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# Partial-state stabilization and optimal feedback control

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## SUMMARY

In this paper, we develop a unified framework to address the problem of optimal nonlinear analysis and feedback control for partial stability and partial-state stabilization. Partial asymptotic stability of the closed-loop nonlinear system is guaranteed by means of a Lyapunov function that is positive definite and decrescent with respect to part of the system state, which can clearly be seen to be the solution to the steady-state form of the Hamilton–Jacobi–Bellman equation and hence guaranteeing both partial stability and optimality. The overall framework provides the foundation for extending optimal linear-quadratic controller synthesis to nonlinear nonquadratic optimal partial-state stabilization. Connections to optimal linear and nonlinear regulation for linear and nonlinear time-varying systems with quadratic and nonlinear nonquadratic cost functionals are also provided. Finally, we also develop optimal feedback controllers for affine nonlinear systems using an inverse optimality framework tailored to the partial-state stabilization problem and use this result to address polynomial and multilinear forms in the performance criterion. Copyright © 2015 John Wiley & Sons, Ltd.

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## 1. INTRODUCTION

In [1], the current status of continuous-time, nonlinear nonquadratic optimal control problems was presented in a simplified and tutorial manner. The basic underlying ideas of the results in [1] are based on the fact that the steady-state solution of the Hamilton–Jacobi–Bellman equation is a Lyapunov function for the nonlinear system and thus guaranteeing both stability and optimality [1, 2]. Specifically, a feedback control problem over an infinite horizon involving a nonlinear nonquadratic performance functional is considered. The performance functional is then evaluated in closed form as long as the nonlinear nonquadratic cost functional considered is related in a specific way to an underlying Lyapunov function that guarantees asymptotic stability of the nonlinear closed-loop system. This Lyapunov function is shown to be the solution of the steady-state Hamilton–Jacobi–Bellman equation. The overall framework provides the foundation for extending linear-quadratic control to nonlinear nonquadratic problems.

In this paper, we extend the framework developed in [1] and [2] to address the problem of optimal *partial-state stabilization*, wherein stabilization with respect to a subset of the system state variables is desired. Partial-state stabilization arises in many engineering applications [3, 4]. Specifically, in spacecraft stabilization via gimballed gyroscopes, asymptotic stability of an equilibrium position of the spacecraft is sought while requiring Lyapunov stability of the axis of the gyroscope relative to

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the spacecraft [4]. Alternatively, in the control of rotating machinery with mass imbalance, spin stabilization about a non-principal axis of inertia requires motion stabilization with respect to a subspace instead of the origin [3]. Perhaps the most common application where partial stabilization is necessary is adaptive control, wherein asymptotic stability of the closed-loop plant states is guaranteed without necessarily achieving parameter error convergence. The need to consider partial stability of the closed-loop system in the aforementioned systems arises from the fact that stability notions involve equilibrium coordinates as well as a manifold of coordinates that is closed but *not* compact. Hence, partial stability involves motion lying in a subspace instead of an equilibrium point.

Even though partial-state stabilization has been considered in the literature [3–5], the problem of optimal partial-state stabilization has received very little attention. In this paper, we consider a notion of optimality that is directly related to a given Lyapunov function that is positive definite and decrescent with respect to part of the system state. Specifically, an optimal partial-state stabilization control problem is stated, and sufficient Hamilton–Jacobi–Bellman conditions are used to characterize an optimal feedback controller. Another important application of partial stability and partial stabilization theory is the unification it provides between time-invariant stability theory and stability theory for time-varying systems [2, 6]. We exploit this unification and specialize our results to address optimal linear and nonlinear regulation for linear and nonlinear time-varying systems with quadratic and nonlinear nonquadratic cost functionals.

The contents of this paper are as follows. In Section 2, we establish notation, definitions, and recall some basic results on partial stability of nonlinear dynamical systems. In Section 3, we consider a nonlinear system with a performance functional evaluated over the infinite horizon. The performance functional is then evaluated in terms of a Lyapunov function that guarantees partial asymptotic stability. We then state an optimal control problem and provide sufficient conditions for characterizing an optimal nonlinear feedback controller guaranteeing partial asymptotic stability of the closed-loop system. These results are then used to address an optimal control problem for uniform asymptotic stabilization of nonlinear time-varying dynamical systems. In Section 4, we specialize the results developed in Section 3 to affine in the control dynamical systems as well as provide connections to the time-varying, linear-quadratic regulator problem [7]. In Section 5, we develop optimal feedback controllers for affine nonlinear systems using an inverse optimality framework tailored to the partial-state stabilization problem. This result is then used to derive time-varying extensions of the results in [8, 9] involving nonlinear feedback controllers minimizing polynomial and multilinear performance criteria. In Section 6, we provide several illustrative numerical examples that highlight the optimal partial-state stabilization framework. Finally, in Section 7, we present conclusions and highlight some future research directions.

## 2. NOTATION, DEFINITIONS, AND MATHEMATICAL PRELIMINARIES

In this section, we establish notation, definitions, and review some basic results on partial stability [2]. Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}^n$ denote the set of $n \times 1$ real column vectors, and $\mathbb{R}^{n\times m}$ denote the set of $n \times m$ real matrices. We write $V'(x) \triangleq \frac{\partial V(x)}{\partial x}$ for the Fréchet derivative of $V$ at $x$, $\|\cdot\|$ for the Euclidean vector norm, $A^{\mathrm{T}}$ for the transpose of the matrix $A$, $\otimes$ for the Kronecker product, $\oplus$ for the Kronecker sum, $I_n$ or $I$ for the $n \times n$ identity matrix, and $0_{n\times m}$ or $0$ for the zero $n \times m$ matrix.

In this paper, we consider nonlinear autonomous dynamical systems of the form

$$\dot{x}_1(t) = f_1(x_1(t), x_2(t)), \qquad x_1(0) = x_{10}, \qquad t \geqslant 0, \tag{1}$$

$$\dot{x}_2(t) = f_2(x_1(t), x_2(t)), \qquad x_2(0) = x_{20}, \tag{2}$$

where, for every $t \geqslant 0$, $x_1(t) \in \mathcal{D} \subseteq \mathbb{R}^{n_1}$ and $x_2(t) \in \mathbb{R}^{n_2}$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $f_1 : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1}$ is such that for every $x_2 \in \mathbb{R}^{n_2}$, $f_1(0, x_2) = 0$ and $f_1(\cdot, x_2)$ is locally Lipschitz continuous in $x_1$; and $f_2 : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$ is such that for every $x_1 \in \mathcal{D}$, $f_2(x_1, \cdot)$ is locally Lipschitz continuous in $x_2$.



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*Definition 2.1 (*[2, Def. 4.1]*)*

(i) The nonlinear dynamical system $\mathcal{G}$ given by (1) and (2) is Lyapunov stable with respect to $x_1$ uniformly in $x_{20}$ if, for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $\|x_{10}\| \leqslant \delta$ implies that $\|x_1(t)\| < \varepsilon$ for all $t \geqslant 0$ and for all $x_{20} \in \mathbb{R}^{n_2}$.

(ii) $\mathcal{G}$ is asymptotically stable with respect to $x_1$ uniformly in $x_{20}$ if $\mathcal{G}$ is Lyapunov stable with respect to $x_1$ uniformly in $x_{20}$, and there exists $\delta > 0$ such that $\|x_{10}\| < \delta$ implies that $\lim_{t\to\infty} x_1(t) = 0$ uniformly in $x_{10}$ and $x_{20}$ for all $x_{20} \in \mathbb{R}^{n_2}$.

(iii) $\mathcal{G}$ is globally asymptotically stable with respect to $x_1$ uniformly in $x_{20}$ if $\mathcal{G}$ is Lyapunov stable with respect to $x_1$ uniformly in $x_{20}$ and $\lim_{t\to\infty} x_1(t) = 0$ uniformly in $x_{10}$ and $x_{20}$ for all $x_{10} \in \mathbb{R}^{n_1}$ and $x_{20} \in \mathbb{R}^{n_2}$.

*Remark 2.1*

It is important to note that there is a key difference between the partial stability definitions given in Definition 2.1 and the definitions of partial stability given in [4]. In particular, the partial stability definitions given in [4] require that both the initial conditions $x_{10}$ and $x_{20}$ lie in a neighborhood of the origin, whereas in Definition 2.1, $x_{20}$ can be arbitrary. As will be seen in the succeeding text, this difference allows us to unify autonomous partial stability theory with time-varying stability theory. An additional difference between our formulation of the partial stability problem and the partial stability problem considered in [4] is in the treatment of the equilibrium of (1) and (2). Specifically, in our formulation, we require the weaker partial equilibrium condition $f_1(0, x_2) = 0$ for every $x_2 \in \mathbb{R}^{n_2}$, whereas in [4], the author requires the stronger equilibrium condition $f_1(0,0) = 0$ and $f_2(0,0) = 0$.

As shown in [2] and [6], an important application of partial stability theory is the unification it provides between time-invariant stability theory and stability theory for time-varying systems. Specifically, consider the nonlinear time-varying dynamical system given by

$$\dot{x}(t) = f(t, x(t)), \qquad x(t_0) = x_0, \qquad t \geqslant t_0, \tag{3}$$

where, for every $t \geqslant t_0$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $f(t,0) = 0$, $f : [t_0,\infty) \times \mathcal{D} \to \mathbb{R}^n$ is jointly continuous in $t$ and $x$, and $f(t,\cdot)$ is locally Lipschitz continuous in $x$ uniformly in $t$ for all $t$ in compact subsets of $[t_0,\infty)$. Now, defining $x_1(\tau) \triangleq x(t)$ and $x_2(\tau) \triangleq t$, where $\tau \triangleq t - t_0$, it follows that the solution $x(t)$, $t \geqslant t_0$, to the nonlinear time-varying dynamical system (3) can be equivalently characterized by the solution $x_1(\tau)$, $\tau \geqslant 0$, to the nonlinear autonomous dynamical system

$$\dot{x}_1(\tau) = f(x_2(\tau), x_1(\tau)), \qquad x_1(0) = x_0, \qquad \tau \geqslant 0, \tag{4}$$

$$\dot{x}_2(\tau) = 1, \qquad x_2(0) = t_0. \tag{5}$$

Note that (4) and (5) are in the same form as the system given by (1) and (2), and Definition 2.1 applied to (4) and (5) specializes to the definitions of uniform Lyapunov stability, uniform asymptotic stability, and global uniform asymptotic stability of (3); for details, see [2, Def. 4.2].

Next, we provide sufficient conditions for partial stability of the nonlinear dynamical system given by (1) and (2). For the statement of the following result, define

$$\dot{V}(x_1, x_2) \triangleq V'(x_1, x_2) f(x_1, x_2),$$

where $f(x_1, x_2) \triangleq [f_1^{\mathrm{T}}(x_1, x_2), f_2^{\mathrm{T}}(x_1, x_2)]^{\mathrm{T}}$, for a continuously differentiable function $V : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$.

 
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*Theorem 2.1* ([2, Th. 4.1])
Consider the nonlinear dynamical system (1) and (2). Then the following statements hold:

(i) If there exist a continuously differentiable function $V : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$ and class $\mathcal{K}$ functions $\alpha(\cdot)$, $\beta(\cdot)$, and $\gamma(\cdot)$ such that

$$\alpha(\|x_1\|) \leqslant V(x_1, x_2) \leqslant \beta(\|x_1\|), \qquad (x_1, x_2) \in \mathcal{D} \times \mathbb{R}^{n_2}, \tag{6}$$

$$\dot{V}(x_1, x_2) \leqslant -\gamma(\|x_1\|), \qquad (x_1, x_2) \in \mathcal{D} \times \mathbb{R}^{n_2}, \tag{7}$$

then the nonlinear dynamical system given by (1) and (2) is asymptotically stable with respect to $x_1$ uniformly in $x_{20}$.

(ii) If there exist a continuously differentiable function $V : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, a class $\mathcal{K}$ function $\gamma(\cdot)$, and class $\mathcal{K}_\infty$ functions $\alpha(\cdot)$ and $\beta(\cdot)$ satisfying (6) and (7), then the nonlinear dynamical system given by (1) and (2) is globally asymptotically stable with respect to $x_1$ uniformly in $x_{20}$.

## 3. OPTIMAL PARTIAL-STATE STABILIZATION

In the first part of this section, we provide connections between Lyapunov functions and nonquadratic cost evaluation. Specifically, we consider the problem of evaluating a nonlinear nonquadratic performance measure that depends on the solution of the nonlinear dynamical system given by (1) and (2). In particular, we show that the nonlinear nonquadratic performance measure

$$J(x_{10}, x_{20}) \triangleq \int_0^\infty L(x_1(t), x_2(t))\mathrm{d}t, \tag{8}$$

where $L : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$ is jointly continuous in $x_1$ and $x_2$, and $x_1(t)$ and $x_2(t)$, $t \geqslant 0$, satisfy (1) and (2), can be evaluated in a convenient form so long as (1) and (2) are related to an underlying Lyapunov function that is positive definite and decrescent with respect to $x_1$ and proves asymptotic stability of (1) and (2) with respect to $x_1$ uniformly in $x_{20}$.

