On the Fuel and Energy Consumption Optimization Problem in Aircraft Path Planning

Andrea L'Afflitto, *Member, IEEE*, Cornel Sultan, *Senior Member, IEEE*

***Abstract*—Optimal trajectories for aircraft in terms of fuel consumption and energy consumption are analyzed and proven to exist by means of necessary and sufficient conditions. In some cases solutions in closed analytical form are found. Environmental models account both for aerodynamic forces and moments acting on aircraft modeled as 6 DOF rigid bodies.**

I. Introduction

FINDING optimal trajectories for formations of aircraft flying in realistic environmental and dynamical conditions can be of paramount complexity or impossible in some circumstances [1]. Feasible approaches to this problem range from purely numerical integration algorithms, whose computational costs are often a constraint to their applicability [2], to model simplifications aimed at achieving analytical closed form solutions, whose parametrization can be further utilized to compute sub-optimal trajectories for cases of increasing intricacy [3] - [5]. Choosing the most suitable integration technique needs a prior careful exploitation of the problem's structure and neglecting this task degrades the performances of the Guidance Navigation and Control subsystems onboard the vehicles [6].

Within the framework of a systematic study aimed at tackling the optimal trajectory and attitude planning problem for formations of aerospace vehicles, the fuel and energy consumption optimization for a 6DOF single aircraft flying in realistic environmental conditions is herein analytically addressed. Specifically, Pontryagin's Minimal Principle (PMP) is extensively employed to reach results not achievable otherwise. As the PMP states a necessary condition on the optimality of trajectories, it defines a set of candidate optimal solutions only. Under particular assumptions the PMP also states sufficient conditions and provides relevant results as shown in [4] but these achievements are not always applicable to some of the cases of interest for aerospace applications. It is herein shown how the set of candidate optimal trajectories can be circumscribed by exploiting the geometrical structure of the set of the costate vectors. Once the set of candidate optimal solutions has been restricted, sufficient conditions of Calculus of Variations can be applied as in [5]. Results provided are propedeutic to studies of formations of 6DOF vehicles flying in realistic environmental conditions.

A. L'Afflitto is with the Aerospace and Ocean Engineering Department, Virginia Tech, VA 24061 USA (e-mail: andrea.lafflitto@vt.edu).

C. Sultan is with the Aerospace and Ocean Engineering Department, Virginia Tech, VA 24061 USA (e-mail: csultan@vt.edu).

II. Fuel and Energy Consumption Cost Indices

The optimal path and attitude planning problem aims at accomplishing an assigned mission while optimizing cost indices such as coverage [7], time [8], robustness [9], or control effort [10]. The control effort can be quantified by the energy and the fuel consumption [1], [3] - [5], [11] - [13]. Let $u(\cdot)$ be the acceleration induced by the controllers. The fuel consumption is given by

$$J_F\left(u(\cdot)\right)=\int_{t_1}^{t_2}\left\|u(t)\right\|_2 dt \tag{1}$$

while the energy consumption is modeled as

$$J_E\left(u(\cdot)\right)=\int_{t_1}^{t_2}\left\|u(t)\right\|_2^2 dt. \tag{2}$$

III. Physical and Mathematical Background

Fixed an inertial reference frame and a principal body reference frame, an aircraft, schematized as a 6 DOF rigid body of constant mass m and matrix of intertia I_{in}, is identified by the *position vector* $r(\cdot):[t_1,t_2]\to\mathbb{R}^3$. Let $v(t):=dr(t)/dt$, $a(t):=dv(t)/dt$, $\omega(\cdot):[t_1,t_2]\to\mathbb{R}^3$ be the angular velocity vector in the body reference frame, and $\sigma(t):=\int_{t_1}^{t}\omega(s)ds$.

The superposition principle holds. Aircraft are modeled as subject to $a_a\left(v(t)\right)=\left\|v(t)\right\|_2^2\left(-\tilde{k}_D\hat{v}(t)+\tilde{k}_L\hat{v}_\perp(t)+\tilde{k}_S\hat{v}_\times(t)\right)$, the aerodynamic acceleration, with $\tilde{k}_{D/L/S}=\dfrac{\rho S C_{D/L/S}}{2m}$, ρ the air density, S the reference area, $C_{D/L/S}$ the drag/lift/side force coefficients, $\hat{v}(\cdot)$, $\hat{v}_\perp(\cdot)$, and $\hat{v}_\times(\cdot)$ the corresponding velocity unit vectors. A constant gravitational acceleration g is accounted. Assuming that at any point of the aircraft the translational velocity is much larger than the tangential one due to the the roational dynamics, it experiences a moment $M_a\left(v(t),\omega(t)\right):=\left\|v(t)\right\|_2^2\left(K_R\hat{\omega}_R(t)+K_P\hat{\omega}_P(t)+K_Y\hat{\omega}_Y(t)\right)$ due to aerosynamics, with $K_{R/P/Y}=\dfrac{\rho S c C_{R/P/Y}}{2}$, c a suitable reference length, $C_{R/P/Y}$ the aerodynamic moment coefficients along the first, second, and third axis of the principal body reference frame, $\hat{\omega}_{R/P/Y}(\cdot)$ the angular velocity unit vectors respectively.

