Calculus of Variations for Guaranteed Optimal Path Planning of Aircraft Formations

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***Abstract*—Constrained trajectories for a formation of *N* aircraft are optimized in terms of kinetic energy, fuel and energy consumption. Each aircraft moves from an initial position, converges to an assigned waypoint, unique for the formation, passes through *m* assigned waypoints, avoids a designated area and finally passes through a given surface. Airplanes are modeled as 3DOF point masses that always maintain a minimum distance between them. Most of the solutions are in closed form and their optimality is guaranteed by necessary and sufficient conditions.**

I. Introduction

In a complex tactical and rescue mission, the members of a formation of aircraft typically have to accomplish four relevant tasks: *T1*) leave assigned initial positions and pass through a waypoint unique for all the vehicles, *T2*) intercept *m* distinct waypoints, *T3*) avoid a no-fly zone between two consecutive waypoints and *T4*) fly through a given surface. The formation must also be kept as tight as possible while minimizing a cost index and avoiding collisions.

This sort of complex mission planning is often addressed with *purely* numerical solutions in which trajectory parameterizations are often selected arbitrarily, with no relevance to the cost index of interest. The approach presented herein instead uses Calculus of Variations (CV) to construct trajectory parameterizations that *guarantee* the optimization of a cost index, thus eliminating the heuristics associated with arbitrary parameterizations. Furthermore, employing the *necessary* and *sufficient* conditions provided by CV, some of the solutions to the path planning problem can result in *analytical closed form* [1].

The optimal path planning problem for formations of aircraft has been widely investigated in the past decades and several approaches have been attempted. Among the most valuable ones it is worth to remind the games theory, used to model conflict scenarios where two or more players compete to achieve a predefined target [2], [3]. Although computationally demanding, genetic algorithms demonstrated to be quite promising in addressing the trajectory generation problem for large formations as shown for instance in [4] and [5]. A more classical approach is represented by dynamic programming, which shares the same foundations with CV although its numerical aspects have been more exploited than the analytical ones [6].

A possible numerical technique to avoid collisions is to wisely select the positions and velocities of waypoints along trajectories [7], [8]. In the formulation exposed herein the waypoints are specified a priori and for synchronous missions they even induce collisions between the members of the formation, as for *T1*. By employing CV, instead, we show how collisions are always guaranteed to be avoided. Moreover, for *T3* we show how to use CV to optimally switch between optimal systems and how to verify that the optimality conditions for switching points are met.

In this paper the problem of planning the trajectory of a formation of *N* aircraft for the tasks *T1-T4* is addressed optimizing the kinetic energy, fuel, and energy consumption. Solutions provided are *modular*, in the sense that a new mission can be created by arranging in a different order the tasks or suppressing some of them.

The paper is structured as follows: par. II and III provide the physical and the mathematical background of the problem, par. IV and V present analytical and numerical results achieved and finally par. VI draws the conclusions.

II. Physical Background

Fixed an inertial reference frame, the generic *i*-th aircraft, schematized as a 3 DOF point mass, can be uniquely identified by the position vector $x_i(\cdot)\in\mathbb{R}^3$. Define $v_i(t)=dx_i(t)/dt$, $a_i(t)=dv_i(t)/dt$. The acceleration induced by the controllers is $a_{ci}(\cdot)$ and $a_{ci}(t)=dv_{ci}(t)/dt$. The formation moves in a constant gravity environment whose acceleration is *g*. Assuming that all aircraft are identical, the accelerations due to aerodynamic forces are $a_a(t)=v^T(t)v(t)\left(-\tilde{k}_D\hat{v}(t)+\tilde{k}_L\hat{v}_\perp(t)+\tilde{k}_S\hat{v}_\times(t)\right)$ where $\tilde{k}_{D/L/S}=\dfrac{\rho SC_{D/L/S}}{2m}$, ρ is the air density, *S* is the reference area, $C_{D/L/S}$ are the drag/lift/side force coefficients, $\hat{v}(\cdot)$, $\hat{v}_\perp(\cdot)$, and $\hat{v}_\times(\cdot)$ are the corresponding velocity unit vectors. Mutual aerodynamic interferences are neglected and superposition yields $a_i(t)=a_{ci}(t)+a_{ai}(t)+g$.