*Theorem 3.1*
Consider the nonlinear dynamical system $\mathcal{G}$ given by (1) and (2) with performance measure (8). Assume that there exist a continuously differentiable function $V : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$ and class $\mathcal{K}$ functions $\alpha(\cdot)$, $\beta(\cdot)$, and $\gamma(\cdot)$ such that

$$\alpha(\|x_1\|) \leqslant V(x_1, x_2) \leqslant \beta(\|x_1\|), \qquad (x_1, x_2) \in \mathcal{D} \times \mathbb{R}^{n_2}, \tag{9}$$

$$\dot{V}(x_1, x_2) \leqslant -\gamma(\|x_1\|), \qquad (x_1, x_2) \in \mathcal{D} \times \mathbb{R}^{n_2}, \tag{10}$$

$$L(x_1, x_2) + V'(x_1, x_2) f(x_1, x_2) = 0, \qquad (x_1, x_2) \in \mathcal{D} \times \mathbb{R}^{n_2}. \tag{11}$$

Then the nonlinear dynamical system $\mathcal{G}$ is asymptotically stable with respect to $x_1$ uniformly in $x_{20}$, and there exists a neighborhood $\mathcal{D}_0 \subseteq \mathcal{D}$ of $x_1 = 0$ such that, for all $(x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}$,

$$J(x_{10}, x_{20}) = V(x_{10}, x_{20}). \tag{12}$$

Finally, if $\mathcal{D} = \mathbb{R}^{n_1}$, and the functions $\alpha(\cdot)$ and $\beta(\cdot)$ satisfying (9) are class $\mathcal{K}_\infty$, then $\mathcal{G}$ is globally asymptotically stable with respect to $x_1$ uniformly in $x_{20}$.

*Proof*
Let $x_1(t)$ and $x_2(t)$, $t \geqslant 0$, satisfy (1) and (2). Then it follows from (10) that

$$\dot{V}(x_1(t), x_2(t)) = V'(x_1(t), x_2(t)) f(x_1(t), x_2(t)) \leqslant -\gamma(\|x_1(t)\|), \qquad t \geqslant 0. \tag{13}$$



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Thus, it follows from (9), (10), and (i) of Theorem 2.1 that $\mathcal{G}$ is asymptotically stable with respect to $x_1$ uniformly in $x_{20}$. Consequently, $x_1(t) \to 0$ as $t \to \infty$ for all initial conditions $(x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}$ for some neighborhood $\mathcal{D}_0 \subseteq \mathcal{D}$ of $x_1 = 0$. Now, because

$$0 = -\dot{V}(x_1(t), x_2(t)) + V'(x_1(t), x_2(t)) f(x_1(t), x_2(t)), \qquad t \geqslant 0, \tag{14}$$

it follows from (11) that

$$\begin{aligned} L(x_1(t), x_2(t)) &= -\dot{V}(x_1(t), x_2(t)) + L(x_1(t), x_2(t)) + V'(x_1(t), x_2(t)) f(x_1(t), x_2(t)) \\ &= -\dot{V}(x_1(t), x_2(t)), \qquad t \geqslant 0. \end{aligned} \tag{15}$$

Next, integrating (15) over $[0, t]$ yields

$$\int_0^t L(x_1(s), x_2(s))\mathrm{d}s = V(x_{10}, x_{20}) - V(x_1(t), x_2(t)), \qquad t \geqslant 0. \tag{16}$$

Now, using (9) and letting $t \to \infty$, it follows from (16) that

$$V(x_{10}, x_{20}) - \beta\left(\lim_{t\to\infty} \|x_1(t)\|\right) \leqslant \int_0^\infty L(x_1(s), x_2(s))\mathrm{d}s \leqslant V(x_{10}, x_{20}) - \alpha\left(\lim_{t\to\infty} \|x_1(t)\|\right), \tag{17}$$

and hence, (12) is a direct consequence of (17) using the fact that $\lim_{t\to\infty} x_1(t) = 0$, and $\alpha(\cdot)$ and $\beta(\cdot)$ are class $\mathcal{K}$ functions. Finally, if $\mathcal{D} = \mathbb{R}^{n_1}$, and $\alpha(\cdot)$ and $\beta(\cdot)$ are class $\mathcal{K}_\infty$ functions, then global asymptotic stability with respect to $x_1$ uniformly in $x_{20}$ is a direct consequence of (ii) of Theorem 2.1. □

The following corollary to Theorem 3.1 considers the non-autonomous dynamical system (3) with performance measure

$$J(t_0, x_0) \triangleq \int_{t_0}^\infty L(t, x(t))\mathrm{d}t, \tag{18}$$

where $L : [t_0, \infty) \times \mathcal{D} \to \mathbb{R}$ is jointly continuous in $t$ and $x$; and $x(t)$, $t \geqslant t_0$, satisfies (3).

*Corollary 3.1*
Consider the nonlinear time-varying dynamical system (3) with performance measure (18). Assume that there exist a continuously differentiable function $V : [t_0, \infty) \times \mathcal{D} \to \mathbb{R}$ and class $\mathcal{K}$ functions $\alpha(\cdot)$, $\beta(\cdot)$, and $\gamma(\cdot)$ such that

$$\alpha(\|x\|) \leqslant V(t, x) \leqslant \beta(\|x\|), \qquad (t, x) \in [t_0, \infty) \times \mathcal{D}, \tag{19}$$

$$\dot{V}(t, x) \leqslant -\gamma(\|x\|), \qquad (t, x) \in [t_0, \infty) \times \mathcal{D}, \tag{20}$$

$$-\frac{\partial V(t, x)}{\partial t} = L(t, x) + \frac{\partial V(t, x)}{\partial x} f(t, x), \qquad (t, x) \in [t_0, \infty) \times \mathcal{D}. \tag{21}$$

Then the nonlinear dynamical system (3) is uniformly asymptotically stable, and there exists a neighborhood of the origin $\mathcal{D}_0 \subseteq \mathcal{D}$ such that, for all $(t_0, x_0) \in [0, \infty) \times \mathcal{D}_0$,

$$J(t_0, x_0) = V(t_0, x_0). \tag{22}$$

Finally, if $\mathcal{D} = \mathbb{R}^n$, and the functions $\alpha(\cdot)$ and $\beta(\cdot)$ satisfying (19) are class $\mathcal{K}_\infty$, then the nonlinear dynamical system (3) is globally uniformly asymptotically stable.

*Proof*
The result is a direct consequence of Theorem 3.1 with $n_1 = n$, $n_2 = 1$, $x_1(t - t_0) = x(t)$, $x_2(t - t_0) = t$, $f_1(x_1, x_2) = f_1(x_2, x_1) = f(t, x)$, $f_2(x_1, x_2) = 1$, and $V(x_1, x_2) = V(x_2, x_1) = V(t, x)$. □



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Next, we use the framework developed in Theorem 3.1 to obtain a characterization of optimal feedback controllers that guarantee closed-loop, partial-state stabilization. Specifically, sufficient conditions for optimality are given in a form that corresponds to a steady-state version of the Hamilton–Jacobi–Bellman equation. To address the problem of characterizing partially stabilizing feedback controllers, consider the controlled nonlinear dynamical system

$$\dot{x}_1(t) = F_1(x_1(t), x_2(t), u(t)), \qquad x_1(0) = x_{10}, \qquad t \geqslant 0, \tag{23}$$

$$\dot{x}_2(t) = F_2(x_1(t), x_2(t), u(t)), \qquad x_2(0) = x_{20}, \tag{24}$$

where, for every $t \geqslant 0$, $x_1(t) \in \mathcal{D} \subseteq \mathbb{R}^{n_1}$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $x_2(t) \in \mathbb{R}^{n_2}$, $u(t) \in U \subseteq \mathbb{R}^m$ with $0 \in U$; $F_1 : \mathcal{D} \times \mathbb{R}^{n_2} \times U \to \mathbb{R}^{n_1}$ and $F_2 : \mathcal{D} \times \mathbb{R}^{n_2} \times U \to \mathbb{R}^{n_2}$ are locally Lipschitz continuous in $x_1$, $x_2$, and $u$; and $F_1(0, x_2, 0) = 0$ for every $x_2 \in \mathbb{R}^{n_2}$. The control $u(\cdot)$ in (23) and (24) is restricted to the class of *admissible* controls consisting of measurable functions $u(\cdot)$ such that $u(t) \in U$, $t \geqslant 0$.

A measurable function $\phi : \mathcal{D} \times \mathbb{R}^{n_2} \to U$ satisfying $\phi(0, x_2) = 0$, $x_2 \in \mathbb{R}^{n_2}$, is called a *control law*. If $u(t) = \phi(x_1(t), x_2(t))$, $t \geqslant 0$, where $\phi(\cdot,\cdot)$ is a control law, and $x_1(t)$ and $x_2(t)$ satisfy (23) and (24), then we call $u(\cdot)$ a *feedback control law*. Note that the feedback control law is an admissible control because $\phi(\cdot,\cdot)$ has values in $U$. Given a control law $\phi(\cdot,\cdot)$ and a feedback control law $u(t) = \phi(x_1(t), x_2(t))$, $t \geqslant 0$, the *closed-loop system* (23) and (24) is given by

$$\dot{x}_1(t) = F_1(x_1(t), x_2(t), \phi(x_1(t), x_2(t))), \qquad x_1(0) = x_{10}, \qquad t \geqslant 0, \tag{25}$$

$$\dot{x}_2(t) = F_2(x_1(t), x_2(t), \phi(x_1(t), x_2(t))), \qquad x_2(0) = x_{20}. \tag{26}$$

We now consider the problem of partial-state stabilization.

*Definition 3.1*
Consider the controlled dynamical system given by (23) and (24). The feedback control law $u = \phi(x_1, x_2)$ is asymptotically stabilizing with respect to $x_1$ uniformly in $x_{20}$ if the closed-loop system (25) and (26) is asymptotically stable with respect to $x_1$ uniformly in $x_{20}$. Furthermore, the feedback control law $u = \phi(x_1, x_2)$ is globally asymptotically stabilizing with respect to $x_1$ uniformly in $x_{20}$ if the closed-loop system (25) and (26) is globally asymptotically stable with respect to $x_1$ uniformly in $x_{20}$.

Next, we present the main theorem for partial-state stabilization characterizing feedback controllers that guarantee partial closed-loop stability and minimize a nonlinear nonquadratic performance functional. For the statement of this result, define $F(x_1, x_2, u) \triangleq [F_1^{\mathrm{T}}(x_1, x_2, u), F_2^{\mathrm{T}}(x_1, x_2, u)]^{\mathrm{T}}$, let $L : \mathcal{D} \times \mathbb{R}^{n_2} \times U \to \mathbb{R}$ be jointly continuous in $x_1$, $x_2$, and $u$; and define the set of partial regulation controllers given by

$$\mathcal{S}(x_{10}, x_{20}) \triangleq \{u(\cdot) : u(\cdot) \text{ is admissible, and } x_1(\cdot) \text{ given by (23) satisfies } x_1(t) \to 0 \text{ as } t \to \infty\}.$$

Note that restricting our minimization problem to $u(\cdot) \in \mathcal{S}(x_{10}, x_{20})$, that is, inputs corresponding to partial-state null convergent solutions, can be interpreted as incorporating a partial-state system detectability condition through the cost.

*Theorem 3.2*
Consider the controlled nonlinear dynamical system $\mathcal{G}$ given by (23) and (24) with

$$J(x_{10}, x_{20}, u(\cdot)) \triangleq \int_0^\infty L(x_1(t), x_2(t), u(t))\mathrm{d}t, \tag{27}$$

where $u(\cdot)$ is an admissible control. Assume that there exist a continuously differentiable function $V : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$; class $\mathcal{K}$ functions $\alpha(\cdot)$, $\beta(\cdot)$, and $\gamma(\cdot)$; and a control law $\phi : \mathcal{D} \times \mathbb{R}^{n_2} \to U$ such that

$$\alpha(\|x_1\|) \leqslant V(x_1, x_2) \leqslant \beta(\|x_1\|), \qquad (x_1, x_2) \in \mathcal{D} \times \mathbb{R}^{n_2}, \tag{28}$$



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$$V'(x_1, x_2)F(x_1, x_2, \phi(x_1, x_2)) \leqslant -\gamma(\|x_1\|), \qquad (x_1, x_2) \in \mathcal{D} \times \mathbb{R}^{n_2}, \tag{29}$$

$$\phi(0, x_2) = 0, \qquad x_2 \in \mathbb{R}^{n_2}, \tag{30}$$

$$L(x_1, x_2, \phi(x_1, x_2)) + V'(x_1, x_2)F(x_1, x_2, \phi(x_1, x_2)) = 0, \qquad (x_1, x_2) \in \mathcal{D} \times \mathbb{R}^{n_2}, \tag{31}$$

$$L(x_1, x_2, u) + V'(x_1, x_2)F(x_1, x_2, u) \geqslant 0, \qquad (x_1, x_2, u) \in \mathcal{D} \times \mathbb{R}^{n_2} \times U. \tag{32}$$

Then, with the feedback control $u = \phi(x_1, x_2)$, the closed-loop system given by (25) and (26) is asymptotically stable with respect to $x_1$ uniformly in $x_{20}$, and there exists a neighborhood $\mathcal{D}_0 \subseteq \mathcal{D}$ of $x_1 = 0$ such that

$$J(x_{10}, x_{20}, \phi(x_1(\cdot), x_2(\cdot))) = V(x_{10}, x_{20}), \qquad (x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}. \tag{33}$$

In addition, if $(x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}$, then the feedback control $u(\cdot) = \phi(x_1(\cdot), x_2(\cdot))$ minimizes $J(x_{10}, x_{20}, u(\cdot))$ in the sense that

$$J(x_{10}, x_{20}, \phi(\cdot, \cdot)) = \min_{u(\cdot) \in \mathcal{S}(x_{10}, x_{20})} J(x_{10}, x_{20}, u(\cdot)). \tag{34}$$

Finally, if $\mathcal{D} = \mathbb{R}^{n_1}$, $U = \mathbb{R}^m$, and the functions $\alpha(\cdot)$ and $\beta(\cdot)$ satisfying (28) are class $\mathcal{K}_\infty$, then the closed-loop system (25) and (26) is globally asymptotically stable with respect to $x_1$ uniformly in $x_{20}$.