Let the *state vector* be $x(\cdot):=\left[x_1^T(\cdot)\ x_2^T(\cdot)\right]^T$, where $x_1(\cdot):=\left[r^T(\cdot)\ v^T(\cdot)\right]^T$ and $x_2(\cdot):=\left[\sigma^T(\cdot)\ \omega^T(\cdot)\right]^T$, and let the *control vector* be $u(\cdot):=\left[u_1^T(\cdot)\ u_2^T(\cdot)\right]^T$, where $u(\cdot):[t_1,t_2]\subset\mathbb{R}\to\Gamma\subset\mathbb{R}^6$. The vectors $u_1(\cdot)$ and $u_2(\cdot)$ respectively represent the linear and the angular accelerations induced by the controllers. Dynamic equations are

$$x'(t)=\begin{bmatrix} v(t) \\ g+a_a(v(t)) \\ \omega(t) \\ -I_{in}^{-1}\omega^{\times}(t)I_{in}\omega(t)+\Omega_a(v(t),\omega(t)) \end{bmatrix}+\begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \quad (3)$$

with boundary conditions $r(t_i)=r_i$, $v(t_i)=v_i$, $\sigma(t_i)=\sigma_i$, and $\omega(t_i)=\omega_i$, $i\in\{1,2\}$, where r_i, v_i, σ_i, and ω_i are given. It holds that $\Omega_a(v(t),\omega(t)):=I_{in}^{-1}M_a(v(t),\omega(t))$, and I is the unit matrix. The right hand side of (3) is defined as $f_0(x(\cdot),u(\cdot))$. To prevent singularities, $\|v(t)\|_2\neq 0$, $\forall t\in[t_1,t_2]$, and $\|\omega(t)\|_2\neq 0$.

Dynamical systems considered herein are autonomous and optimized on a *fixed* time interval $[t_1,t_2]\subset\mathbb{R}$.

In order to optimize (1), Γ is open and bounded by two concentric hyperspheres in $\mathbb{R}^6$ of suitable radii ρ_{F1} and ρ_{F2}, $\rho_{F1}<\rho_{F2}$, centered in the origin. To optimize (2), Γ is an open hypersphere of radius ρ_E centered in the origin.

In the following the asterix (*) used as subscript indicates optimal or candidate optimal vectors. The apex (') denotes the first derivative with respect to the independent variable. Integration constants are denoted by k.

IV. Fuel Consumption Optimization

Lemma 1 Any $u_*(\cdot)\in\Gamma$, such that $u_*(\cdot)/\|u_*(\cdot)\|_2$ satisfies (3), its boundary conditions, and is solution of

$$\begin{cases} \begin{bmatrix} 0 & 0 & 0 & 0 \\ I & \mathcal{A}(v_*(t)) & 0 & \mathcal{E}(v_*(t),\omega_*(t)) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & \mathcal{M}(\omega_*(t))+\mathcal{Z}(v_*(t),\omega_*(t)) \end{bmatrix}\lambda(t)=-\lambda'(t) \\ \lambda_0\dfrac{u_*(t)}{\|u_*(t)\|_2}+\begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}\lambda(t)=0 \end{cases} \quad (4)$$

with arbitrary boundary conditions on $\lambda(t_1)$, $\lambda(t_2)$, is a candidate optimal solution for the cost index (1).

The matrices $\mathcal{Z}(v(\cdot),\omega(\cdot))$, $\mathcal{A}(v(\cdot))$, $\mathcal{M}(\omega(\cdot))$, $\omega^{\times}(\cdot)$ (skew symmetric), and $\mathcal{E}(v(\cdot),\omega(\cdot))$ are in Appendix A.