III. Mathematical Background

Kinetic energy consumption is modeled as $J_i=\int_{t_1}^{t_2}v_{ci}^T(t)v_{ci}(t)dt$, energy consumption as

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$J_i = \int_{t_1}^{t_2} a_{ci}^T(t)\, a_{ci}(t)\, dt$, and fuel consumption as $J_i = \int_{t_1}^{t_2} \sqrt{a_{ci}^T(t)\, a_{ci}(t)}\, dt$, where t_1 and t_2 define a fixed time interval to move between waypoints. Minimization of the cost functionals J_i belongs to the family of unconstrained endpoint problems known as the *problem of Lagrange* [9].

Given $\Omega = \left\{ y(\cdot) \in C^p(t_1, t_2) \mid y^{(i)}(t_1) = y_1^{(i)}, y^{(i)}(t_2) = y_2^{(i)} \right\}$, where $i \in \{0, \ldots, p-1\}$, $y(\cdot): [t_1, t_2] \subset \mathbb{R} \to \mathbb{R}^n$, $y_1^{(i)}$ and $y_2^{(i)}$ are given, define the cost index $J(y(\cdot)) = \int_{t_1}^{t_2} f\left(t, y(t), y'(t), \ldots, y^{(p)}(t)\right) dt$, where $y^{(p)}(\cdot)$ is the p-th derivative of $y(\cdot)$ with respect to t. The Problem of Lagrange aims at finding $y_*(\cdot) \in \Omega$ such that $J(y_*(\cdot)) \le J(y(\cdot))$. Euler Necessary Condition (ENC), Legendre Necessary Condition (LNC), Weierstrass Necessary Condition (WNC) and Jacobi Necessary Condition (JNC) and associated sufficient conditions contribute in determining *local* and *global* minima [1].

Collision avoidance can be accounted for by the theorem of multipliers as in par. A. Avoiding a no-fly zone can be modeled as an *endpoint problem with inequality constraints*, as in par. B. Finally, assuming that an endpoint lies on a given surface is a *point to surface problem* as in par. C.

A. *The Theorem of Multipliers*

This theorem restates the ENC [1] in integral form for the constrained Problem of Lagrange [9], [10]. Consider the problem of Lagrange and constrain it with $q < n$ functions $\phi_i(t, y(t))$ such that $\phi_i(t, y_*(t)) = 0$, i=1,…q. If $y_*(\cdot)$ is a local minimum for $J(\cdot)$ on Ω, then there exist multipliers $\lambda_0, \lambda_1(\cdot), \ldots, \lambda_q(\cdot)$ such that $|\lambda_0| + \sum_{i=1}^{q} |\lambda_i(t)| \neq 0$, $\forall t \in [t_1, t_2]$, and $y_*(\cdot)$ is also an extremal for the cost index

$$\mathcal{L} = \int_{t_1}^{t_2} \left(\lambda_0 f\left(s, y(s), \ldots, y^{(p)}(s)\right) + \sum_{i=1}^{q} \lambda_i(s) \phi_i(s, y(s)) \right) ds. \quad (1)$$

B. *Endpoint Problem with Inequality Constraints*

Let $\varphi(\cdot): [t_1, t_2] \subset \mathbb{R} \to \mathbb{R}^n$ be a constraint function of class C^p. Finding $\hat{y}_*(\cdot) \in \Omega$ such that $J(\hat{y}_*(\cdot)) \le J(y(\cdot))$, $\forall y(\cdot) \in \Omega$ and such that $\hat{y}_*(\cdot) \ge \varphi(\cdot)$, where the inequality is meant component-wise, is the scope of the endpoint problem with inequality *holonomic* constraints. A useful result to address this problem is given next [11].

Lemma 1 If $y_*(\cdot)$ is an extremal for the corresponding unconstrained endpoint problem and violates $\varphi(\cdot)$ for some $t \in (t_3, t_4) \subset [t_1, t_2]$, then $\hat{y}_*(t) = \varphi(t)$ in $(t_5, t_6) \subseteq (t_3, t_4)$ and $\hat{y}_*(t) = y_*(t)$ elsewhere [11].