*Proof*

Local and global asymptotic stability with respect to $x_1$ uniformly in $x_{20}$ are a direct consequence of (28) and (29) by applying Theorem 2.1 to the closed-loop system given by (25) and (26). Furthermore, using (31), condition (33) is a restatement of (12) as applied to the closed-loop system.

Next, let $(x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}$, let $u(\cdot) \in \mathcal{S}(x_{10}, x_{20})$, and let $x_1(t)$ and $x_2(t)$, $t \geqslant 0$, be solutions of (23) and (24). Then, it follows that

$$0 = -\dot{V}(x_1(t), x_2(t)) + V'(x_1(t), x_2(t))F(x_1(t), x_2(t), u(t)), \qquad t \geqslant 0. \tag{35}$$

Hence,

$$\begin{aligned} L(x_1(t), x_2(t), u(t)) = &-\dot{V}(x_1(t), x_2(t)) + L(x_1(t), x_2(t), u(t)) \\ &+ V'(x_1(t), x_2(t))F(x_1(t), x_2(t), u(t)), \qquad t \geqslant 0. \end{aligned} \tag{36}$$

Now, using (28) and the fact that $u(\cdot) \in \mathcal{S}(x_{10}, x_{20})$, it follows that

$$0 = \lim_{t\to\infty} \alpha(\|x_1(t)\|) \leqslant \lim_{t\to\infty} V(x_1(t), x_2(t)) \leqslant \lim_{t\to\infty} \beta(\|x_1(t)\|) = 0. \tag{37}$$

Thus, it follows from (36), (37), (32), (33), and the fact that $u(\cdot) \in \mathcal{S}(x_{10}, x_{20})$, that

$$\begin{aligned} \int_0^\infty L(x_1(t), x_2(t), u(t))\mathrm{d}t &= \int_0^\infty -\dot{V}(x_1(t), x_2(t))\mathrm{d}t + \int_0^\infty L(x_1(t), x_2(t), u(t))\mathrm{d}t \\ &\quad + \int_0^\infty \left( \frac{\partial V(x_1, x_2)}{\partial x_1} F_1(x_1(t), x_2(t), u(t)) \right. \\ &\qquad \left. + \frac{\partial V(x_1, x_2)}{\partial x_2} F_2(x_1(t), x_2(t), u(t)) \right) \mathrm{d}t \\ &\geqslant \int_0^\infty -\dot{V}(x_1(t), x_2(t))\mathrm{d}t \\ &= -\lim_{t\to\infty} V(x_1(t), x_2(t)) + V(x_{10}, x_{20}) \\ &= J(x_{10}, x_{20}, \phi(x_1(\cdot), x_2(\cdot))), \end{aligned} \tag{38}$$

which yields (34). □



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Note that (31) is the steady-state, Hamilton–Jacobi–Bellman equation for the nonlinear controlled dynamical system (23) and (24) with performance criterion (27). Furthermore, conditions (31) and (32) guarantee optimality with respect to the set of admissible partially asymptotically stabilizing controllers $\mathcal{S}(x_{10}, x_{20})$. However, it is important to note that an explicit characterization of $\mathcal{S}(x_{10}, x_{20})$ is not required. In addition, the optimal asymptotically stabilizing with respect to $x_1$ uniformly in $x_{20}$ *feedback* control law $u = \phi(x_1, x_2)$ is independent of the initial condition $(x_{10}, x_{20})$ and, using (31) and (32), is given by

$$\phi(x_1, x_2) = \operatorname{argmin}_{u \in \mathcal{S}(x_{10}, x_{20})} \left[ L(x_1, x_2, u) + \frac{\partial V(x_1, x_2)}{\partial x_1} F_1(x_1, x_2, u) + \frac{\partial V(x_1, x_2)}{\partial x_2} F_2(x_1, x_2, u) \right]. \tag{39}$$

*Remark 3.1*
Setting $n_1 = n$ and $n_2 = 0$, the nonlinear controlled dynamical system given by (23) and (24) reduces to

$$\dot{x}(t) = F(x(t), u(t)), \qquad x(0) = x_0, \qquad t \geqslant 0. \tag{40}$$

In this case, (28) implies that $V(\cdot)$ is positive definite with respect to $x$, and the conditions of Theorem 3.2 reduce to the conditions of Theorem 8.2 of Haddad and Chellaboina [2] characterizing the classical optimal control problem for time-invariant systems on an infinite interval.

Finally, we use Theorem 3.2 to provide a unification between optimal partial-state stabilization and optimal control for nonlinear time-varying systems. Specifically, consider the nonlinear time-varying controlled dynamical system

$$\dot{x}(t) = F(t, x(t), u(t)), \qquad x(t_0) = x_0, \qquad t \geqslant t_0, \tag{41}$$

with performance measure

$$J(t_0, x_0, u(\cdot)) \triangleq \int_{t_0}^{\infty} L(t, x(t), u(t))\mathrm{d}t, \tag{42}$$

where, for every $t \geqslant t_0$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $u(t) \in U \subseteq \mathbb{R}^m$ with $0 \in U$; $L : [t_0, \infty) \times \mathcal{D} \times U \to \mathbb{R}$ and $F : [t_0, \infty) \times \mathcal{D} \times U \to \mathbb{R}^n$ are jointly continuous in $t$, $x$, and $u$; $F(t, \cdot, u)$ is Lipschitz continuous in $x$ for every $(t, u) \in [t_0, \infty) \times U$; and $F(t, x, \cdot)$ is Lipschitz continuous in $u$ for every $(t, x) \in [t_0, \infty) \times \mathcal{D}$. For the statement of the next result, define the set of regulation controllers

$$\mathcal{S}(t_0, x_0) \triangleq \{u(\cdot) : u(\cdot) \text{ is admissible and } x(\cdot) \text{ given by (41) satisfies } x(t) \to 0 \text{ as } t \to \infty\}.$$

*Corollary 3.2*
Consider the controlled nonlinear time-varying dynamical system (41) with performance measure (42) where $u(\cdot)$ is an admissible control. Assume that there exist a continuously differentiable function $V : [t_0, \infty) \times \mathcal{D} \to \mathbb{R}$, class $\mathcal{K}$ functions $\alpha(\cdot)$, $\beta(\cdot)$, and $\gamma(\cdot)$, and a control law $\phi : [t_0, \infty) \times \mathcal{D} \to U$ such that

$$\alpha(\|x\|) \leqslant V(t, x) \leqslant \beta(\|x\|), \quad (t, x) \in [t_0, \infty) \times \mathcal{D}, \tag{43}$$

$$\frac{\partial V(t, x)}{\partial t} + \frac{\partial V(t, x)}{\partial x} F(t, x, \phi(t, x)) \leqslant -\gamma(\|x\|), \quad (t, x) \in [t_0, \infty) \times \mathcal{D}, \tag{44}$$

$$\phi(t, 0) = 0, \quad t \in [t_0, \infty), \tag{45}$$

$$-\frac{\partial V(t, x)}{\partial t} = L(t, x, \phi(t, x)) + \frac{\partial V(t, x)}{\partial x} F(t, x, \phi(t, x)), \quad (t, x) \in [t_0, \infty) \times \mathcal{D}, \tag{46}$$

$$L(t, x, u) + \frac{\partial V(t, x)}{\partial t} + \frac{\partial V(t, x)}{\partial x} F(t, x, u) \geqslant 0, \quad (t, x, u) \in [t_0, \infty) \times \mathcal{D} \times U. \tag{47}$$

 

Then, with the feedback control $u = \phi(t, x)$, the closed-loop system

$$\dot{x}(t) = F(t, x(t), \phi(x(t))), \qquad x(t_0) = x_0, \qquad t \geqslant t_0, \tag{48}$$

is uniformly asymptotically stable, and there exists a neighborhood of the origin $\mathcal{D}_0 \subseteq \mathcal{D}$ such that

$$J(t_0, x_0, \phi(\cdot,\cdot)) = V(t_0, x_0), \qquad (t_0, x_0) \in [0, \infty) \times \mathcal{D}_0. \tag{49}$$

In addition, if $(t_0, x_0) \in [0, \infty) \times \mathcal{D}_0$, then the feedback control $u(\cdot) = \phi(\cdot, x(\cdot))$ minimizes $J(t_0, x_0, u(\cdot))$ in the sense that

$$J(t_0, x_0, \phi(\cdot,\cdot)) = \min_{u(\cdot)\in\mathcal{S}(t_0,x_0)} J(t_0, x_0, u(\cdot)). \tag{50}$$

Finally, if $\mathcal{D} = \mathbb{R}^n$, $U = \mathbb{R}^m$, and the functions $\alpha(\cdot)$ and $\beta(\cdot)$ satisfying (43) are class $\mathcal{K}_\infty$, then the nonlinear dynamical system $\mathcal{G}$ is globally uniformly asymptotically stable.

*Proof*
The proof is a direct consequence of Theorem 3.2 with $n_1 = n$, $n_2 = 1$, $x_1(t - t_0) = x(t)$, $x_2(t - t_0) = t$, $F_1(x_1, x_2, u) = F_1(x_2, x_1, u) = F(t, x, u)$, $F_2(x_1, x_2, u) = 1$, $\phi(x_1, x_2) = \phi(x_2, x_1) = \phi(t, x)$, and $V(x_1, x_2) = V(x_2, x_1) = V(t, x)$. □

Note that (46) and (47) give the Hamilton–Jacobi–Bellman equation

$$-\frac{\partial V(t,x)}{\partial t} = \min_{u\in\mathcal{S}(t_0,x_0)} \left[ L(t,x,u) + \frac{\partial V(t,x)}{\partial x} F(t,x,u) \right], \qquad (t,x) \in [t_0, \infty) \times \mathcal{D}, \tag{51}$$

which characterizes the optimal control

$$\phi(t,x) = \operatorname{argmin}_{u\in\mathcal{S}(t_0,x_0)} \left[ L(t,x,u) + \frac{\partial V(t,x)}{\partial x} F(t,x,u) \right] \tag{52}$$

for time-varying systems on a finite or infinite interval.

## 4. PARTIAL-STATE STABILIZATION FOR AFFINE DYNAMICAL SYSTEMS AND CONNECTIONS TO THE TIME-VARYING LINEAR-QUADRATIC REGULATOR PROBLEM

In this section, we specialize the results of Section 3 to nonlinear affine dynamical systems of the form

$$\dot{x}_1(t) = f_1(x_1(t), x_2(t)) + G_1(x_1(t), x_2(t))u(t), \quad x_1(0) = x_{10}, \quad t \geqslant 0, \tag{53}$$

$$\dot{x}_2(t) = f_2(x_1(t), x_2(t)) + G_2(x_1(t), x_2(t))u(t), \quad x_2(0) = x_{20}, \tag{54}$$

where, for every $t \geqslant 0$, $x_1(t) \in \mathbb{R}^{n_1}$ and $x_2(t) \in \mathbb{R}^{n_2}$, $u(t) \in \mathbb{R}^m$; and $f_1 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1}$, $f_2 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$, $G_1 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1\times m}$, and $G_2 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_2\times m}$ are such that $f_1(0, x_2) = 0$ for all $x_2 \in \mathbb{R}^{n_2}$; $f_1(\cdot, x_2)$, $f_2(\cdot, x_2)$, $G_1(\cdot, x_2)$, and $G_2(\cdot, x_2)$ are locally Lipschitz continuous in $x_1$; and $f_1(x_1, \cdot)$, $f_2(x_1, \cdot)$, $G_1(x_1, \cdot)$, and $G_2(x_1, \cdot)$ are locally Lipschitz continuous in $x_2$. Furthermore, we consider performance integrands $L(x_1, x_2, u)$ of the form

$$L(x_1, x_2, u) = L_1(x_1, x_2) + L_2(x_1, x_2)u + u^{\mathrm{T}} R_2(x_1, x_2)u, \qquad (x_1, x_2, u) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^m, \tag{55}$$

where $L_1 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, $L_2 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{1\times m}$, and $R_2 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{m\times m}$ are such that $R_2(x_1, x_2) \geqslant N(x_1) > 0$, $(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, so that (27) becomes

$$J(x_{10}, x_{20}, u(\cdot)) = \int_0^\infty \big[L_1(x_1(t), x_2(t)) + L_2(x_1(t), x_2(t))u(t) + u^{\mathrm{T}}(t)R_2(x_1(t), x_2(t))u(t)\big]\,\mathrm{d}t. \tag{56}$$

For the statement of the next result, define

$$f(x_1, x_2) \triangleq \left[f_1^{\mathrm{T}}(x_1, x_2), f_2^{\mathrm{T}}(x_1, x_2)\right]^{\mathrm{T}}, \qquad G(x_1, x_2) \triangleq \left[G_1^{\mathrm{T}}(x_1, x_2), G_2^{\mathrm{T}}(x_1, x_2)\right]^{\mathrm{T}}.$$

