37) is semistable and the performance measure

$$J_0(z_0, v(\cdot)) = \lim_{\varepsilon\to 0}\int_0^\infty [(y(t)-y_{\rm e})^{\rm T}(y(t)-y_{\rm e}) + \varepsilon^2(v(t)-v_{\rm e})^{\rm T}(v(t)-v_{\rm e})]{\rm d}t \qquad (44)$$

is minimized in the sense that

$$J_0(z_0, K) = \min_{v(\cdot)\in\mathcal{S}(z_0)} J_0(z_0, v(\cdot)), \qquad (45)$$

where $0 = Az_{\rm e} + Bv_{\rm e}$ for some $v_{\rm e} \in \mathbb{R}^m$, $y_{\rm e} = Cz_{\rm e}$, and

$$\mathcal{S}(z_0) \triangleq \{v(\cdot) : v(\cdot) \text{ is admissible and } z(\cdot) \text{ given by (37) is bounded and satisfies } z(t) \to z_{\rm e} \text{ as } t \to \infty\}.$$

Theorem 4.3: Consider the nonlinear dynamical system (2) with $u(\cdot) \in \mathcal{S}(x_0)$, measured output (30), performance output (39), and performance measure (7). If the hypothesis of Theorem 4.2 hold, then, with

$$\phi(x) = \mathcal{L}^{-1}(x)\left(-b(z) + \psi(z) - \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}B^{\rm T}\begin{bmatrix} I_m & 0\\ 0 & 0\end{bmatrix} z\right), \qquad (46)$$

where $\mathcal{L}(x)$ is given by (32), $b(z)$ is given by (41), $\psi(z) = [\psi_1(z), \ldots, \psi_m(z)]^{\rm T}$, and $\psi_j(z)$, $j = 1, \ldots, m$, is given by (42), the solution $x(t) = x_{\rm e}$, $t \ge 0$, of the closed-loop system (4) and (5) is semistable and

$$\min_{u(\cdot)\in\mathcal{S}(x_0)} J_0(x_0, u(\cdot)) = J_0(x_0, \phi(x(\cdot))) = 0 \qquad (47)$$

for all $x_0 \in \mathcal{N}$, where $\mathcal{N}$ is a neighborhood of the set $f^{-1}(0) = \{x \in \mathcal{D} : f(x) = 0\}$. Furthermore, the feedback control $\phi(\cdot)$ minimizes $J_0(x_0, u(\cdot))$ in the sense that

$$J_0(x_0, \phi(x(\cdot))) = \min_{u(\cdot)\in\mathcal{S}(x_0)} J_0(x_0, u(\cdot)). \qquad (48)$$

V. Illustrative Numerical Example

In this section, we provide a numerical example to highlight the optimal singular semistabilization frameworks developed in this paper. Let $\theta \triangleq [\theta_x, \theta_y, \theta_z]^{\rm T} \in \mathbb{R}^3$ and $\eta \in \mathbb{R}$ denote the vector and scalar Euler parameters respectively, let ω_1, ω_2, and $\omega_3 \in \mathbb{R}$ denote the components of the angular velocity vector with respect to a given inertial reference frame expressed in a central body reference frame, and let u_1, u_2, and $u_3 \in \mathbb{R}$ denote the control torques about the body center of mass. The Euler parameters are such that

$$\theta_x^2(t) + \theta_y^2(t) + \theta_z^2(t) + \eta^2(t) = 1, \qquad t \ge 0, \qquad (49)$$

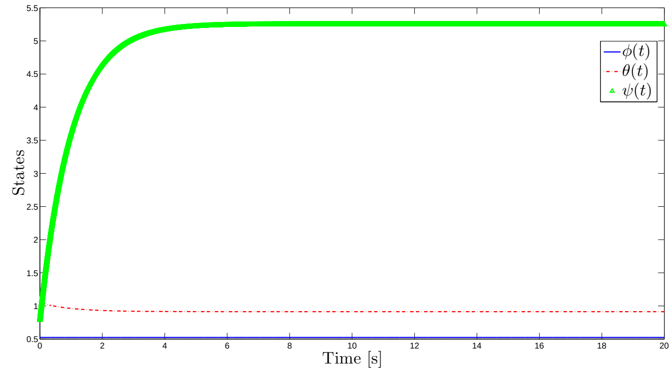

Fig. 1. Closed-loop system trajectories versus time.