Points t_5 and t_6 determine the location of the *switching points*. Lemma 1 defines the switching points but not how to choose them. The following theorem is useful for this issue.

Theorem 1 Let $y_*(\cdot)$ be an extremal of $J(y(\cdot))$, then the optimal switching point is located at t_s such that

$$y_*(t_s) = \varphi(t_s) \quad \text{and}$$
$$f\left(t_s, y_*(t_s), \ldots, y_*^{(p)}(t_s)\right) - f\left(t_s, \varphi(t_s), \ldots, \varphi^{(p)}(t_s)\right) = \\ = -\sum_{i=0}^{p-1} (-1)^{(i-1)} \left[y_*^{(i)}(t_s) - \varphi^{(i)}(t_s) \right]^T f_{(i+1)}\left(t_s, y_*(t_s), \ldots, y_*^{(p)}(t_s)\right) \quad (2)$$

where $f_{(i)}(\cdot, \ldots, \cdot)$ is the derivative of $f(\cdot, \ldots, \cdot)$ with respect to the i-th argument [11].

Both for Lemma 1 and Theorem 1, $y_*(\cdot)$ just needs to be an extremal for $J(y(\cdot))$, i.e. it needs to satisfy the ENC [1].

C. *Point to Surface Problem*

This is a variant of the problem of Lagrange and only the relevant differences from the endpoint problem [1] will be highlighted. Assume one of the endpoints lies on a smooth surface $S(z(t)) = 0$, where $S(\cdot): \mathbb{R}^n \to \mathbb{R}$. For the problem addressed herein let p=2 and n=3. Therefore define $\Omega_S = \left\{ y(\cdot) \in C_\infty^2 \mid y^{(i)}(t_1) = y_1^{(i)}, S\left(y(t_2^*)\right) = 0, y'(t_2^*) = y_2' \right\}$, $i \in \{0, 1\}$, $C_\infty^2 = \left\{ y(\cdot): [t_1, +\infty) \to \mathbb{R}^n \mid y(\cdot) \in C^2(t_1, T), \forall T > t_1 \right\}$, and $t_2^* > t_1$. The scope of the problem of Lagrange now, is to find $y_*(\cdot) \in \Omega_S$, such that $J(y_*(\cdot)) \le J(y(\cdot))$. ENC, LNC, JNC, and WNC developed for the original problem of Lagrange, still hold but their boundary conditions become

$$y_*^{(i)}(t_1) = y_1^{(i)}, \; S\left(y_*(t_2^*)\right) = 0, \; y_*'(t_2^*) = y_2',$$
$$\Upsilon_j = \lambda \left(\sum_{l=1}^{n} y_l' \left. \frac{\partial^2 S(y_*(t))}{\partial y_j \partial y_l} \right|_{t=t_2^*} \right) + \mu \left. \frac{\partial S(y_*(t))}{\partial y_j} \right|_{t=t_2^*}$$
$$\left. \frac{\partial f\left(t, y_*(t), y_*'(t), y_*''(t)\right)}{\partial y_j''} \right|_{t=t_2^*} = \lambda \left. \frac{\partial S(y_*(t))}{\partial y_j} \right|_{t=t_2^*} \quad (3)$$
$$f\left(t_2^*, y_*(t_2^*), y_*'(t_2^*), y_*''(t_2^*)\right) - \sum_{k=1}^{n} y_{*k}'(t_2^*) \Upsilon_k = \\ = \sum_{k=1}^{n} y_k''(t_2^*) \left. \frac{\partial f\left(t, y_*(t), y_*'(t), y_*''(t)\right)}{\partial y_k''} \right|_{t=t_2^*} + \lambda \left(\sum_{l=1}^{n} y_l' \left. \frac{\partial^2 S(y_*(t))}{\partial t \partial y_l} \right|_{t=t_2^*} \right) \\ + \mu \left. \frac{\partial S(y_*(t))}{\partial t} \right|_{t=t_2^*}$$

where $j \in \{1,...,n\}$, λ and μ are arbitrary constants, y_i is the i-th component of the vector $y(\cdot)$, $\Upsilon_j = \left.\frac{\partial f\left(t, y_*(t), y_*'(t), y_*''(t)\right)}{\partial y_{*j}'}\right|_{t=t_2^*} - \frac{d}{dt}\left.\frac{\partial f\left(t, y_*(t), y_*'(t), y_*''(t)\right)}{\partial y_j''}\right|_{t=t_2^*}$

The final time t_2^* is determined from (3).