*Theorem 4.1*
Consider the controlled nonlinear affine dynamical system (53) and (54) with performance measure (56). Assume that there exist a continuously differentiable function $V : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$; class $\mathcal{K}_\infty$ functions $\alpha(\cdot)$ and $\beta(\cdot)$; and a class $\mathcal{K}$ function $\gamma(\cdot)$ such that

$$\alpha(\|x_1\|) \leqslant V(x_1, x_2) \leqslant \beta(\|x_1\|), \quad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \tag{57}$$

$$V'(x_1, x_2)\left[f(x_1, x_2) - \frac{1}{2}G(x_1, x_2)R_2^{-1}(x_1, x_2)L_2^{\mathrm{T}}(x_1, x_2) - \frac{1}{2}G(x_1, x_2)R_2^{-1}(x_1, x_2)G^{\mathrm{T}}(x_1, x_2)V'^{\mathrm{T}}(x_1, x_2)\right] \leqslant -\gamma(\|x_1\|), \quad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \tag{58}$$

$$L_2(0, x_2) = 0, \quad x_2 \in \mathbb{R}^{n_2}, \tag{59}$$

$$0 = L_1(x_1, x_2) + V'(x_1, x_2)f(x_1, x_2) - \frac{1}{4}\left[V'(x_1, x_2)G(x_1, x_2) + L_2(x_1, x_2)\right] \cdot R_2^{-1}(x_1, x_2)\left[V'(x_1, x_2)G(x_1, x_2) + L_2(x_1, x_2)\right]^{\mathrm{T}}, \qquad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}. \tag{60}$$

Then, with the feedback control

$$u = \phi(x_1, x_2) = -\frac{1}{2}R_2^{-1}(x_1, x_2)\left[L_2(x_1, x_2) + V'(x_1, x_2)G(x_1, x_2)\right]^{\mathrm{T}}, \tag{61}$$

the closed-loop system

$$\dot{x}_1(t) = f_1(x_1(t), x_2(t)) + G_1(x_1(t), x_2(t))\phi(x_1(t), x_2(t)), \quad x_1(0) = x_{10}, \quad t \geqslant 0, \tag{62}$$

$$\dot{x}_2(t) = f_2(x_1(t), x_2(t)) + G_2(x_1(t), x_2(t))\phi(x_1(t), x_2(t)), \quad x_2(0) = x_{20}, \tag{63}$$

is globally asymptotically stable with respect to $x_1$ uniformly in $x_{20}$, and the performance measure (56) is minimized in the sense of (34). Finally,

$$J(x_{10}, x_{20}, \phi(x_1(\cdot), x_2(\cdot))) = V(x_{10}, x_{20}), \qquad (x_{10}, x_{20}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}. \tag{64}$$

*Proof*
The result is a consequence of Theorem 3.2 with $\mathcal{D} = \mathbb{R}^{n_1}$, $U = \mathbb{R}^m$, $F(x_1, x_2, u) = f(x_1, x_2) + G(x_1, x_2)u$, and $L(x_1, x_2, u) = L_1(x_1, x_2) + L_2(x_1, x_2)u + u^{\mathrm{T}}R_2(x_1, x_2)u$. Specifically, the feedback control law (61) follows from (39) by setting

$$\frac{\partial}{\partial u}\left[L_1(x_1, x_2) + L_2(x_1, x_2)u + u^{\mathrm{T}}R_2(x_1, x_2)u + V'(x_1, x_2)\left(f(x_1, x_2) + G(x_1, x_2)u\right)\right] = 0. \tag{65}$$

Now, with $u = \phi(x_1, x_2)$ given by (61), conditions (57), (58), and (60) imply (28), (29), and (31), respectively.

Next, because $V(\cdot,\cdot)$ is continuously differentiable and, by (57), $V(0,x_2)$, $x_2 \in \mathbb{R}^{n_2}$, is a local minimum of $V(\cdot,\cdot)$, it follows that $V'(0,x_2) = 0$, $x_2 \in \mathbb{R}^{n_2}$, and hence, it follows from (59) and (61) that $\phi(0,x_2) = 0$, which implies (30). Finally, because

$$\begin{aligned} &L(x_1,x_2,u) + V'(x_1,x_2)[f(x_1,x_2) + G(x_1,x_2)u] \\ &= L(x_1,x_2,u) + V'(x_1,x_2)[f(x_1,x_2) + G(x_1,x_2)u] - L(x_1,x_2,\phi(x_1,x_2)) \\ &\quad - V'(x_1,x_2)[f(x_1,x_2) + G(x_1,x_2)\phi(x_1,x_2)] \\ &= [u - \phi(x_1,x_2)]^{\mathrm{T}} R_2(x_1,x_2)[u - \phi(x_1,x_2)] \\ &\geqslant 0, \qquad (x_1,x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \end{aligned} \tag{66}$$

condition (32) holds. The result now follows as a direct consequence of Theorem 3.2. □

Next, we use Theorem 4.1 to address the classical time-varying, linear-quadratic optimal control problem. Specifically, consider the linear time-varying dynamical system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \qquad x(t_0) = x_0, \qquad t \geqslant t_0, \tag{67}$$

with performance measure

$$J(t_0,x_0,u(\cdot)) = \int_{t_0}^{\infty} \left[x^{\mathrm{T}}(t)R_1(t)x(t) + u^{\mathrm{T}}(t)R_2(t)u(t)\right] \mathrm{d}t, \tag{68}$$

where, for all $t \geqslant t_0$, $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$; $A : [t_0,\infty) \to \mathbb{R}^{n\times n}$ and $B : [t_0,\infty) \to \mathbb{R}^{n\times m}$ are continuous and uniformly bounded; and $R_1 : [t_0,\infty) \to \mathbb{R}^{n\times n}$ and $R_2 : [t_0,\infty) \to \mathbb{R}^{m\times m}$ are continuous, uniformly bounded, and positive definite, and hence, there exist $\gamma, \sigma > 0$ such that $R_1(t) \geqslant \gamma I_n > 0$ and $R_2(t) \geqslant \sigma I_m > 0$ for all $t \geqslant t_0$.

*Corollary 4.1*
Consider the linear time-varying dynamical system (67) with quadratic performance measure (68), and let $P : [t_0,\infty) \to \mathbb{R}^{n\times n}$ be a continuously differentiable, uniformly bounded, positive definite solution of

$$\begin{aligned} -\dot{P}(t) = A^{\mathrm{T}}(t)P(t) + P(t)A(t) + R_1(t) - P(t)B(t)R_2^{-1}(t)B^{\mathrm{T}}(t)P(t), \\ \lim_{t_{\mathrm{f}}\to\infty} P(t_{\mathrm{f}}) = 0, \qquad t \in [t_0,\infty). \end{aligned} \tag{69}$$

Then, with the feedback control

$$u = \phi(t,x) = -R_2^{-1}(t)B^{\mathrm{T}}(t)P(t)x, \qquad (t,x) \in [t_0,\infty) \times \mathbb{R}^n, \tag{70}$$

the dynamical system (67) is globally uniformly asymptotically stable and

$$J(t_0,x_0,\phi(\cdot,\cdot)) = x_0^{\mathrm{T}} P(t_0)x_0, \qquad (t_0,x_0) \in [0,\infty) \times \mathbb{R}^n. \tag{71}$$

Furthermore, the feedback control $u(\cdot) = \phi(\cdot,x(\cdot))$ minimizes (68) in the sense of (50).

*Proof*
The result is a consequence of Theorem 4.1 with $n_1 = n$, $n_2 = 1$, $x_1(t-t_0) = x(t)$, $x_2(t-t_0) = t$, $f_1(x_1,x_2) = f_1(x_2,x_1) = A(t)x$, $f_2(x_1,x_2) = 1$, $G_1(x_1,x_2) = G_1(x_2,x_1) = B(t)$, $G_2(x_1,x_2) = 0$, $L_1(x_1,x_2) = L_1(x_2,x_1) = x^{\mathrm{T}}R_1(t)x$, $L_2(x_1,x_2) = 0$, $R_2(x_1,x_2) = R_2(x_2,x_1) = R_2(t)$, $V(x_1,x_2) = V(x_2,x_1) = x^{\mathrm{T}}P(t)x$, $\alpha(\|x_1\|) = \alpha\|x\|^2$, $\beta(\|x_1\|) = \beta\|x\|^2$, and $\gamma(\|x_1\|) = \gamma\|x\|^2$, for some $\alpha, \beta, \gamma > 0$. Specifically, because $P(\cdot)$ is uniformly bounded and positive definite, there exist constants $\alpha > 0$ and $\beta > 0$ such that $\alpha I_n \leqslant P(t) \leqslant \beta I_n$, $t \geqslant t_0$, and hence,

$$\alpha\|x\|^2 \leqslant V(t,x) \leqslant \beta\|x\|^2, \qquad (t,x) \in [t_0,\infty) \times \mathbb{R}^n, \tag{72}$$

which verifies (57).

Next, (70) is a restatement of (61). Now, note that, with $\tilde{A}(t) \triangleq A(t) + B(t)K(t)$, $K(t) \triangleq -R_2^{-1}(t)B^{\mathrm{T}}(t)P(t)$, and $\tilde{R}(t) \triangleq R_1(t) + P(t)B(t)R_2^{-1}(t)B^{\mathrm{T}}(t)P(t)$, (69) can be equivalently written as

$$-\dot{P}(t) = \tilde{A}^{\mathrm{T}}(t)P(t) + P(t)\tilde{A}(t) + \tilde{R}(t), \qquad \lim_{t_{\mathrm{f}}\to\infty} P(t_{\mathrm{f}}) = 0, \qquad t \in [t_0, \infty), \tag{73}$$

where $\tilde{A}(t)$, $t \geqslant t_0$, characterizes the closed-loop dynamics of the closed-loop system (67) and (70) given by

$$\dot{x}(t) = \tilde{A}(t)x(t), \qquad x(t_0) = x_0, \qquad t \geqslant t_0. \tag{74}$$

Next, computing the derivative of $V(t,x)$ along the trajectories of the closed-loop system (74) gives

$$\begin{aligned}\dot{V}(t,x) &= x^{\mathrm{T}}\dot{P}(t)x + 2x^{\mathrm{T}}P(t)\tilde{A}(t)x \\ &= x^{\mathrm{T}}\left[\dot{P}(t) + \tilde{A}^{\mathrm{T}}(t)P(t) + P(t)\tilde{A}(t)\right]x \\ &= -x^{\mathrm{T}}\tilde{R}(t)x, \qquad (t,x) \in [t_0,\infty)\times\mathbb{R}^n \\ &\leqslant -\gamma\|x\|^2, \qquad (t,x) \in [t_0,\infty)\times\mathbb{R}^n,\end{aligned} \tag{75}$$

which verifies (58).

Finally, it follows from (69) that

$$\begin{aligned}&x^{\mathrm{T}}R_1(t)x + \phi^{\mathrm{T}}(t,x)R_2(t)\phi(t,x) + \frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x}\left[A(t)x + B(t)\phi(t,x)\right] \\ &= x^{\mathrm{T}}\left[\dot{P}(t) + A^{\mathrm{T}}(t)P(t) + P(t)A(t) + R_1(t) - P(t)B(t)R_2^{-1}(t)B^{\mathrm{T}}(t)P(t)\right]x \\ &= 0, \qquad (t,x) \in [t_0,\infty)\times\mathbb{R}^n,\end{aligned} \tag{76}$$

which verifies (60). The result now follows as a direct consequence of Theorem 4.1. □

Corollary 4.1 gives sufficient conditions for global uniform asymptotic stability and optimality of the linear dynamical system (67) with the state feedback control law (70). Because the closed-loop linear dynamical system (74) is globally uniformly asymptotically stable, (74) is globally (uniformly) exponentially stable [10]. Corollary 4.1 assumes the existence of a continuously differentiable, uniformly bounded, positive definite $P : [t_0,\infty) \to \mathbb{R}^{n\times n}$ satisfying the differential Riccati Equation 69. However, if (67) is completely controllable and completely observable (through the cost), then there exists a unique continuously differentiable, uniformly bounded, non-negative definite solution $P : [t_0,\infty) \to \mathbb{R}^{n\times n}$ to (69) such that the linear dynamical system (67), with state feedback control law (70), is globally (uniformly) exponentially stable [11, Th. 3.5, 3.6].

## 5. INVERSE OPTIMAL CONTROL

In this section, we construct state feedback controllers for nonlinear affine in the control dynamical systems that are predicated on an *inverse optimal control problem* [12–16]. In particular, to avoid the complexity in solving the steady-state, Hamilton–Jacobi–Bellman Equation (60), we do not attempt to minimize a given cost functional, but rather, we parameterize a family of stabilizing controllers that minimize some derived cost functional that provides flexibility in specifying the control law. The performance integrand is shown to explicitly depend on the nonlinear system dynamics, the Lyapunov function of the closed-loop system, and the stabilizing feedback control law, wherein the coupling is introduced via the Hamilton–Jacobi–Bellman equation. Hence, by varying the parameters in the Lyapunov function and the performance integrand, the proposed framework can be used to characterize a class of globally partial-state stabilizing controllers that can meet closed-loop system response constraints.