Proof (brief): The Hamiltonian function is

$$H=\lambda_0\|u(t)\|_2+\left[\lambda_1^T(t)\lambda_2^T(t)\lambda_3^T(t)\lambda_4^T(t)\right]f_0(x(t),u(t)) \quad (5)$$

with $\lambda_i(\cdot)$, $i\in\{0,1,2,3,4\}$, *costate vectors*, and $\lambda(t):=\left[\lambda_1^T(t)\lambda_2^T(t)\lambda_3^T(t)\lambda_4^T(t)\right]^T$. By imposing

$$\frac{\partial H(\lambda(t),x(t),u)}{\partial u}=0,\ \frac{\partial H(\lambda(t),x(t),u)}{\partial x}=-\lambda'(t), \quad (6)$$

the system (4) is achieved. The problem is *normal* [18], i.e. $\lambda_0\neq 0$: assume that $\lambda_0=0$, then $\lambda(t)\equiv 0$ for all $t\in[t_1,t_2]$, which is not admissible by PMP. Thus it is herein assumed that $\lambda_0=1$. It holds that $\dfrac{\partial^2}{\partial u^2}\|u(t)\|_2=\dfrac{\|u(t)\|_2^2 I-u(t)u^T(t)}{\|u(t)\|_2^{3/2}}$, whose eigenvalues are $\left(0,\|u(t)\|_2^{-1},\|u(t)\|_2^{-1}\right)$ and whose determinant is always zero for all $t\in[t_1,t_2]$. Thus $\dfrac{\partial^2}{\partial u^2}\|u(t)\|_2\succeq 0$ where the inequality indicates positive semi definiteness. Thus, $u_*(\cdot)$ is a candidate minimizer for (1) □.

It is remarkable that the direction of $u_*(\cdot)$, i.e. $u_*(\cdot)/\|u_*(\cdot)\|_2$, is provided by (4) but *not* its magnitude. Therefore, accounting for the convexity of $\|u(\cdot)\|_2$ on $\mathbb{R}$, $u_*(\cdot)\in\partial\Gamma$, the closure of Γ. This result does not depend on the dynamic equations but only on the cost index addressed.

The matrix $\mathcal{M}(\omega(\cdot))$ is invertible if $\omega_i(\cdot)\neq\omega_j(\cdot)$ and $I_i\neq I_j$, $(i,j)\in\{R,P,Y\}^2$. Symmetric vehicles can be similarly addressed but it is beyond the scopes of this work.

Theorem 1 The candidate optimal solution $u_*(\cdot)$ for (1) subject to (3) is such that

$$\left\|\mathcal{A}^{-1}(v_*(t))\left[\|u_*(t)\|_2 u_{1*}'(t_1)+\left(u_*'^T(t)u_*(t)\right)u_{1*}(t)+\right.\right. \\ \left.\left.+\|u_*(t)\|_2^2 k_4-\|u_*(t)\|_2\,\mathcal{E}(v_*(t),\omega_*(t))u_{2*}(t)\right]\right\|_2\leq\|u_*(t)\|_2^2 \quad (7)$$

and such that

$$\left\|\left(\mathcal{M}(\omega_*(t))+\mathcal{Z}(v_*(t),\omega_*(t))\right)^{-1}\left[\|u_*(t)\|_2 u_{2*}'(t)+\right.\right. \\ \left.\left.+\left(u_*'^T(t)u_*(t)\right)u_{2*}(t)+\|u_*(t)\|_2^2 k_2\right)\right\|_2\leq\|u_*(t)\|_2^2. \quad (8)$$

Proof (brief): From (4) it follows that, $\lambda_1(t)=k_1$,

$\lambda_3(t)=k_2$, $\lambda_4^{'}(t)=-k_2-\left(\mathcal{M}\left(\omega_*(t)\right)+\mathcal{Z}\left(v_*(t),\omega_*(t)\right)\right)\lambda_4(t)$, $\lambda_2^{'}(t)=-k_1-\mathcal{A}\left(v_*(t)\right)\lambda_2(t)-\mathcal{E}\left(v_*(t),\omega_*(t)\right)\lambda_4(t)$, with k_1 and k_2 constants. In addition it holds that $\frac{u_{1*}(t)}{\left\|u_*(t)\right\|_2}=-\lambda_2(t)$, and $\frac{u_{2*}(t)}{\left\|u_*(t)\right\|_2}=-\lambda_4(t)$. Because $\left\|\lambda_2(t)\right\|_2^2\le 1$, and $\left\|\lambda_4(t)\right\|_2^2\le 1$, by exploiting $\lambda_2^{'}(\cdot)$ and $\lambda_4^{'}(\cdot)$ as functions of $u_*(\cdot)$, (7) and (8) are achieved □.