IV. Formation Path Planning

To keep the formation tight and model the collision avoidance constraint, we impose that $\left\|x_i(t) - x_j(t)\right\|_2 = r_{ij}$, where $r_{ij} \in \mathbb{R}$ is given. For the present study, the only way to accomplish *T1* is to let all the aircraft pass through the unique assigned waypoint one at the time; the remaining tasks do not require the elements of the formation to alter this initial order. Then the i-th aircraft has to keep a given distance $r_{i(i-1)}$ from the *(i-1)*-th aircraft impling a chief-deputy logic between two consecutive aircraft as justified in par. A. Applying the theorem of multipliers the path planning problem reduces to finding the minimizers for

$$I_i = \lambda_{0i} J_i + \lambda_i \int_{t_1}^{t_2} \left(\left\| x_i(t) - x_{i-1}(t) \right\|_2^2 - r_{i(i-1)}^2 \right) dt \qquad (4)$$

where we assume that λ_i is a nonzero constant. A discussion about the choice of the multipliers is beyond the scope of the present paper but a survey is provided by [15]. Another way to address collision avoidance is presented in [2], where the unconstrained problem is considered and collision avoidance is achieved once the cost index J_i is optimized.

A. *Optimal Trajectories Through m+1 Waypoints*

Tasks *T1-T2* are addressed in this paragraph.

Theorem 2 (Consumed Kinetic Energy Optimization) Assuming that g=0 and $a_{ai}(t) = 0$, the index I_i associated to $J_i = \int_{t_1}^{t_2} v_{ci}^T(t) v_{ci}(t) dt$ through (4), is optimized by

$$a_{ci}^*(t) = \lambda_i \left(x_i^*(t) - x_{i-1}^*(t) \right) \qquad (5)$$

where the * denotes the candidate minimizer for I_i.

Proof (brief): Employing the theorem of multipliers one can specialize (4) and then prove that $\lambda_{0i} \neq 0$. As in [1], the ENC applied to $I_i(x_i(\cdot))$ leads to (5). The *Strengthen* LNC is verified because the second variation of the integrand of I_i at $x_i(\cdot)$ is $2\Im \succ 0$, where $\Im$ is the identity matrix and the curly inequality indicates that the matrix is positive definite on $\mathbb{R}^n$. Consequently, the integrand of J_i, $v_{ci}^T(\cdot) v_{ci}(\cdot)$, as well as any extremal of I_i, $x_i(\cdot)$, are regular.

The accessory problem consists in finding $t_c \neq t_1$ and $\eta(\cdot)$ not identically zero such that

$$\eta''(t) = \lambda_i \eta(t),\ \eta(t_1) = 0,\ \eta(t_c) = 0. \qquad (6)$$

If $\lambda_i > 0$, t_c does not exist and therefore the strengthen JNC holds for any interval $[t_1, t_2]$. If $\lambda_i < 0$, then $t_2 - t_1 \leq \pi / \sqrt{-\lambda_i}$ for the strengthen JNC to hold.

Thus, if the strengthen JNC holds, the sufficient condition for local minima [1] is verified and therefore the chosen control law is proven to actually provide a local minimum for the assigned cost index. Since the integrand of I_i is convex, the trajectory found is a global minimum [1], [16].

Theorem 3 (Consumed Energy Optimization) Assuming that g=0, $a_{ai}(t) = 0$, and $J_i = \int_{t_1}^{t_2} a_{ci}^T(t) a_{ci}(t) dt$, according to the boundary conditions I_i can be globally optimized by

$$a_{ci}^*(t) = k_{1i} t + k_{2i} + \lambda_i \int_{t_1}^{t} \int_{t_1}^{\tau} \left(x_i^*(s) - x_{i-1}^*(s) \right) ds d\tau \qquad (7)$$

where k_{1i} and k_{2i} are integration constants.