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*Theorem 5.1*

Consider the controlled nonlinear affine dynamical system (53) and (54) with performance measure (56). Assume there exist a continuously differentiable function $V : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, class $\mathcal{K}_\infty$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, and a class $\mathcal{K}$ function $\gamma(\cdot)$ such that

$$\alpha(\|x_1\|) \leqslant V(x_1, x_2) \leqslant \beta(\|x_1\|), \quad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \tag{77}$$

$$V'(x_1, x_2)\left[f(x_1, x_2) - \frac{1}{2}G(x_1, x_2)R_2^{-1}(x_1, x_2)L_2^{\mathrm{T}}(x_1, x_2) - \frac{1}{2}G(x_1, x_2)R_2^{-1}(x_1, x_2)G^{\mathrm{T}}(x_1, x_2)V'^{\mathrm{T}}(x_1, x_2)\right] \leqslant -\gamma(\|x_1\|), \quad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \tag{78}$$

$$L_2(0, x_2) = 0, \quad x_2 \in \mathbb{R}^{n_2}. \tag{79}$$

Then, with the feedback control

$$u = \phi(x_1, x_2) = -\frac{1}{2}R_2^{-1}(x_1, x_2)\left[L_2(x_1, x_2) + V'(x_1, x_2)G(x_1, x_2)\right]^{\mathrm{T}}, \tag{80}$$

the closed-loop system given by (62) and (63) is globally asymptotically stable with respect to $x_1$ uniformly in $x_{20}$, and the performance functional (56), with

$$L_1(x_1, x_2) = \phi^{\mathrm{T}}(x_1, x_2)R_2(x_1, x_2)\phi(x_1, x_2) - V'(x_1, x_2)f(x_1, x_2), \tag{81}$$

is minimized in the sense that

$$J(x_{10}, x_{20}, \phi(\cdot, \cdot)) = \min_{u(\cdot) \in \mathcal{S}(x_{10}, x_{20})} J(x_{10}, x_{20}, u(\cdot)). \tag{82}$$

Finally,

$$J(x_{10}, x_{20}, \phi(x_1(\cdot), x_2(\cdot)) = V(x_{10}, x_{20}), \qquad (x_{10}, x_{20}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}. \tag{83}$$

*Proof*

The proof is identical to the proof of Theorem 4.1. □

Next, we specialize Theorem 5.1 to linear time-varying systems controlled by nonlinear controllers that minimize a polynomial cost functional. For the following result, let $R_1 : [t_0, \infty) \to \mathbb{R}^{n \times n}$, $R_2 : [t_0, \infty) \to \mathbb{R}^{m \times m}$, and $\hat{R}_q : [t_0, \infty) \to \mathbb{R}^{n \times n}$, $q = 2, \ldots, r$, where $r$ is a positive integer, be continuous, uniformly bounded, and positive definite matrices, that is, there exist $\gamma$, $\sigma$, $\hat{\sigma}_q > 0$, $q = 2, \ldots, r$, such that $R_1(t) \geqslant \gamma I_n > 0$, $R_2(t) \geqslant \sigma I_m > 0$, and $\hat{R}_q(t) \geqslant \hat{\sigma}_q I_m > 0$, for all $t \geqslant t_0$. Furthermore, for the following result, we consider performance integrands in (42) of the form

$$L(t, x, u) = L_1(t, x) + L_2(t, x)u + u^{\mathrm{T}}R_2(t, x)u, \qquad (t, x, u) \in [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m, \tag{84}$$

where $L_1 : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}$, $L_2 : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^{1 \times m}$, and $R_2(t, x) \geqslant N(x) > 0$, $(t, x) \in [t_0, \infty) \times \mathbb{R}^n$, so that (42) becomes

$$J(t_0, x_0, u(\cdot)) = \int_{t_0}^{\infty} \left[L_1(t, x(t)) + L_2(t, x(t))u(t) + u^{\mathrm{T}}(t)R_2(t, x(t))u(t)\right] \mathrm{d}t. \tag{85}$$

*Corollary 5.1*

Consider the controlled linear time-varying dynamical system (67), where $u(\cdot)$ is admissible. Assume that there exist a continuously differentiable, uniformly bounded, positive definite



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$P : [t_0, \infty) \to \mathbb{R}^{n\times n}$ and continuously differentiable, uniformly bounded, non-negative definite $M_q : [t_0, \infty) \to \mathbb{R}^{n\times n}$, $q = 2, \ldots, r$, such that

$$-\dot{P}(t) = A^{\mathrm{T}}(t)P(t) + P(t)A(t) + R_1(t) - P(t)S(t)P(t), \quad \lim_{t_f\to\infty} P(t_f) = 0, \quad t \in [t_0, \infty) \tag{86}$$

and

$$-\dot{M}_q(t) = (A(t) - S(t)P(t))^{\mathrm{T}}M_q(t) + M_q(t)(A(t) - S(t)P(t)) + \hat{R}_q(t), \quad \lim_{t_f\to\infty} M_q(t_f) = 0, \quad q = 2, \ldots, r, \quad t \in [t_0, \infty), \tag{87}$$

where $S(t) \triangleq B(t)R_2^{-1}(t)B^{\mathrm{T}}(t)$. Then the zero solution $x(t) \equiv 0$ of the closed-loop system

$$\dot{x}(t) = A(t)x(t) + B(t)\phi(t, x), \qquad x(t_0) = x_0, \qquad t \geqslant t_0, \tag{88}$$

is globally uniformly asymptotically stable with feedback control

$$u = \phi(t, x) = -R_2^{-1}(t)B^{\mathrm{T}}(t)\left(P(t) + \sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}M_q(t)\right)x, \tag{89}$$

and the performance functional (85) with $R_2(t, x) = R_2(t)$, $L_2(t, x) = 0$, and

$$L_1(t, x) = x^{\mathrm{T}}\left(R_1(t) + \sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}\hat{R}_q(t) + \left[\sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}M_q(t)\right]^{\mathrm{T}} S(t)\left[\sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}M_q(t)\right]\right)x \tag{90}$$

is minimized in the sense that

$$J(t_0, x_0, \phi(\cdot, \cdot)) = \min_{u(\cdot)\in\mathcal{S}(t_0,x_0)} J(t_0, x_0, u(\cdot)), \qquad (t_0, x_0) \in [0, \infty) \times \mathbb{R}^n. \tag{91}$$

Finally,

$$J(t_0, x_0, \phi(\cdot, \cdot)) = x_0^{\mathrm{T}}P(t_0)x_0 + \sum_{q=2}^{r}\frac{1}{q}\left(x_0^{\mathrm{T}}M_q(t_0)x_0\right)^q, \qquad (t_0, x_0) \in [0, \infty) \times \mathbb{R}^n. \tag{92}$$

*Proof*

The result is a consequence of Theorem 5.1 with $n_1 = n$, $n_2 = 1$, $x_1(t - t_0) = x(t)$, $x_2(t - t_0) = t$, $f_1(x_1, x_2) = f_1(x_2, x_1) = A(t)x$, $f_2(x_1, x_2) = 1$, $G_1(x_1, x_2) = G_1(x_2, x_1) = B(t)$, $G_2(x_1, x_2) = 0$, $L_1(x_1, x_2) = L_1(x_2, x_1) = L_1(t, x)$, where $L_1(t, x)$ is given by (90), $L_2(x_1, x_2) = 0$, $R_2(x_1, x_2) = R_2(x_2, x_1) = R_2(t)$, $V(x_1, x_2) = V(x_2, x_1) = x^{\mathrm{T}}P(t)x + \sum_{q=2}^{r}\frac{1}{q}(x^{\mathrm{T}}M_q(t)x)^q$, $\alpha(\|x_1\|) = \alpha\|x\|^2$, $\beta(\|x_1\|) = \beta\|x\|^2 + \sum_{q=2}^{r}\frac{1}{q}\hat{\beta}_q^q\|x\|^{2q}$, and $\gamma(\|x_1\|) = -\gamma\|x\|^2 - \sum_{q=2}^{r}\hat{\sigma}_q\hat{\beta}_q^{q-1}\|x\|^{2q}$, for some $\alpha$, $\beta$, $\gamma$, $\hat{\beta}_q$, and $\hat{\sigma}_q > 0$, $q = 2, \ldots, r$. Specifically, because $P(\cdot)$ and $M_q(\cdot)$ are uniformly bounded and, respectively, positive and non-negative definite, there exist constants $\alpha$, $\beta$, and $\hat{\beta}_q > 0$, $q = 2, \ldots, r$, such that $\alpha I_n \leqslant P(t) \leqslant \beta I_n$ and $0 \leqslant M_q(t) \leqslant \hat{\beta}_q I_n$, $t \geqslant t_0$, and hence,

$$\alpha\|x\|^2 \leqslant V(t, x) \leqslant \beta\|x\|^2 + \sum_{q=2}^{r}\frac{1}{q}\hat{\beta}_q^q\|x\|^{2q}, \qquad (t, x) \in [t_0, \infty) \times \mathbb{R}^n, \tag{93}$$

which verifies (77).



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Next, (89) is a restatement of (80). Now, let $\phi(t,x) = \phi_1(t,x) + \phi_2(t,x)$, where

$$\phi_1(t,x) \triangleq -R_2^{-1}(t)B^{\mathrm{T}}(t)P(t)x, \tag{94}$$

$$\phi_2(t,x) \triangleq -R_2^{-1}(t)B^{\mathrm{T}}(t)\sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}M_q(t)x. \tag{95}$$

Computing the derivative of $V(t,x)$ along the trajectories of the closed-loop system (88) gives

$$\begin{aligned}
\dot V(t,x) &= x^{\mathrm{T}}\left(\dot P(t)x + P(t)A(t) + A^{\mathrm{T}}(t)P(t)\right)x + 2x^{\mathrm{T}}P(t)B(t)\phi(t,x)\\
&\quad + \sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}\left[x^{\mathrm{T}}\left(\dot M_q(t) + M_q(t)A(t) + A^{\mathrm{T}}(t)M_q(t)\right)x\right.\\
&\qquad\qquad \left. + 2x^{\mathrm{T}}M_q(t)B(t)\phi(t,x)\right]\\
&= x^{\mathrm{T}}\left(\dot P(t)x + P(t)A(t) + A^{\mathrm{T}}(t)P(t) - P(t)S(t)P(t)\right)x - x^{\mathrm{T}}P(t)S(t)P(t)x\\
&\quad + 2x^{\mathrm{T}}P(t)B(t)\phi_2(t,x) + \sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}\left[x^{\mathrm{T}}(\dot M_q(t) + M_q(t)(A(t) - S(t)P(t))\right.\\
&\quad \left. + (A - S(t)P(t))^{\mathrm{T}}M_q(t))\,x + 2x^{\mathrm{T}}M_q(t)B(t)\phi_2(t,x)\right], \qquad (t,x)\in[t_0,\infty)\times\mathbb{R}^n.
\end{aligned} \tag{96}$$

Now, using (86) and (87), (96) yields

$$\begin{aligned}
\dot V(t,x) &= -x^{\mathrm{T}}\left(R_1(t) + \sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}\hat R_q(t)\right)x - x^{\mathrm{T}}P(t)S(t)P(t)x\\
&\quad - 2x^{\mathrm{T}}\left[\sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}M_q(t)\right]^{\mathrm{T}}S(t)\left[\sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}M_q(t)\right]x\\
&\quad - 2x^{\mathrm{T}}P(t)S(t)\sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}M_q(t)x\\
&\leqslant -x^{\mathrm{T}}R_1(t)x - x^{\mathrm{T}}\sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}\hat R_q(t)x\\
&\leqslant -\gamma\|x\|^2 - \sum_{q=2}^{r}(\hat\beta_q\|x\|^2)^{q-1}\hat\sigma_q\|x\|^2\\
&\leqslant -\gamma\|x\|^2 - \sum_{q=2}^{r}\hat\sigma_q\hat\beta_q^{q-1}\|x\|^{2q}, \qquad (t,x)\in[t_0,\infty)\times\mathbb{R}^n,
\end{aligned} \tag{97}$$

and hence, (78) holds.

Finally, note that

$$\begin{aligned}
\phi^{\mathrm{T}}(t,x)R_2(t)\phi(t,x) &= x^{\mathrm{T}}P(t)S(t)P(t)x + 2x^{\mathrm{T}}P(t)S(t)\sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}M_q(t)x\\
&\quad + x^{\mathrm{T}}\left[\sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}M_q(t)\right]^{\mathrm{T}}S(t)\left[\sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}M_q(t)\right]x,
\end{aligned} \tag{98}$$



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which, using the first equality in (97), implies

$$
\begin{aligned}
\dot V(t,x) = & -x^{\mathrm T}R_1(t)x - x^{\mathrm T}\sum_{q=2}^{r}(x^{\mathrm T}M_q(t)x)^{q-1}\hat R_q(t)x - \phi(t,x)R_2(t)\phi(t,x) \\
& - x^{\mathrm T}\left[\sum_{q=2}^{r}(x^{\mathrm T}M_q(t)x)^{q-1}M_q(t)\right]^{\mathrm T} S(t)\left[\sum_{q=2}^{r}(x^{\mathrm T}M_q(t)x)^{q-1}M_q(t)\right]x \\
= & -L_1(t,x) - \phi^{\mathrm T}(t,x)R_2(t)\phi(t,x),
\end{aligned} \tag{99}
$$

where $L_1(t,x)$ is given by (90), and thus, (81) is verified. The result now follows as a direct consequence of Theorem 5.1. □

Finally, we specialize Theorem 5.1 to linear time-varying systems controlled by nonlinear controllers that minimize a multilinear cost functional. For the following result, define $x^{[k]} \triangleq x \otimes x \otimes \cdots \otimes x$ and $\overset{k}{\oplus} A \triangleq A \oplus A \oplus \cdots \oplus A$, with $x$ and $A$ appearing $k$ times, where $k$ is a positive integer. Furthermore, define $\mathcal{N}^{(k,n)} \triangleq \{\Psi \in \mathbb{R}^{1\times n^k} : \Psi x^{[k]} \geqslant 0,\ x \in \mathbb{R}^n\}$ and let $\hat P_q : [t_0,\infty) \to \mathbb{R}^{1\times n^{2q}}$, $\hat R_{2q} : [t_0,\infty) \to \mathbb{R}^{1\times n^{2q}}$, $q = 2,\ldots,r$, where $r$ is a positive integer, and $R_2 : [t_0,\infty) \to \mathbb{R}^{m\times m}$ be continuous and uniformly bounded, $\hat R_{2q}(t)$, $\hat P_q(t) \in \mathcal{N}^{(2q,n)}$, and $R_2(t) \geqslant \sigma I_m > 0$, for some $\sigma > 0$ and for all $t \geqslant t_0$.