Whether the matrices $\mathcal{M}\left(\omega(\cdot)\right)+\mathcal{Z}\left(v(\cdot),\omega(\cdot)\right)$ and $\mathcal{A}\left(v(\cdot)\right)$ can be inverted needs to be studied case by case.

A direct consequence of Theorem 1 is:

Corollary 1 Assume $u_*(\cdot)$ is constrained on a hypersphere of radius ρ_{F1} centered in the origin of the control space, then the following relations hold

$$\begin{aligned}&\left\|\mathcal{A}^{-1}\left(v_*(t)\right)\left[\rho_{F1}u_{1*}^{'}(t_1)+\left(u_*^{'T}(t)u_*(t)\right)u_{1*}(t)+\right.\right.\\&\left.\left.+\rho_{F1}^2k_4-\rho_{F1}\mathcal{E}\left(v_*(t),\omega_*(t)\right)u_{2*}(t)\right]\right\|_2=\rho_{F1}^2,\end{aligned}\tag{9}$$

$$\begin{aligned}&\left\|\left(\mathcal{M}\left(\omega_*(t)\right)+\mathcal{Z}\left(v_*(t),\omega_*(t)\right)\right)^{-1}\left[\rho_{F1}u_{2*}^{'}(t)+\right.\right.\\&\left.\left.+\left(u_*^{'T}(t)u_*(t)\right)u_{2*}(t)+\rho_{F1}^2k_2\right]\right\|_2=\rho_{F1}^2.\end{aligned}\tag{10}$$

Under simplified assumptions, the fact that $u_*(\cdot)\in\partial\Gamma$ can be proven by differential geometry for vehicles whose lift does not depend on $v(\cdot)$, e.g. aerostats:

Theorem 2 Let $u_2(t)\equiv 0$ and $\omega(t_1)=0$, then the optimal control vector for (1) is such that $u_*(\cdot)\in\partial\Gamma$.

Proof (brief): Under these assumptions $\sigma(t)=const$ and, as $u_1(t)\equiv u(t)$, from Lemma 1 it follows that

$$\begin{cases}\frac{u_*(t)}{\left\|u_*(t)\right\|_2}=-\lambda_2(t),\ \lambda_2^{'}(t)=-\mathcal{D}\left(v_*(t)\right)\lambda_2(t)\\ \lambda_2(t_2)\ \text{arbitrary.}\end{cases}\tag{11}$$

Consequently it must hold that

$$\left\|\lambda_2(t)\right\|_2^2=1\ \text{and}\ \lambda_2^T(t)\mathcal{D}\left(v_*(t)\right)\lambda_2(t)=0.\tag{12}$$

The first of (12) is a sphere centered at the origin of the space of the costate vector $\lambda_2(\cdot)$ while the second of (12) is a non-rectangular non-orthogonal cone whose normal section is non-circular and whose vertex is at the origin of the costate space (Theorem 7 - Appendix B) [19]. Therefore the direction of the optimal control vector $\frac{u_*(\cdot)}{\left\|u_*(\cdot)\right\|_2}$ is described by the intersection of a sphere and a cone in the space of $\lambda_2(\cdot)$. By diagonalizing $\mathcal{D}\left(v(\cdot)\right)$ and accounting for its eigenvalues (Appendix A), it follows that there is no $\lambda_2(\cdot)$ such that (12) has a real solution. Thus there not exist any $u(\cdot)\in\Gamma$ that minimizes (1), hence $u_*(\cdot)\in\partial\Gamma$ □.

A. *Negligible Aerodynamic Forces and Moments*

By simplifying the environmental model, further analytical results can be achieved.

Lemma 2 Let $a(t)=u_1(t)+g$ and $\Omega_a\left(v(t),\omega(t)\right)\equiv 0$, then

$$\begin{aligned}&\left(k_3^Tk_4\right)^2\ge k_3^Tk_3\left(1-k_4^Tk_4\right),\ \text{if}\ -\frac{k_3^Tk_4}{k_3^Tk_3}\in(t_1,t_2)\\&k_3^Tk_3t_i^2+2k_3^Tk_4t_i+k_4^Tk_4\le 1,\begin{cases}i=1\ \text{if}\ t_1\ge-\frac{k_3^Tk_4}{k_3^Tk_3}\\ i=2\ \text{if}\ t_2\le-\frac{k_3^Tk_4}{k_3^Tk_3}\end{cases}\end{aligned}\tag{13}$$

where $\frac{u_{1*}(t)}{\left\|u_*(t)\right\|_2}=k_3t+k_4$, and

$$\begin{aligned}\rho_{F1}^2=&\left\|\mathcal{M}^{-1}\left(\omega_*(t)\right)\left[\rho_{F1}u_{2*}^{'}(t)+\right.\right.\\&\left.\left.+\left(u_*^{'T}(t)u_*(t)\right)u_{2*}(t)+\rho_{F1}^2k_5\right]\right\|_2.\end{aligned}\tag{14}$$