Proof (brief): By the strong analogies between the energy consumption and the kinetic energy cost indices, eq. (7) is deduced as (5) in Theorem 2. The strengthen LNC holds as well. A solution to the accessory problem depends on λ_i and on the velocity and acceleration of the i-th aircraft at t_1. If the strengthen JNC holds, then the sufficient condition for local and global minima are verified.

Theorem 4 (Fuel Consumption Optimization) Assuming that g=0 and $a_{ai}(t) = 0$, candidate optimizers for the cost index $I_i = \int_{t_1}^{t_2} \left[\sqrt{a_{ci}^T(t) a_{ci}(t)} + \lambda_i \left(\left\| x_i(t) - x_{i-1}(t) \right\|_2^2 - r_{i(i-1)}^2 \right) \right] dt$ are solutions of

$$\hat{a}_{ci}^*(t) = k_{1i} t + k_{2i} + \lambda_i \int_{t_1}^{t} \int_{t_1}^{\tau} \left(x_i^*(s) - x_{i-1}^*(s) \right) ds d\tau \qquad (8)$$

where $\hat{a}_{ci}(\cdot)$ is the unit vector of $a_{ci}(\cdot)$.

Proof (brief): Employing the theorem of multipliers one can prove that $\lambda_{0i} \neq 0$. The ENC applied to I_i leads to (8). The LNC, but not the strengthen LNC, holds as the second variation of the integrand of I_i with respect to $a_{ci}(\cdot)$ is positive semidefinite for any interval $[t_1, t_2]$. Sufficient conditions for weak local minima require the strengthen LNC to be verified. Thus any solution to (8) can be a candidate optimal trajectory only.

Further sufficient conditions still represent an open field of research in Calculus of Variations [17], [18].

B. *No-Fly Zone Avoidance*

The beginning and the end of the no-fly zone for each aircraft of the formation are characterized by the m-th endpoint from task *T2* and the first endpoint of task *T4* respectively. Our approach imposes selecting analytical shapes to model no-fly zones. We assume that the no-fly zone is an infinite cylinder of radius *r*: $\hat{\varphi}(t) = r\left[\cos(t)\ \sin(t)\ h\right]^T$, where *h* is a suitably large constant. For simplicity the constraint can be normalized as $\varphi(\cdot) = \hat{\varphi}(\cdot)/r$. Using eq. (2) in the numerical simulations of par. V it is possible to find the location of the optimal swiching points reminding that p=1 for the kinetic energy optimization and p=2 for the other indices.

C. *Optimal Trajectories for the Point to Surface Problem*

The optimization of the trajectory of the formation for the task *T4* is addressed solving the problem of Lagrange for the Point to Surface problem, where the surface is a plane.

Theorem 5 (Consumed Kinetic Energy Optimization) Assuming that g=0 and $a_{ai}(t) = 0$, eq. (5) globally optimizes the trajectory of the *i*-th element of the formation in terms of the kinetic energy while performing task *T4*.

Proof (brief): ENC, the strengthen LNC, and the strengthen JNC have been proven to hold in Theorem 2. Therefore eq. (5) and the following considerations still hold but the boundary conditions have to be modified imposing (3). Because the equation of motion cannot be integrated analytically, t_2^* has to be evaluated numerically. Numerical simulations for the mission scenario considered in par. V show that $t_2^* \le t_2$, where t_2 is the maximum final time to accomplish task *T4*.

Theorem 6 (Consumed Energy Optimization) Assuming that g=0 and $a_{ai}(t) = 0$, eq. (7) may optimize the trajectory of the *i*-th element of the formation for the energy consumption while performing task *T4*.

Proof (brief): As for Theorem 3, ENC, the strengthen LNC, and the strengthen JNC hold and therefore eq. (7) can be the global optimizer also for task *T4* using as boundary conditions (3). Numerical simulations show that $t_* \le t_2$.

Theorem 7 (Fuel Consumption Optimization) Assuming that g=0 and $a_{ai}(t) = 0$, eq. (8) provides candidate optimizers for the trajectory of the *i*-th aircraft in terms of fuel consumption while performing task *T4*.

Proof (brief): By applying the ENC eq. (8) is obtained as in Theorem 4. The LNC holds but not its strengthen formulation as in Theorem 4. Therefore any solution to (8) is a candidate optimal solution only. Tackling the Point to Surface Problem, the boundary conditions are given by (3).