*Corollary 5.2*
Consider the controlled linear time-varying dynamical system (67), where $u(\cdot)$ is admissible. Assume that there exist a continuously differentiable, uniformly bounded, positive definite $P : [t_0,\infty) \to \mathbb{R}^{n\times n}$ and continuously differentiable, uniformly bounded $\hat P_q : [t_0,\infty) \to \mathbb{R}^{1\times n^{2q}}$, $q = 2,\ldots,r$, such that, $\hat P_q \in \mathcal{N}^{(k,n)}$,

$$
-\dot P(t) = A^{\mathrm T}(t)P(t)+P(t)A(t)+R_1(t)-P(t)S(t)P(t), \quad \lim_{t_f\to\infty} P(t_f) = 0, \quad t \in [t_0,\infty) \tag{100}
$$

and

$$
\begin{aligned}
-\dot{\hat P}_q(t) = \hat P_q(t)\left[\overset{2q}{\oplus}(A(t) - S(t)P(t))\right] + \hat R_{2q}(t), \quad & \lim_{t_f\to\infty} \hat P_q(t_f) = 0, \\
& q = 2,\ldots,r, \qquad t \in [t_0,\infty),
\end{aligned} \tag{101}
$$

where $S(t) \triangleq B(t)R_2^{-1}(t)B^{\mathrm T}(t)$. Then the zero solution $x(t) \equiv 0$ of the closed-loop system (88) is globally uniformly asymptotically stable with the feedback control law

$$
\phi(t,x) = -R_2^{-1}(t)B^{\mathrm T}(t)\left(P(t)x + \frac{1}{2}g'^{\mathrm T}(t,x)\right), \tag{102}
$$

where $g(t,x) \triangleq \sum_{q=2}^{r}\hat P_q(t)x^{[2q]}$, and the performance functional (85) with $R_2(t,x) = R_2(t)$, $L_2(t,x) = 0$, and

$$
L_1(t,x) = x^{\mathrm T}R_1(t)x + \sum_{q=2}^{r}\hat R_{2q}(t)x^{[2q]} + \frac{1}{4}g'(t,x)S(t)g'^{\mathrm T}(t,x) \tag{103}
$$

is minimized in the sense of (91). Finally,

$$
J(t_0,x_0,\phi(\cdot,\cdot)) = x_0^{\mathrm T}P(t_0)x_0 + \sum_{q=2}^{r}\hat P_q(t_0)x_0^{[2q]}, \qquad (t_0,x_0) \in [0,\infty)\times\mathbb{R}^n. \tag{104}
$$



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*Proof*

The result is a consequence of Theorem 5.1 with $n_1 = n$, $n_2 = 1$, $x_1(t - t_0) = x(t)$, $x_2(t - t_0) = t$, $f_1(x_1, x_2) = f_1(x_2, x_1) = A(t)x$, $f_2(x_1, x_2) = 1$, $G_1(x_1, x_2) = G_1(x_2, x_1) = B(t)$, $G_2(x_1, x_2) = 0$, and $L_1(x_1, x_2) = L_1(x_2, x_1) = L_1(t, x)$, where $L_1(t, x)$ is given by (103), $L_2(x_1, x_2) = 0$, $R_2(x_1, x_2) = R_2(x_2, x_1) = R_2(t)$, $V(x_1, x_2) = V(x_2, x_1) = x^{\mathrm{T}}P(t)x + \sum_{q=2}^{r} \hat{P}_q(t)x^{[2q]}$, $\alpha(\|x_1\|) = \alpha\|x\|^2$, $\beta(\|x_1\|) = \beta\|x\|^2$, and $\gamma(\|x_1\|) = -\gamma\|x\|^2$, for some $\alpha$, $\beta$, $\gamma > 0$. Specifically, because $P(\cdot)$ is uniformly bounded and positive definite, there exist constants $\alpha, \beta > 0$ such that $\alpha I_n \leqslant P(t) \leqslant \beta I_n$. In addition, because $\hat{P}_q(t) \in \mathcal{N}^{(2q,n)}$, $q = 2, \ldots, n$, for all $t \geqslant t_0$, it follows that

$$\alpha\|x\|^2 \leqslant V(t, x) \leqslant \beta\|x\|^2, \qquad (t, x) \in [t_0, \infty) \times \mathbb{R}^n, \tag{105}$$

which verifies (77).

Computing the derivative of $V(t, x)$ along the trajectories of the closed-loop system (88) gives

$$\begin{aligned}
\dot{V}(t, x) &= x^{\mathrm{T}}\left(\dot{P}(t)x + P(t)A(t) + A^{\mathrm{T}}(t)P(t)\right)x + 2x^{\mathrm{T}}P(t)B(t)\phi(t, x) \\
&\quad + \sum_{q=2}^{r} \dot{\hat{P}}_q(t)x^{[2q]} + \frac{\partial}{\partial x}\left[\sum_{q=2}^{r} \hat{P}_q(t)x^{[2q]}\right](A(t)x + B(t)\phi(t, x)) \\
&= x^{\mathrm{T}}\left(\dot{P}(t)x + P(t)A(t) + A^{\mathrm{T}}(t)P(t) - P(t)S(t)P(t)\right)x \\
&\quad - x^{\mathrm{T}}P(t)S(t)P(t)x - x^{\mathrm{T}}P(t)S(t)g'^{\mathrm{T}}(t, x) \\
&\quad + \sum_{q=2}^{r} \dot{\hat{P}}_q(t)x^{[2q]} + g'(t, x)\left[(A(t) - S(t)P(t))x - \frac{1}{2}S(t)g'^{\mathrm{T}}(t, x)\right]
\end{aligned} \tag{106}$$

for all $(t, x) \in [t_0, \infty) \times \mathbb{R}^n$. Next, noting that

$$\begin{aligned}
g'(t, x)(A(t) - S(t)P(t))x &= \sum_{q=2}^{r} \hat{P}_q(t)\frac{\partial}{\partial x}\left[x^{[2q]}\right](A(t) - S(t)P(t))x \\
&= \sum_{q=2}^{r} \hat{P}_q(t)\frac{\partial}{\partial x}[x \otimes \cdots \otimes x](A(t) - S(t)P(t))x \\
&= \sum_{q=2}^{r} \hat{P}_q(t)[I_n \otimes \cdots \otimes x + x \otimes \cdots \otimes I_n](A(t) - S(t)P(t))x \\
&= \sum_{q=2}^{r} \hat{P}_q(t)[(A(t) - S(t)P(t))x \otimes \cdots \otimes x \\
&\quad + x \otimes \cdots \otimes (A(t) - S(t)P(t))x] \\
&= \sum_{q=2}^{r} \hat{P}_q(t)[(A(t) - S(t)P(t)) \otimes \cdots \otimes I_n \\
&\quad + I_n \otimes \cdots \otimes (A(t) - S(t)P(t))]\, x^{[2q]} \\
&= \sum_{q=2}^{r} \hat{P}_q(t)\left[\overset{2q}{\otimes}(A(t) - S(t)P(t))\right]x^{[2q]},
\end{aligned} \tag{107}$$



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it follows from (100), (101), and (107) that

$$\begin{aligned}\dot{V}(t,x) =& -x^{\mathrm{T}}R_1(t)x - x^{\mathrm{T}}P(t)S(t)P(t)x - x^{\mathrm{T}}P(t)S(t)g'^{\mathrm{T}}(t,x) \\ &+ \sum_{q=2}^{r}\left(\dot{\hat{P}}_q(t) + \hat{P}_q(t)\left[\overset{2q}{\otimes}(A(t)-S(t)P(t))\right]\right)x^{[2q]} - \frac{1}{2}g'(t,x)S(t)g'^{\mathrm{T}}(t,x) \\ =& -x^{\mathrm{T}}R_1(t)x - x^{\mathrm{T}}P(t)S(t)P(t)x - x^{\mathrm{T}}P(t)S(t)g'^{\mathrm{T}}(t,x) \\ &- \sum_{q=2}^{r}\hat{R}_{2q}(t)x^{[2q]} - \frac{1}{2}g'(t,x)S(t)g'^{\mathrm{T}}(t,x).\end{aligned} \tag{108}$$

Finally, note that

$$\begin{aligned}\phi^{\mathrm{T}}(t,x)R_2(t)\phi(t,x) &= \left(x^{\mathrm{T}}P(t) + \frac{1}{2}g'(t,x)\right)S(t)\left(P(t)x + \frac{1}{2}g'^{\mathrm{T}}(t,x)\right) \\ &= x^{\mathrm{T}}P(t)S(t)P(t)x + \frac{1}{4}g'(t,x)S(t)g'^{\mathrm{T}}(t,x) + x^{\mathrm{T}}P(t)S(t)g'^{\mathrm{T}}(t,x),\end{aligned} \tag{109}$$

which, using (108), implies that

$$\dot{V}(t,x) = -x^{\mathrm{T}}R_1(t)x - \sum_{q=2}^{r}\hat{R}_{2q}(t)x^{[2q]} - \frac{1}{4}g'(t,x)S(t)g'^{\mathrm{T}}(t,x) - \phi^{\mathrm{T}}(t,x)R_2(t)\phi(t,x) \tag{110}$$

for all $(t,x) \in [t_0,\infty) \times \mathbb{R}^n$, and hence, (78) holds with $\gamma(\|x\|) = -\gamma\|x\|^2$. In addition, writing (110) as

$$\dot{V}(t,x) = -L_1(t,x) - \phi^{\mathrm{T}}(t,x)R_2(t)\phi(t,x), \tag{111}$$

where $L_1(t,x)$ is given by (103), (82) holds. The result now follows as a direct consequence of Theorem 5.1. □

## 6. ILLUSTRATIVE NUMERICAL EXAMPLES

In this section, we provide several numerical examples to highlight the optimal and inverse optimal partial-state asymptotic stabilization framework developed in the paper.

### 6.1. *Optimal partial stabilization of a flexible spacecraft*

Consider the flexible spacecraft given by Haddad and Chellaboina [2]

$$\dot{\omega}_1(t) = I_{23}\omega_2(t)\omega_3(t) - \alpha_1\omega_1(t) + u_1(t), \qquad \omega_1(0) = \omega_{10}, \qquad t \geqslant 0, \tag{112}$$

$$\dot{\omega}_2(t) = I_{31}\omega_3(t)\omega_1(t) - \alpha_2\omega_2(t) + u_2(t), \qquad \omega_2(0) = \omega_{20}, \tag{113}$$

$$\dot{\omega}_3(t) = I_{12}\omega_1(t)\omega_2(t), \qquad \omega_3(0) = \omega_{30}, \tag{114}$$

where $I_{23} \triangleq (I_2 - I_3)/I_1$; $I_{31} \triangleq (I_3 - I_1)/I_2$; and $I_{12} \triangleq (I_1 - I_2)/I_3$, $I_1$, $I_2$, and $I_3$ are the principal moments of inertia of the spacecraft such that $I_1 > I_2 > I_3 > 0$, $\alpha_1 \geqslant 0$, and $\alpha_2 \geqslant 0$ reflect dissipation in the $\omega_1$ and $\omega_2$ coordinates of the spacecraft; and $u_1$ and $u_2$ are the spacecraft control moments. For this example, we seek a state feedback controller $u = [u_1, u_2]^{\mathrm{T}} = \phi(x_1, x_2)$, where $x_1 = [\omega_1,\ \omega_2]^{\mathrm{T}}$ and $x_2 = \omega_3$, such that the performance measure

$$J(x_1(0), x_2(0), u(\cdot)) = \int_0^{\infty}\left[x_1^{\mathrm{T}}(t)R_1x_1(t) + u^{\mathrm{T}}(t)u(t)\right]\mathrm{d}t, \tag{115}$$



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where $R_1 > 0$, is minimized in the sense of (34); and (112)–(114) globally asymptotically stable with respect to $x_1$ uniformly in $x_2(0)$.