Proof (brief): From Lemma 1 it follows that $\lambda_1(t)=-k_3$, $\frac{u_{1*}(t)}{\left\|u_*(t)\right\|_2}=k_3t+k_4$, $\lambda_3(t)=k_5$, $\frac{u_{2*}(t)}{\left\|u_*(t)\right\|_2}=-\lambda_4(t)$, and $\lambda_4^{'}(t)=-k_5-\mathcal{M}\left(\omega_*(t)\right)\lambda_4(t)$. As $\left\|\lambda_2(t)\right\|_2\le 1\quad\forall t\in[t_1,t_2]$, then it needs to hold that $\min_{t\in[t_1,t_2]}\left\|\lambda_2(t)\right\|_2^2\le 1$ which results in (13). Differentiating $\frac{u_{2*}(\cdot)}{\left\|u_*(\cdot)\right\|_2}$, reminding that $\left\|\lambda_4(t)\right\|_2\le 1$, and that $u_*(\cdot)\in\partial\Gamma$, (14) is obtained □.

Theorem 3 Let $u_2(t)\equiv 0$ and $\omega(t_1)=0$, then

$$r_*(t)=\frac{1}{2}\tilde{g}(t-t_1)^2+v_1(t-t_1)+r_1,\ u_*(t)=\rho_{F1}k_4\tag{15}$$

where $\tilde{g}=g+\rho_{F1}k_4$ is the pseudo gravitational acceleration.

Proof (brief): Under these assumptions $\sigma(t)=const$ and as $u_1(t)\equiv u(t)$, from Lemma 2 and Theorem 2 it follows that $\frac{u_*(t)}{\|u_*(t)\|_2}=k_3t+k_4$. As $\|k_3t+k_4\|_2=1$ for any $t\in[t_1,t_2]$, it must hold that $k_3=0$ and (15) is obtained □.

Corollary 1 If $u_1(\cdot):[t_1,t_2]\subset\mathbb{R}\to\Gamma_1\subset\mathbb{R}^3$, with Γ_1 open convex compact and such that $\Gamma_1\cap\{0\}=\varnothing$, then (1) is minimized if and only if (15) holds.

Proof (brief): The proof of necessity is the same as in Theorem 3. As the integrand of (1) is convex on Γ_1 and as $f_0(\cdot,\cdot)$ is linear in its components, then also the sufficiency of the PMP for the optimality of (15) holds [18].

V. Energy Consumption Optimization

Theorem 4 Necessary condition to optimize (2) subject to (3) is that the following hold:

$$\begin{cases} a_*(t)=g+a_a(v_*(t))+u_{1*}(t) \\ u_{1*}'(t)=k_6-\left(\mathcal{A}(v_*(t))u_{1*}(t)+\mathcal{E}(v_*(t),\omega_*(t))\right)u_{2*}(t) \\ \omega_*'(t)=-I_{in}^{-1}\omega_*^{\times}(t)I_{in}\omega_*(t)+\Omega_a(v_*(t),\omega_*(t))+u_{2*}(t) \\ u_{2*}'(t)=k_7-\left(\mathcal{M}(\omega_*(t))+\mathcal{Z}(v_*(t),\omega_*(t))\right)u_{2*}(t) \\ r_*(t_i)=r_i,\sigma_*(t_i)=\sigma_i, i\in\{1,2\};\ u_*(t_1)\ \text{arbitrary.} \end{cases} \quad (16)$$

Proof (brief): The Hamiltonian is given by (5). By imposing (6), as the environmental conditions are the same as in Lemma 1, the first of (4) still holds. As $x(t_1)$ and $x(t_2)$ are assigned, then $\lambda(t_1)$ and $\lambda(t_2)$ are arbitrary. The second of (4) is replaced by

$$2\lambda_0 u_*(t)+\begin{bmatrix}0 & I & 0 & 0\\ 0 & 0 & 0 & I\end{bmatrix}\lambda(t)=0. \quad (17)$$