V. Numerical Simulations

For the sake of clarity, a formation of only three aircraft is considered: the *chief* (C), the *first deputy* (FD) and the *second deputy* (SD). All quantities are non-dimensional. For each cost index, the vehicles leave from the positions (-1,-1,1), (-2,-2,3) and (-3,-3,2) respectively, pass through (0.1,0.1,0.1) of the inertial reference system (Task T1), pass through the waypoints (3,3,3), (2,2,2) and (2,2,1) respectively (Task T2), avoid a no-fly zone represented by an infinite cylinder of radius 1 parallel to the *z* axis and passing trough (4.5,2,0) (Task T3) and finally reach the planar surface defined by *x*=10 from (7,2,-2), (7,2,0) and (7,3,1) respectively (Task T4). Hereafter are reported the trajectories of all the tasks optimizing cost indices selected. The time interval [*T1 T2*] assumed for each task is always 5.

The minimum distance imposed between the airplanes is 0.5 and this constraint is always respected.

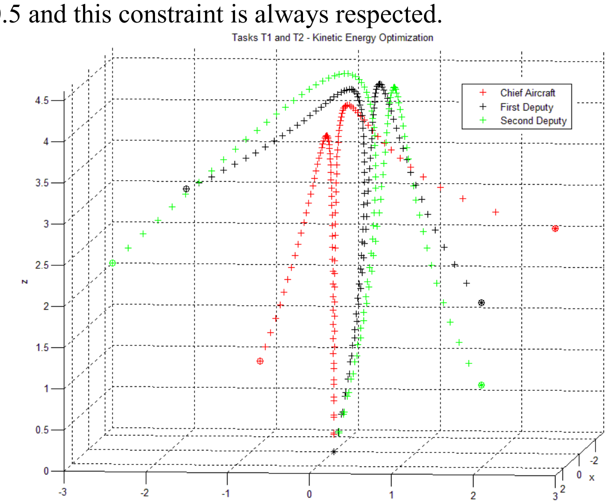

Fig. 1:Kinetic Energy Optimization – Tasks 1 and 2

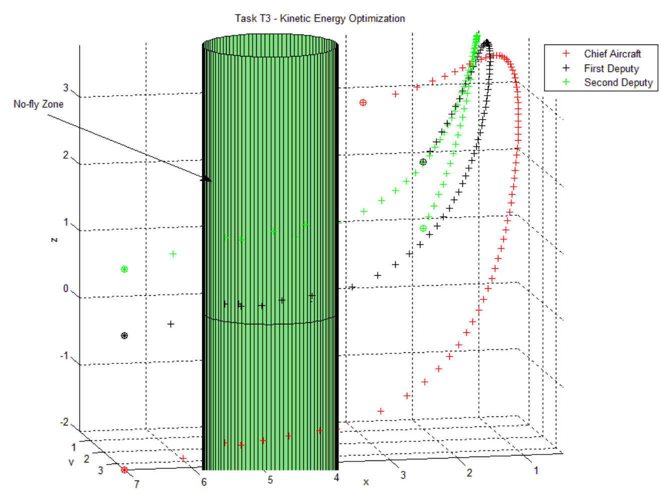

Fig. 2: Kinetic Energy Optimization – Task 3

For the consumed kinetic energy optimization it was numerically established that the trajectory of the Chief Aircraft has switching points at (4.31, 3.12, -1.22) and (5.45, 3.23, -1.64), the trajectory of the FD has switching points (4.34, 3.21, 0.13) and (5.43, 3.17, -0.05), the trajectory of the SD has switching points (4.13, 3.03, 1.02) and (5.49, 3.01, 0.97). The corresponding trajectories are depicted in Fig. 1 - Fig. 3. Note that the trajectories lie on the surface of the cylinder that represents the no-fly zone (Fig. 2).