Note that (112)–(114) with performance measure (115) can be casted in the form of (53) and (54) with performance measure (56). In this case, Theorem 4.1 can be applied with $n_1 = 2, n_2 = 1, m = 2$, $f(x_1, x_2) = [I_{23}\omega_2\omega_3,\ I_{31}\omega_3\omega_1,\ I_{12}\omega_1\omega_2]^{\mathrm{T}} - Ax_1$, $A \triangleq \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \end{bmatrix}^{\mathrm{T}}$, $G(x_1, x_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{\mathrm{T}}$, $L_1(x_1, x_2) = x_1^{\mathrm{T}} R_1 x_1$, $L_2(x_1, x_2) = 0$, and $R_2(x_1, x_2) = I_2$ to characterize the optimal partially stabilizing controller. Specifically, in this case, (60) reduces to

$$\begin{aligned} 0 &= x_1^{\mathrm{T}} R_1 x_1 + V'(x_1, x_2) f(x_1, x_2) - V'(x_1, x_2) A x_1 \\ &\quad - \frac{1}{4} V'(x_1, x_2) G(x_1, x_2) G^{\mathrm{T}}(x_1, x_2) V'^{\mathrm{T}}(x_1, x_2), \qquad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}. \end{aligned} \tag{116}$$

Now, choosing $V(x_1, x_2) = x_1^{\mathrm{T}} P x_1$, where $P > 0$, it follows from (116) that

$$0 = x_1^{\mathrm{T}} R_1 x_1 + V'(x_1, x_2) f(x_1, x_2) - 2x_1^{\mathrm{T}} P H x_1 - x_1^{\mathrm{T}} P P x_1, \qquad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \tag{117}$$

where $H \triangleq \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix}$, and $V'(x_1, x_2) f(x_1, x_2) = 0$ only if

$$P = \rho J, \tag{118}$$

where $\rho > 0$ and $J \triangleq \begin{bmatrix} -I_{31} & 0 \\ 0 & I_{23} \end{bmatrix}$. In this case, (117) and (118) imply that

$$0 = R_1 - 2\rho J H - \rho^2 J^2. \tag{119}$$

Hence, (57) holds with $\alpha(\|x_1\|) = \rho\,\lambda_{\min}(J)\|x_1\|^2$ and $\beta(\|x_1\|) = \rho\,\lambda_{\max}(J)\|x_1\|^2$, where $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the minimum and maximum eigenvalues, respectively, and (58) holds with $\gamma(\|x_1\|) = \lambda_{\min}(R_1)\|x_1\|^2$.

Because all of the conditions of Theorem 4.1 hold, it follows that the feedback control law (60) given by

$$\begin{aligned} \phi(x_1, x_2) &= -\frac{1}{2} R_2^{-1}(x_1, x_2) G^{\mathrm{T}}(x_1, x_2) V'^{\mathrm{T}}(x_1, x_2) \\ &= -\rho J x_1, \qquad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \end{aligned} \tag{120}$$

guarantees that the dynamical system (112)–(114) is globally asymptotically stable with respect to $x_1$ uniformly in $x_2(0)$ and, for all $(x_1(0), x_2(0)) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$,

$$J(x_1(0), x_2(0), \phi(x_1(\cdot), x_2(\cdot))) = x_1^{\mathrm{T}}(0) P x_1(0). \tag{121}$$

Let $I_1 = 20\,\mathrm{kg{\cdot}m^2}$, $I_2 = 15\,\mathrm{kg{\cdot}m^2}$, $I_3 = 10\,\mathrm{kg{\cdot}m^2}$, $\omega_{10} = \pi/3\,\mathrm{Hz}$, $\omega_{20} = \pi/4\,\mathrm{Hz}$, $\omega_{30} = \pi/5\,\mathrm{Hz}$, $\alpha_1 = 1.1668\,\mathrm{Hz}$, $\alpha_2 = 0.2\,\mathrm{Hz}$, and $R_1 = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}\mathrm{Hz}^2$. Figure 1 shows the state trajectories of the controlled system versus time for $\rho = 1.81\,\mathrm{Hz/(N{\cdot}m^2)}$. Note that $x_1(t) = [\omega_1(t),\ \omega_2(t)]^{\mathrm{T}} \to 0$ as $t \to \infty$, whereas $x_2(t) = \omega_3(t)$ does not converge to zero. Figure 2 shows the control signal versus time. Finally, $J(x_1(0), x_2(0), \phi(x_1(\cdot), x_2(\cdot))) = 1.6024\,\mathrm{Hz}^3$.

### 6.2. *Thermoacoustic combustion model*

In this example, we consider the control of thermoacoustic instabilities in combustion processes. Engineering applications involving steam and gas turbines and jet and ramjet engines for power generation and propulsion technology involve combustion processes. Because of the inherent coupling



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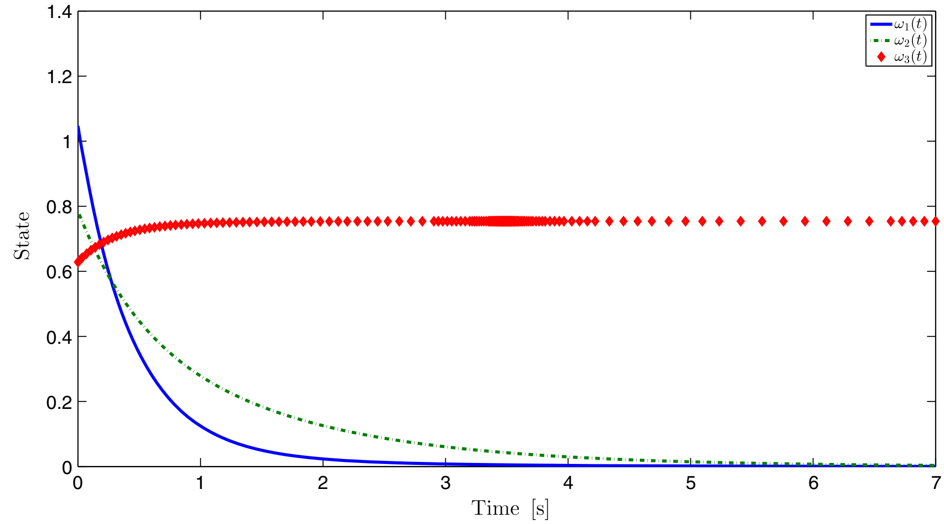


Figure 1. Closed-loop system trajectories versus time.

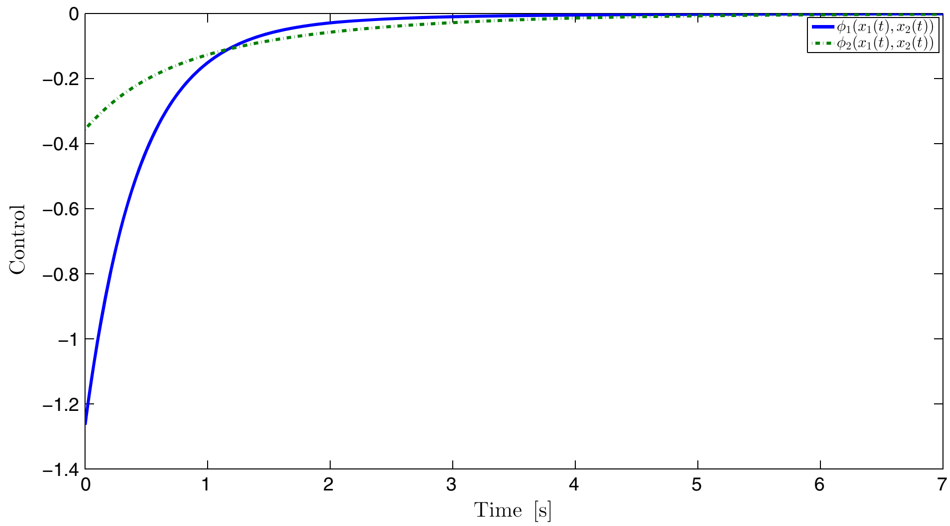


Figure 2. Control signal versus time.

between several intricate physical phenomena in these processes involving acoustics, thermodynamics, fluid mechanics, and chemical kinetics, the dynamic behavior of combustion systems is characterized by highly complex nonlinear models [17, 18]. The unstable dynamic coupling between heat release in combustion processes generated by reacting mixtures releasing chemical energy and unsteady motions in the combustor develops acoustic pressure and velocity oscillations that can severely affect operating conditions and system performance.

Consider the nonlinear dynamical system adopted from [2, 17] given by

$$\dot{q}_1(t) = -\alpha_1 q_1(t) - \beta q_1(t) q_2(t) \cos q_3(t) + u(t), \qquad q_1(0) = q_{10}, \qquad t \geqslant 0, \tag{122}$$

$$\dot{q}_2(t) = -\alpha_2 q_2(t) + \beta q_1^2(t) \cos q_3(t) + u(t), \qquad q_2(0) = q_{20} \neq 0, \tag{123}$$

$$\dot{q}_3(t) = 2\theta_1 - \theta_2 - \beta \left( \frac{q_1^2(t)}{q_2(t)} - 2q_2(t) \right) \sin q_3(t), \qquad q_3(0) = q_{30}, \tag{124}$$

 
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representing a time-averaged, two-mode thermoacoustic combustion model, where $\alpha_1 > 0$ and $\alpha_2 > 0$ represent decay constants; $\theta_1$ and $\theta_2 \in \mathbb{R}$ represent frequency shift constants; $\beta = ((\gamma+1)/8\gamma)\,\omega_1$, where $\gamma$ denotes the ratio of specific heats and $\omega_1$ is the frequency of the fundamental mode; and $u$ is the control input signal. As shown in [17] and [18], only the first two states $q_1$ and $q_2$ representing the modal amplitudes of a two-mode thermoacoustic combustion model are relevant in characterizing system instabilities because the third state $q_3$ represents the phase difference between the two modes [19]. Hence, we require asymptotic stability of $q_1(t)$, $t \geqslant 0$, and $q_2(t)$, $t \geqslant 0$, which necessitates partial stabilization.

For this example, we seek a state feedback controller $u = \phi(x_1, x_2)$, where $x_1 = [q_1,\ q_2]^{\mathrm{T}}$ and $x_2 = q_3$, such that the performance measure

$$J(x_1(0), x_2(0), u(\cdot)) = \int_0^\infty \left[x_1^{\mathrm{T}}(t) R_1 x_1(t) + u^2(t)\right] \mathrm{d}t, \tag{125}$$

where

$$R_1 = \rho \begin{bmatrix} 2\alpha_1 + \rho & \rho \\ \rho & 2\alpha_2 + \rho \end{bmatrix}, \qquad \rho > 0, \tag{126}$$

is minimized in the sense of (34), and (122)–(124) is globally asymptotically stable with respect to $x_1$ uniformly in $x_2(0)$.

Note that (122)–(124) with performance measure (125) can be casted in the form of (53) and (54) with performance measure (56). In this case, Theorem 4.1 can be applied with $n_1 = 2$, $n_2 = 1$, $m = 1$, $f(x_1, x_2) = \left[-\alpha_1 q_1 - \beta q_1 q_2 \cos q_3,\ -\alpha_2 q_2 + \beta q_1^2 \cos q_3,\ 2\theta_1 - \theta_2 - \beta \left(\frac{q_1^2}{q_2} - 2q_2\right) \sin q_3\right]^{\mathrm{T}}$, $G(x_1, x_2) = \begin{bmatrix} 1 & 1 & 0\end{bmatrix}^{\mathrm{T}}$, $L_1(x_1, x_2) = x_1^{\mathrm{T}} R_1 x_1$, $L_2(x_1, x_2) = 0$, and $R_2(x_1, x_2) = 1$ to characterize the optimal partially stabilizing controller. Specifically, (60) reduces to

$$\begin{aligned} 0 = x_1^{\mathrm{T}} R_1 x_1 + V'(x_1, x_2) f(x_1, x_2) - \frac{1}{4} V'(x_1, x_2) G(x_1, x_2) G^{\mathrm{T}}(x_1, x_2) V'^{\mathrm{T}}(x_1, x_2), \\ (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \end{aligned} \tag{127}$$

which implies that

$$V'(x_1, x_2) = 2\rho\,[q_1,\ q_2,\ 0]. \tag{128}$$

Furthermore, because $V(0, x_2) = 0$, $x_2 \in \mathbb{R}$,

$$V(x_1, x_2) = \rho\, x_1^{\mathrm{T}} x_1, \tag{129}$$

which is positive definite with respect to $x_1$, and hence, (57) holds.

Because all of the conditions of Theorem 4.1 hold, it follows that the feedback control (61) given by

$$\begin{aligned} \phi(x_1, x_2) &= -\frac{1}{2} R_2^{-1}(x_1, x_2) G^{\mathrm{T}}(x_1, x_2) V'^{\mathrm{T}}(x_1, x_2) \\ &= -\rho \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} q_1 & q_2 & 0 \end{bmatrix}^{\mathrm{T}} \\ &= -\rho \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \end{aligned} \tag{130}$$

guarantees that the dynamical systems (122)–(124) is globally asymptotically stable with respect to $x_1$ uniformly in $x_2(0)$ and, for all $(x_1(0), x_2(0)) \in \mathbb{R}^2 \times \mathbb{R}$,

$$J(x_1(0), x_2(0), \phi(x_1(\cdot), x_2(\cdot))) = \rho\, x_1^{\mathrm{T}}(0) x_1(0). \tag{131}$$



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Let $\alpha_1 = 5\,\text{Hz}$, $\alpha_2 = 45\,\text{Hz}$, $\gamma = 1.4$, $\omega_1 = 1\,\text{Hz}$, $\theta_1 = 4\,\text{Hz}$, $\theta_2 = 32\,\text{Hz}$, $\rho = 1\,\text{Hz}$, $q_{10} = 2$, $q_{20} = 1$, and $q_{30} = 3$. Figure 3 shows the state trajectories of the controlled system versus time. Note that $x_1(t) = [q_1(t), q_2(t)]^{\mathrm{T}} \to 0$ as $t \to \infty$, whereas $x_2(t) = q_3(t)$ is unstable. Figure 4 shows the control signal versus time. Finally, $J(x_1(0), x_2(0), \phi(x_1(\cdot), x_2(\cdot))) = 5\,\text{Hz}$.