As for the fuel consumption optimization, the problem is normal, thus it is imposed that $\lambda_0=1$. From the first of (4) and (17) it follows that

$$\begin{cases} \lambda_1(t)=k_6;\ \lambda_3(t)=k_7 \\ \lambda_2'(t)=-k_6-\mathcal{A}(v_*(t))\lambda_2(t)-\mathcal{E}(v_*(t),\omega_*(t))\lambda_4(t) \\ \lambda_4'(t)=-k_7-\left(\mathcal{M}(\omega_*(t))+\mathcal{Z}(v_*(t),\omega_*(t))\right)\lambda_4(t) \\ u_{1*}(t)=-\lambda_2(t);\ u_{2*}(t)=-\lambda_4(t) \\ \lambda(t_1),\ \lambda(t_2)\ \text{arbitrary} \end{cases} \quad (18)$$

where k_6 and k_7 are constants. As $\frac{\partial^2}{\partial u^2}\|u(t)\|_2^2\succ 0$, any $u_*(\cdot)$ is a candidate minimizer for (2). From (18), by accounting for (3), the necessary condition (16) is proven □.

It is important to notice that Theorem 4 yields on Γ. For the sake of completeness, assigned ρ_E, (2) subject to (3) needs to be optimized also on the set $\partial\Gamma$.

Corollary 2 Let $\tilde{v}(\cdot)$ and $\tilde{\omega}(\cdot)$ be the linear and angular velocities of the aircraft for $u_1(t)\equiv 0$ and $u_2(t)\equiv 0$ respectively. Then, for any vector norm, $\|v_*(t)-\tilde{v}(t)\|$ and $\|\omega_*(t)-\tilde{\omega}(t)\|$ are bounded for all $t\in[t_1,t_2]$ and for all $(x_1(\cdot),x_2(\cdot))\in D_1\times D_2\subset\mathbb{R}^{12}$, open connected set.

Proof (brief): As $\tilde{v}'(t)=g+a_a(t)$ is differentiable on $[t_1,t_2]$ and for any $x_1(\cdot)\in D_1$, let L_1 be the Lipschitz constant for $\tilde{v}'(\cdot)$. Similarly as $\tilde{\omega}'(t):=\Omega_a(v(t),\omega(t))$ is differentiable on $[t_1,t_2]$ and for any $x_2(\cdot)\in D_2$, let L_2 be the Lipschitz constant for $\tilde{\omega}'(\cdot)$. By the PMP $u(\cdot)$ is bounded on Γ, hence $\|u_1(t)\|\le\mu_1$ and $\|u_2(t)\|\le\mu_2$ for any norm. By the Bounded Perturbation of State Equation theorem [20],

$$\begin{aligned} \|v_*(t)-\tilde{v}(t)\| &\le\left(\|v_*(t_1)-\tilde{v}(t_1)\|+\frac{\mu_1}{L_1}\right)e^{L_1(t-t_1)}-\frac{\mu_1}{L_1} \\ \|\omega_*(t)-\tilde{\omega}(t)\| &\le\left(\|\omega_*(t_1)-\tilde{\omega}(t_1)\|+\frac{\mu_2}{L_2}\right)e^{L_2(t-t_1)}-\frac{\mu_2}{L_2}\ \square. \end{aligned} \quad (19)$$

Because of the coupling in (16), analytical solutions, if any, are difficult to find [4] but boundness of $v_*(\cdot)$ and $\omega_*(\cdot)$ restricts the set of candidate solutions.

A. *Negligible Aerodynamic Forces and Moments*

By simplifing the environmental conditions, further results for the consumed energy optimization are achieved:

Theorem 5 Let $a(t)=u_1(t)+g$ and $\Omega_a(v(t),\omega(t))\equiv 0$. Then necessary and sufficient conditions to minimize (2) are

$$\begin{cases} r_*(t)=\frac{1}{6}k_8(t-t_1)^3+\frac{1}{2}k_9(t-t_1)^2+v_1(t-t_1)+r_1 \\ u_{1*}(t)=k_8t+k_9 \\ \omega_*'(t)=-I_{in}^{-1}\omega_*^{\times}(t)I_{in}\omega_*(t)+u_{2*}(t) \\ u_{2*}'(t)=k_{11}-\mathcal{M}(\omega_*(t))u_{2*}(t) \\ \sigma_*(t_1)=\sigma_1,\ \sigma_*(t_2)=\sigma_2,\ u_{2*}(t_1)\ \text{arbitrary} \end{cases} \quad (20)$$

where k_i, $i\in\{8,9\}$, are reported in Appendix A.