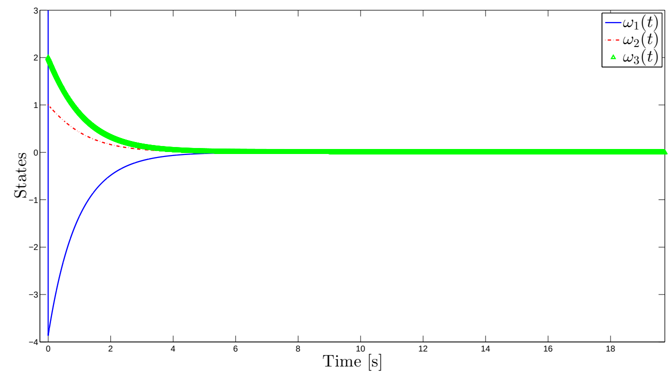

Fig. 2. Closed-loop system angular velocities versus time.

and, for all $t \ge 0$, the rotational equations of motion for the rigid body are given by [23]

$$\begin{bmatrix}\dot\theta_x(t)\\ \dot\theta_y(t)\\ \dot\theta_z(t)\\ \dot\eta(t)\\ \dot\omega_1(t)\\ \dot\omega_2(t)\\ \dot\omega_3(t)\end{bmatrix} = \frac{1}{2}\begin{bmatrix}\theta_y(t)\omega_3(t) - \theta_z(t)\omega_2(t) + \eta(t)\omega_1(t)\\ \theta_z(t)\omega_1(t) - \theta_x(t)\omega_3(t) + \eta(t)\omega_2(t)\\ \theta_x(t)\omega_2(t) - \theta_y(t)\omega_1(t) + \eta(t)\omega_3(t)\\ -\theta_x(t)\omega_1(t) - \theta_y(t)\omega_2(t) - \theta_z(t)\omega_3(t)\\ I_{23}\omega_2(t)\omega_3(t)\\ -I_{31}\omega_1(t)\omega_3(t)\\ I_{12}\omega_1(t)\omega_2(t)\end{bmatrix} + \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\\0&0&0\\ \frac{1}{I_1}&0&0\\ 0&\frac{1}{I_2}&0\\ 0&0&\frac{1}{I_3}\end{bmatrix}u(t), \quad \begin{bmatrix}\theta_x(0)\\ \theta_y(0)\\ \theta_z(0)\\ \eta(0)\\ \omega_1(0)\\ \omega_2(0)\\ \omega_3(0)\end{bmatrix} = \begin{bmatrix}\theta_{x0}\\ \theta_{y0}\\ \theta_{z0}\\ \eta_0\\ \omega_{10}\\ \omega_{20}\\ \omega_{30}\end{bmatrix}, \qquad (50)$$

where I_1, I_2, and I_3 are the principal moments of inertia, $I_1 \ge I_2 \ge I_3 > 0$, $u = [u_1, u_2, u_3]^{\rm T}$, $I_{23} = (I_2 - I_3)/I_1$, $I_{31} = (I_3 - I_1)/I_2$, and $I_{12} = (I_1 - I_2)/I_3$.