### *6.3. Inverse optimal control of an axisymmetric spacecraft*

For our final example, we consider a spacecraft with one axis of symmetry [20, p. 753] given by

$$\dot{\omega}_1(t) = I_{23}\omega_2(t)\omega_3(t) + \alpha_1 u_1(t), \qquad \omega_1(0) = \omega_{10}, \qquad t \geq 0, \tag{132}$$

$$\dot{\omega}_2(t) = -I_{23}\omega_3(t)\omega_1(t) + \alpha_2 u_2(t), \qquad \omega_2(0) = \omega_{20}, \tag{133}$$

$$\dot{\omega}_3(t) = \alpha_3 u_1(t) + \alpha_4 u_2(t), \qquad \omega_3(0) = \omega_{30}, \tag{134}$$

where $I_{23} \triangleq (I_2 - I_3)/I_1$, $I_1$, $I_2$, and $I_3$ are the principal moments of inertia of the spacecraft such that $0 < I_1 = I_2 < I_3$, $\alpha_1$, $\alpha_2$, $\alpha_3$, and $\alpha_4 \in \mathbb{R}$; $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$; and $u_1$ and $u_2$ are

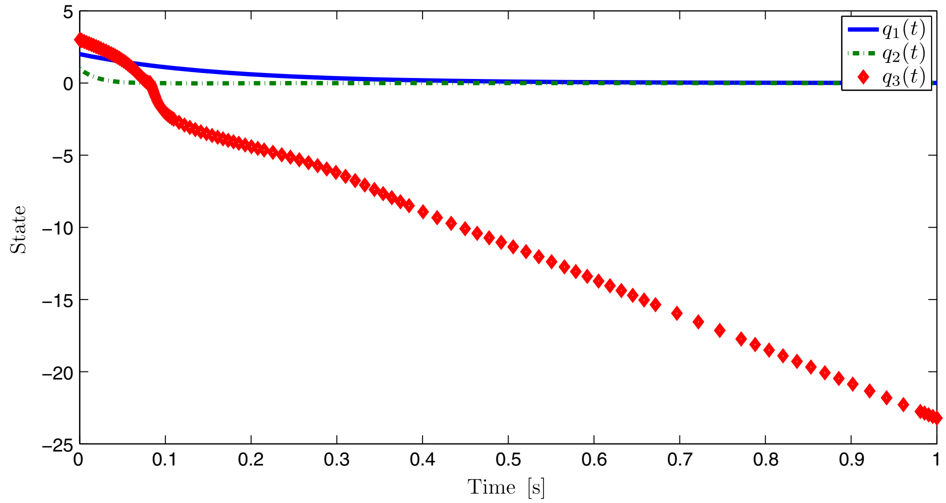


Figure 3. Closed-loop system trajectories versus time.

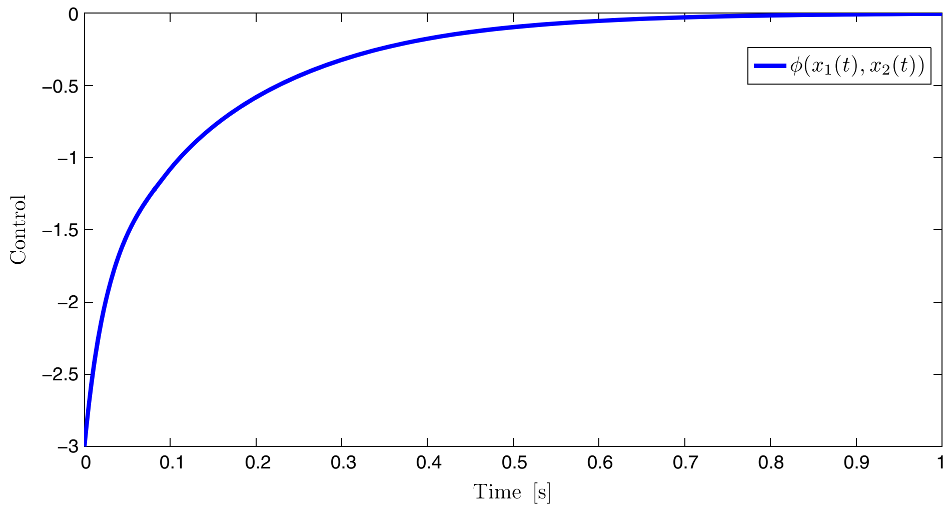


Figure 4. Control signal versus time.



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the spacecraft control moments. In this example, we apply Theorem 5.1 to find an *inverse optimal* globally partial-state stabilizing control law $u = [u_1, u_2]^{\mathrm{T}} = \phi(x_1, x_2)$, where $x_1 = [\omega_1, \omega_2]^{\mathrm{T}}$ and $x_2 = \omega_3$, such that the spacecraft is spin-stabilized about its third principle axis of inertia; that is, the dynamical system (132)–(134) is globally asymptotically stable with respect to $x_1$ uniformly in $x_2(0)$. Note that (132)–(134) can be casted in the form of (53) and (54), with $n_1 = 2$, $n_2 = 1$, $m = 2$, $f(x_1, x_2) = [I_{23}\omega_2\omega_3, -I_{23}\omega_3\omega_1, 0]^{\mathrm{T}}$, and $G(x_1, x_2) = \begin{bmatrix} \alpha_1 & 0 & \alpha_3 \\ 0 & \alpha_2 & \alpha_4 \end{bmatrix}^{\mathrm{T}}$.

To construct an inverse optimal controller for (132)–(134), let

$$V(x_1, x_2) = x_1^{\mathrm{T}} \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix} x_1, \tag{135}$$

where $p_1$ and $p_2 > 0$, $L(x_1, x_2, u) = L_1(x_1, x_2) + L_2(x_1, x_2)u + u^{\mathrm{T}}u$, and let

$$L_2(x_1, x_2) = 2\left[-\frac{I_{23}}{\alpha_1}\omega_2\omega_3, \frac{I_{23}}{\alpha_2}\omega_1\omega_3\right]. \tag{136}$$

Now, the inverse optimal control law (80) is given by

$$\phi(x_1, x_2) = \left[-\alpha_1 p_1\omega_1 - \frac{I_{23}}{\alpha_1}\omega_2\omega_3, -\alpha_2 p_2\omega_2 + \frac{I_{23}}{\alpha_1}\omega_1\omega_3\right]^{\mathrm{T}}, \tag{137}$$

and, in this case, the performance functional (56), with

$$L_1(x_1, x_2) = \omega_1^2\left(\alpha_1^2 p_1^2 + \frac{\omega_3^2}{\alpha_2^2} I_{23}^2\right)^2 + (\alpha_2 p_2\omega_2)^2 + \left(I_{23}\frac{\omega_2\omega_3}{\alpha_1}\right), \tag{138}$$

is minimized in the sense of (82). Furthermore, because (77) holds with $\alpha(\|x_1\|) = \beta(\|x_1\|) = p_1\omega_1^2 + p_2\omega_2^2$ and because

$$\begin{aligned} &V'(x_1, x_2)\left[f(x_1, x_2) - \frac{1}{2}G(x_1, x_2)L_2^{\mathrm{T}}(x_1, x_2) - \frac{1}{2}G(x_1, x_2)G^{\mathrm{T}}(x_1, x_2)V'^{\mathrm{T}}(x_1, x_2)\right] \\ &= -2\alpha_1^2 p_1^2\omega_1^2 - 2\alpha_2^2 p_2^2\omega_2^2, \qquad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \end{aligned} \tag{139}$$

(78) holds with $\gamma(\|x_1\|) = 2\alpha_1^2 p_1^2\omega_1^2 + 2\alpha_2^2 p_2^2\omega_2^2$. Therefore, with the feedback control law $\phi(x_1, x_2)$ given by (135), the closed-loop system (132)–(134) is globally asymptotically stable with

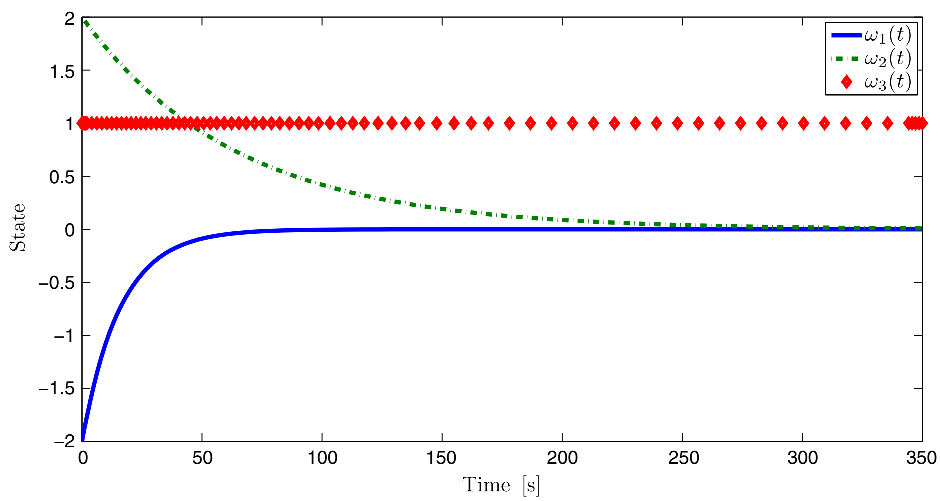


Figure 5. Closed-loop system trajectories versus time.



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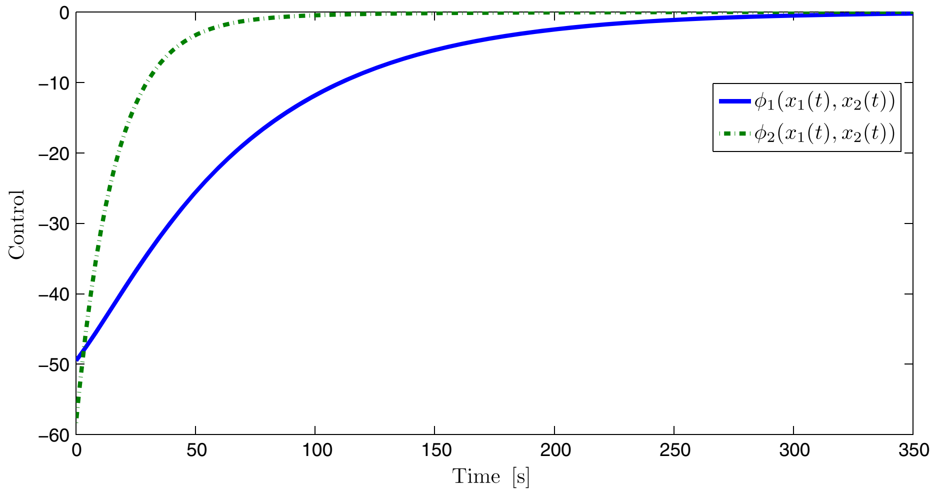


Figure 6. Control signal versus time.

respect to $x_1$ uniformly in $x_2(0)$. Note that $\phi(x_1, x_2)$, $L_1(x_1, x_2)$, and $\gamma(\|x_1\|)$ do not depend on $\alpha_3$ or $\alpha_4$.

Let $p_1 = 200$, $p_2 = 50$, $I_1 = I_2 = 4\,\mathrm{kg \cdot m^2}$, $I_3 = 20\,\mathrm{kg \cdot m^2}$, $\alpha_1 = \frac{\sqrt{2}}{2I_1}$, $\alpha_2 = \frac{\sqrt{2}}{2I_2}$, $\alpha_3 = \alpha_4 = 0$, $\omega_{10} = -2\,\mathrm{Hz}$, $\omega_{20} = 2\,\mathrm{Hz}$, and $\omega_{30} = 1\,\mathrm{Hz}$, Figure 5 shows the state trajectories of the controlled system versus time. Note that $x_1(t) = [\omega_1(t),\ \omega_2(t)]^{\mathrm{T}} \to 0$ as $t \to \infty$ and $x_2(t) = \omega_3(t) = \omega_{30}$, $t \geqslant 0$. Figure 6 shows the control signal versus time. Finally, $J(x_1(0), x_2(0), \phi(x_1(\cdot), x_2(\cdot))) = 1000\,\mathrm{Hz}^2$.

## 7. CONCLUSION

In this paper, an optimal control problem for partial-state stabilization is stated, and sufficient conditions are derived to characterize an optimal nonlinear feedback controller that guarantees asymptotic stability of part of the closed-loop system state. Specifically, we utilized a steady-state Hamilton–Jacobi–Bellman framework to characterize optimal nonlinear feedback controllers with a notion of optimality that is directly related to a given Lyapunov function that is positive definite and decrescent with respect to part of the system state. This result was then used to address optimal linear and nonlinear regulation for linear and nonlinear time-varying systems with quadratic and nonlinear nonquadratic performance measures. In addition, we developed inverse optimal feedback controllers for affine nonlinear systems and linear time-varying systems with polynomial and multilinear performance criteria. Extensions of this framework for addressing optimal adaptive controllers are currently under development.

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