Proof (brief): Specializing (18) according to the hypothesis, $\lambda_1(t)=k_8$, $u_{1*}(t)=-\lambda_2(t)=k_8t+k_9$, $\lambda_3(t)=k_{11}$,

$\lambda_4^{'}(t) = -k_{11} - \mathcal{M}(\omega_*(t))\lambda_4(t)$, and $u_{2*}(t) = -\lambda_4(t)$. By integrating $a_*(\cdot)$, (20) is achieved. As Γ for the energy consumption optimization is an hypersphere in the space of the control vector and, as $f_0(\cdot,\cdot)$ is linear in its components, (20) provides a sufficient condition $u_*(\cdot)$ on Γ □.

It is worth to notice that, unlike in Lemma 2, the translational and the rotational dynamics are decoupled.

VI. Conclusions

In the context of a systematic study aimed at exploiting analytically the optimal path planning problem for aerospace vehicles, some aspects related to the optimization of the fuel and of the energy consumption for single aircraft flying in realistic environmental conditions have been analytically discussed. Advances presented are relevant both to further deepen previous achievements [1], [3] - [5] and to provide an adequate mathematical background to an analytical study of the fuel and energy consumption optimization problem for aircraft formations in realistic environmental conditions.

By modeling the vehicles as 6DOF rigid bodies subject to constant gravity acceleration and to aerodynamic forces and moments, necessary and sufficient conditions for optimality of the control actions and of the corresponding trajectories have been proven. Simplified dynamical models have also been addressed to further exploit analytical solutions.

It has been proven that the control vector that optimizes the fuel consumption lays on an hypersphere of assigned suitable radius. This result has been proven both by applying Calculus of Variations, i.e. Pontryagin's Principle, and by applying analytical differential geometry.

Future extensions of the results exposed concern both increasing the degree of complexity of the environmental and of the dynamical models, and the study of the fuel and energy optimization problem for formations of vehicles performing complex missions.

Appendix A - Matrices

Let the subscript (i,j) determine the element on the i-th row and j-th column of a matrix and let the subscript k identify the k-th component of a vector, with $(i,j,k) \in \{1,2,3\}^3$.

$$\mathcal{M}(\omega(t)) = \begin{bmatrix} 0 & -\omega_Y(t)\tilde{I}_2 & -\omega_P(t)\tilde{I}_3 \\ -\omega_Y(t)\tilde{I}_1 & 0 & -\omega_R(t)\tilde{I}_3 \\ -\omega_P(t)\tilde{I}_1 & -\omega_R(t)\tilde{I}_2 & 0 \end{bmatrix}$$

with $\tilde{I}_1 = \frac{I_P - I_Y}{I_R}$, $\tilde{I}_2 = \frac{I_R - I_Y}{I_P}$, $\tilde{I}_3 = \frac{I_P - I_R}{I_Y}$, and $I_{R/P/Y}$ the moments of inertia in the principal body reference frame.

$$\omega^\times(t) = \begin{bmatrix} 0 & -\omega_Y(t) & \omega_P(t) \\ \omega_Y(t) & 0 & -\omega_R(t) \\ -\omega_P(t) & \omega_R(t) & 0 \end{bmatrix}$$

Let $R(\cdot)$ be a rotation matrix such that $\hat{v}_\perp(t) = R(t)\hat{v}(t)$

$$\mathcal{A}(v(t)) = \mathcal{D}(v(t)) + \mathcal{L}(v(t)) + S(v(t))$$

$$\mathcal{D}_{(i,i)}(v(t)) = -\tilde{k}_D \frac{v_i^2(t) + \|v(t)\|_2^2}{\|v(t)\|_2}$$

$$\mathcal{D}_{(i,j)}(v(t)) = -\tilde{k}_D \frac{v_i(t)v_j(t)}{\|v(t)\|_2}$$

$$\mathcal{L}_{(i,i)}(v(t)) = \tilde{k}_L \left(R_{(i,i)} \|v(t)\|_2 + \sum_{k=1}^{3} R_{(i,k)}(t) v_i(t) v_k(t) \Big/ \|v(t)\|_2 \right)$$

$$\mathcal{L}_{(i,j)}(v(t)) = k_L \left(R_{(j,i)} \|v(t)\|_2 + \sum_{k=1}^{3} R_{(j,k)}(t) v_i(t) v_k(t) \Big/ \|v(t)\|_2 \right)$$

$$S(v(t)) = \tilde{k}_S \frac{\partial}{\partial v}\left[\|v(t)\|_2 \left(R(t)v(t) \right) \wedge \hat{v}(t) \right]$$

The eigenvalues of $\mathcal{D}(v(\cdot))$ are $-\|v(t)\|_2 (1,1,2)$. The eigenvectors $\left[-\frac{v_2(t)}{v_1(t)}\ 1\ 0 \right]^T$, $\left[-\frac{v_3(t)}{v_1(t)}\ 0\ 1 \right]^T$, and $v(t)$ respectively.