For $x = [\theta_x, \theta_y, \theta_z, \omega_1, \omega_2, \omega_3]^{\rm T}$ and measured output

$$\begin{bmatrix}\hat y_1(t)\\ \hat y_2(t)\\ \hat y_3(t)\end{bmatrix} = \begin{bmatrix}\theta_x(t)\\ \theta_y(t)\\ \theta_z(t)\end{bmatrix}, \qquad (51)$$

the affine nonlinear dynamical system given by (50), (49), and (51) is in the same form of (2) and (30) with $n = 6$

and $m = 3$. In this case, $L_{G_i}\hat{h}_j = 0$, $i, j \in \{1, 2, 3\}$, (32) specializes to

$$\mathcal{L}(x) = \frac{1}{2}\begin{bmatrix} \frac{\eta}{I_1} & -\frac{\theta_z}{I_2} & \frac{\theta_y}{I_3} \\ \frac{\theta_z}{I_1} & \frac{\eta}{I_2} & -\frac{\theta_x}{I_3} \\ -\frac{\theta_y}{I_1} & \frac{\theta_x}{I_2} & \frac{\eta}{I_3} \end{bmatrix}, \tag{52}$$

the dynamical system given by (50), (49), and (51) has vector relative degree $\{2, 2, 2\}$ on $\{x \in \mathbb{R}^6 : \eta \neq 0\}$ [24], and $r = \sum_{i=1}^{3} r_i = 6$. Since all of the conditions of Theorem 4.2 are satisfied, if $\eta \neq 0$, then the nonlinear dynamical system given by (50), (49), and (51) with feedback (40) is equivalent to the linear dynamical system (37), where $k_{i,j} < 0$, $i = 1, 2$ and $j = 1, 2, 3$, $z = [\hat{y}_1, \dot{\hat{y}}_1, \hat{y}_2, \dot{\hat{y}}_2, \hat{y}_3, \dot{\hat{y}}_3]^{\mathrm{T}}$, $v \in \mathbb{R}^3$, $A^j = \begin{bmatrix} 0 & 1 \\ k_{1,j} & k_{2,j} \end{bmatrix}$, and

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{6\times 3}. \tag{53}$$

The vector $b(z)$ given by (41) is omitted for conciseness.

Setting $k_{1,1} = k_{1,2} = k_{1,3} = 0$ and $k_{2,1} = k_{2,2} = k_{2,3} = -1$, the pair (A, B) is controllable and setting $C = B^{\mathrm{T}}$, the pair (A, C) is semiobservable. In this case, the transfer function of the linear dynamical system (37) and (39), where

$$y(t) = h(x(t)) = \left[\dot{\theta}_x(t), \dot{\theta}_y(t), \dot{\theta}_z(t)\right]^{\mathrm{T}}, \tag{54}$$

is given by

$$G(s) = C(sI_6 - A)B = \frac{1}{s+1} I_3, \qquad s \in \mathbb{C}, \tag{55}$$

which is minimum phase and right invertible. Hence, it follows from Theorem 4.3 that with $u = \phi(x)$ given by (46), the solution $x(t) = x_{\mathrm{e}}$, $t \geq 0$, of the closed-loop system (50) and (54) is semistable, and (47) is satisfied.

Let $I_1 = 20\,\mathrm{kg{\cdot}m^2}$, $I_2 = 15\,\mathrm{kg{\cdot}m^2}$, $I_3 = 10\,\mathrm{kg{\cdot}m^2}$, $\theta_{x0} = 0.20$, $\theta_{y0} = 0.53$, $\theta_{z0} = 0.02$, $\omega_{10} = 3\,\mathrm{Hz}$, $\omega_{20} = 1\,\mathrm{Hz}$, and $\omega_{30} = 2\,\mathrm{Hz}$. Figures 1 and 2 show the state trajectories of the controlled system versus time. Note that $x(t) \to x_{\mathrm{e}} = [0.2007,\, 0.7758,\, 0.1468\, 0,\, 0,\, 0]^{\mathrm{T}}$ as $t \to \infty$.

VI. Conclusion

In this paper, we presented two approaches to address the optimal singular control problem for semistabilization of affine nonlinear dynamical systems. Specifically, using the fact that the cost-to-go function that solves the Hamilton-Jacobi-Bellman equation for semistabilization is not required to be sign definite, we solved the nonlinear semistable optimal singular control problem applying the results in [16], [17] for optimal semistabilization. Finally, we addressed the optimal singular control problem for semistabilization using differential geometric methods and the concepts of state-feedback linearization and feedback equivalence.

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