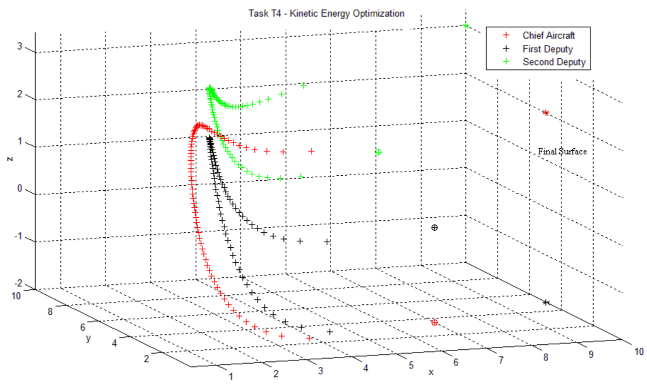

Fig. 3: Kinetic Energy Optimization – Task 4

For the consumed energy optimization, the simulations provided the following results.

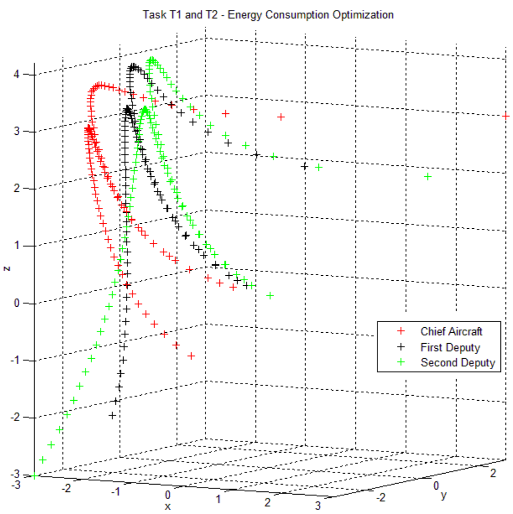

Fig. 4:Energy Consumption Optimization – Tasks 1 and 2

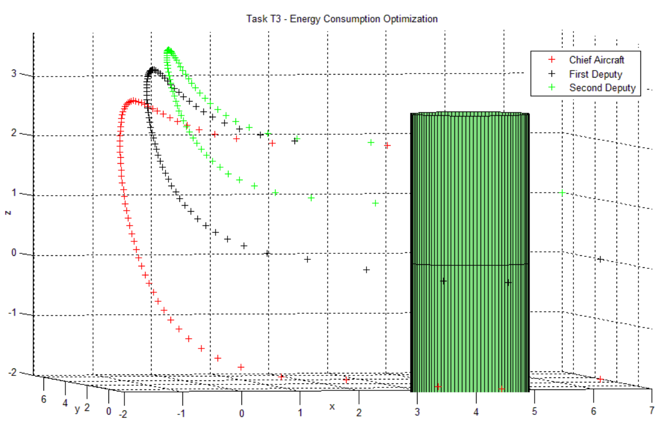

Fig. 5: Energy Consumption Optimization – Task 3

The trajectory of the Chief Aircraft has switching points at (3.57, 1.12, -1.87) and (4.46, 1.94, -1.93), the trajectory of the FD has switching points (3.71, 1.28, -0.33) and (4.48, 1.95, -0.24). However the trajectory of the SD does not have switching points. Fig. 4 - Fig. 6 illustrate the trajectories.

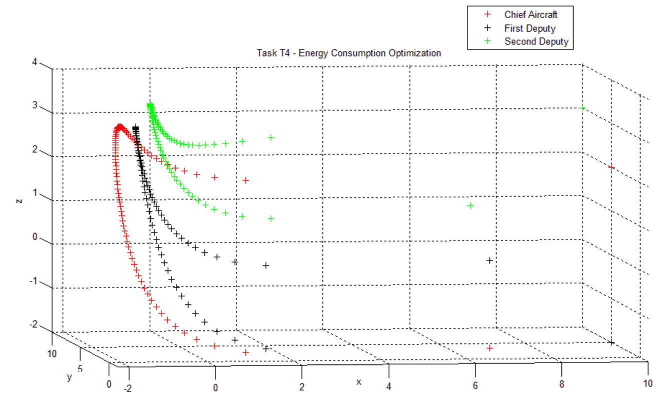

Fig. 6: Energy Consumption Optimization – Task 4

Lastly, for the optimization of fuel consumption we obtained the following results.