$$\mathcal{Z}(v(t),\omega(t)) = 2\tilde{K}\|v(t)\|_2^2 \left(\omega(t)\omega^T(t) - \|\omega(t)\|_2^2 I \right) \Big/ \|\omega(t)\|_2^3$$

where $\tilde{K}_{R/P/Y} = \frac{K_{R/P/Y}}{I_{R/P/Y}}$.

$$\mathcal{E}(v(t),\omega(t)) = -2\frac{\tilde{K}v(t)\omega^T(t)}{\|\omega(t)\|_2}$$

The integration constants in Theorem 5 are the following

$$k_8 = 6\frac{v_1 + v_2}{(t_2 - t_1)^2} - 12\frac{r_2 - r_1}{(t_2 - t_1)^3}$$

$$k_9 = 6\frac{r_2 - r_1}{(t_2 - t_1)^2} - 2\frac{2v_1 + v_2}{t_2 - t_1}$$

Appendix B

Theorem 6 Under the assumptions of Theorem 2,

$$\mathcal{D}'(v(t)) = \mathcal{D}(v(t))\mathcal{D}(v(t)). \quad (21)$$

Proof (brief): From the first of (11), it holds that

$\frac{d}{dt}\left(\lambda_2^T(t)\lambda_2(t)\right)=2\lambda_2^{'T}(t)\lambda_2(t)=0$. Accounting for the second of (11), (21) is achieved □.

Theorem 7 Under the assumptions of Theorem 2,

$$\lambda_2^T(t)\mathcal{D}\left(v_*(t)\right)\lambda_2(t)=0 \qquad (22)$$

is a non-rectangular non-orthogonal quadric cone in the space of λ_2, whose normal section is non-circular and its vertex is in the origin of the space of costate vectors.

Proof (brief): The $\mathbb{R}^{4\times 4}$ matrix associated to (22) is [19]

$$M\left(\mathcal{D}\left(v_*(t)\right)\right):=\begin{bmatrix}\mathcal{D}\left(v_*(t)\right) & 0\\ 0 & 0\end{bmatrix}.$$

Accounting for the fact that $\mathcal{D}\left(v(\cdot)\right)$ is positive definite, $rank\left(M\left(\mathcal{D}\left(v_*(t)\right)\right)\right)=3$. Therefore (22) is a quadric cone.

The origin clearly belongs to this quadric cone. By applying the same convention on subscripts as in Appendix A, let the components of $\mathcal{D}\left(v(\cdot)\right)$ be briefly identified by a_{ij}, $(i,j)\in\{1,2,3\}^2$. Eq. (22) is in the form $a_{11}x^2+2a_{12}xy+2a_{13}xz+a_{22}y^2+2a_{23}yz+a_{33}z^2=0$, where (x,y,z) represent the components of $\lambda_2(\cdot)$. As (22) is a cone, this form ensures the vertex to be in the origin [19].

Let the directions of the axis of the quadric cone be $(\lambda,\mu,\nu)\in\mathbb{R}^3$ (22), it is *circular* if $rank\left(\mathcal{D}\left(v_*(t)\right)-\lambda I\right)=1$, where I is the unit matrix. Accounting for the eigenvalues of $\mathcal{D}\left(v(\cdot)\right)$, $rank\left(\mathcal{D}\left(v_*(t)\right)-\lambda I\right)=2$, $\forall t\in[t_1,t_2]$, thus (22) has not a circular normal section.

The cone (22) is *rectangular* if $trace\left(\mathcal{D}\left(v_*(t)\right)\right)=0$ but under the assumptions given in par. III it holds that $trace\left(\mathcal{D}\left(v_*(t)\right)\right)=-4\tilde{k}_D\left\|v(t)\right\|_2\neq 0$.

The quadric cone (22) is *orthogonal* if $\varphi\left(\mathcal{D}\left(v_*(t)\right)\right):=a_{22}a_{33}+a_{33}a_{11}+a_{22}a_{11}-a_{23}^2-a_{13}^2-a_{12}^2=0$. In this case $\varphi\left(\mathcal{D}\left(v_*(t)\right)\right)=7\left(v_{1*}^2(t)v_{2*}^2(t)+v_{3*}^2(t)v_{2*}^2(t)+v_{1*}^2(t)v_{3*}^2(t)\right)+$ $+5\left(v_{1*}^4(t)+v_{2*}^4(t)+v_{3*}^4(t)\right)>0$ □.

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