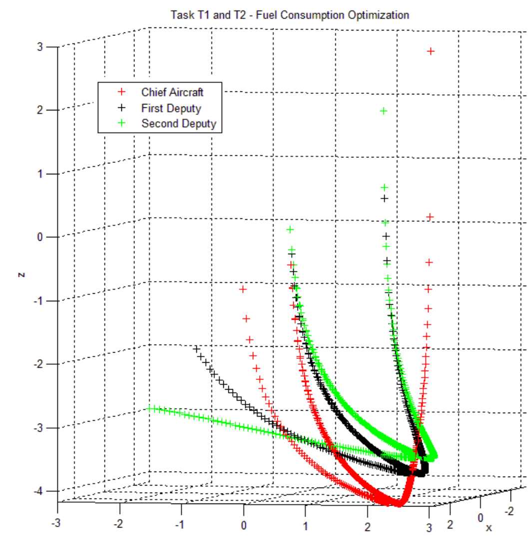

Fig. 7: Fuel Consumption Optimization – Tasks 1 and 2

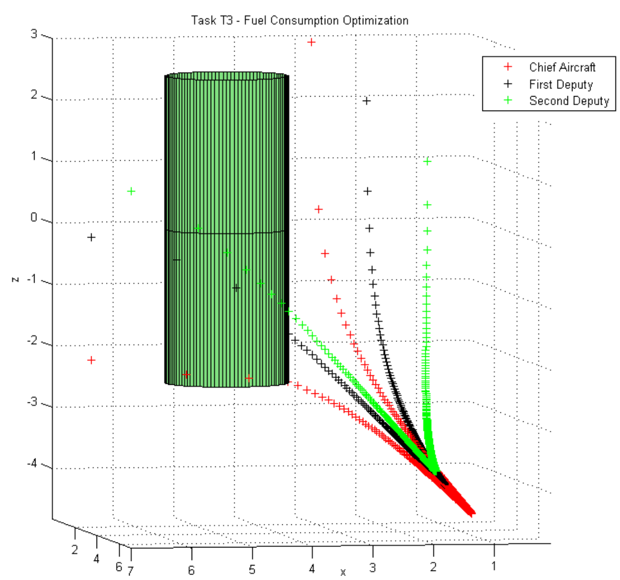

Fig. 8: Fuel Consumption Optimization – Task 3

The trajectory of the Chief Aircraft has switching points at (3.69, 2.23, -2.43) and (5.37, 2.91, -2.25), FD has switching points (4.21, 2.31, -1.27) and (5.47, 2.11, -0.68), the trajectory of the SD does not have any switching point (see Fig. 7 - Fig. 9 for the trajectories).

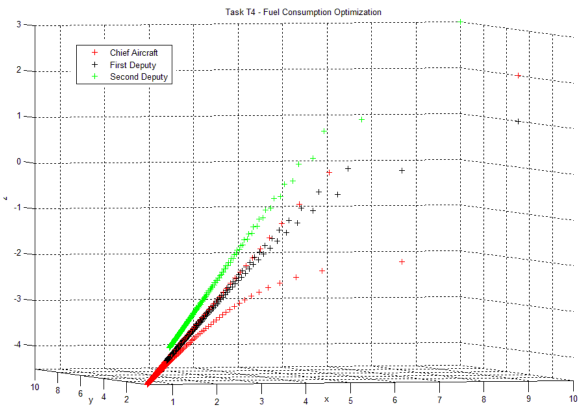

Fig. 9: Fuel Consumption Optimization – Task 4

VI. Conclusions

Necessary and sufficient conditions provided by the Calculus of Variations have been employed to uniquely determine the trajectories that minimize the consumed kinetic energy, the energy consumption, and the fuel consumption of a formation of *N* aircraft executing a complex tactical and rescue mission. Calculus of Variations actually guarantees their optimality. Each aircraft was modeled as a 3DOF point mass moving in a constant gravity environment and subject to aerodynamic forces. Avoidance of collision is guaranteed by Calculus of Variations imposing a chief-deputy logic in the control law. The problem of optimizing the trajectory of the formation while avoiding the no-fly zone has been addressed introducing the concept of switching between optimal systems. Conditions for optimal switching have been considered and numerically verified. The feasibility of the approach has been verified and illustrated by numerical simulations.

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