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Finite-time partial stability and stabilization, and optimal feedback control ☆

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Abstract

In this paper, we develop a unified framework to address the problem of optimal nonlinear analysis and feedback control design for finite-time partial stability and finite-time, partial-state stabilization. Finite-time partial stability of the closed-loop nonlinear system is guaranteed by means of a Lyapunov function that is positive definite and decrescent with respect to part of the system state, satisfies a differential inequality involving fractional powers, and can clearly be seen to be the solution to the steady-state form of the Hamilton-Jacobi-Bellman equation guaranteeing both partial stability and optimality. The overall framework provides the foundation for extending optimal linear-quadratic controller synthesis to nonlinear-nonquadratic optimal finite-time, partial-state stabilization. In addition, we specialize our results to address the problem of optimal finite-time control for nonlinear time-varying systems. Finally, we develop optimal feedback controllers for affine nonlinear systems using an inverse optimality framework tailored to the finite-time, partial-state stabilization problem and use this result to address finite-time, partial-state stabilizing sublinear controllers that minimize a derived performance criterion involving subquadratic terms.

1. Introduction

In [1] the current status of continuous-time, nonlinear nonquadratic optimal control problems was presented in a simplified and tutorial manner. The basic underlying ideas of the results in [1]

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are based on the fact that the steady-state solution of the Hamilton-Jacobi-Bellman equation is a Lyapunov function for the nonlinear system and thus guaranteeing both stability and optimality [2,1]. Specifically, a feedback control problem over an infinite horizon involving a nonlinear-nonquadratic performance functional is considered. The performance functional is then evaluated in closed form as long as the nonlinear nonquadratic cost functional considered is related in a specific way to an underlying Lyapunov function that guarantees asymptotic stability of the nonlinear closed-loop system. This Lyapunov function is shown to be the solution of the steady-state Hamilton-Jacobi-Bellman equation. The overall framework provides the foundation for extending linear-quadratic control to nonlinear-nonquadratic problems.

In this paper, we extend the framework developed in [1,2] to address the problem of optimal *finite-time stabilization*, that is, the problem of finding state-feedback control laws that minimize a given performance measure and guarantee finite-time stability of the closed-loop system. In addition, we address the problem of optimal *partial-state stabilization*, wherein stabilization with respect to a subset of the system state variables is desired. Even though finite-time stabilization [3–11] and partial-state stabilization [12,13] have been considered in the literature as separate problems as well as a combined problem [14–16], the problem of *optimal finite-time*, *partial-state stabilization* has not been addressed in the literature.

Finite-time stabilization of second-order systems was considered in [3,4], whereas the authors in [5,6] consider finite-time stabilization of higher-order systems as well as finite-time stabilization using output feedback. Design of globally strongly stabilizing continuous controllers for linear and nonlinear systems using the theory of homogeneous systems was studied in [7,8]. Finite-time partial stabilization of chained systems are considered in [14,15], whereas finite-time partial stabilizability using continuous and discontinuous homogeneous state feedback controllers is considered in [16]. Discontinuous finite-time stabilizing feedback controllers have also been developed in the literature [9–11]. Alternatively, sliding mode (typically discontinuous) control design has also been used to guarantee finite-time convergence and more recently finite-time stability; see [17] and the numerous references therein. However, for practical implementation, discontinuous feedback controllers can lead to chattering due to system uncertainty or measurement noise, and hence, may excite unmodeled high-frequency system dynamics.

The problem of partial stabilization has also been considered in the literature. Specifically, in [13,18] the authors construct controllers for spacecraft stabilization, wherein asymptotic stability of an equilibrium point is sought while requiring Lyapunov stability of the remaining closed-loop system states of the spacecraft. In [12], the authors consider partial stabilization of rotating machinery with mass imbalance, wherein motion stabilization with respect to a subspace instead of the origin is sought.

In this paper, we consider a notion of optimality that is directly related to a given Lyapunov function that is positive definite and decrescent with respect to part of the system state, and satisfies a differential inequality involving fractional powers. Specifically, an *optimal* finite-time, partial-state stabilization control problem is stated and sufficient Hamilton-Jacobi-Bellman conditions are used to characterize an optimal feedback controller. The steady-state solution of the Hamilton-Jacobi-Bellman equation is clearly shown to be a Lyapunov function for part of the closed-loop system state that guarantees both finite-time partial stability and optimality. In addition, we explore connections of our approach with inverse optimal control [19–22], wherein we parametrize a family of finite-time, partial-state stabilizing sublinear controllers that minimize a derived cost functional involving subquadratic terms. Subquadratic performance criteria have been studied in [23,24] and have been shown to permit a unified treatment of a broad range of design goals. Another important application of partial stability and partial stabilization theory is the unification it provides between time-invariant stability theory and stability theory for time-varying systems [2,25]. We exploit this unification and specialize our results to address the problem of optimal finite-time control for nonlinear time-varying dynamical systems.

The contents of this paper are as follows. In Section 2, we establish notation, definitions, and develop some basic results on finite-time partial stability of nonlinear dynamical systems. In Section 3, we present sufficient conditions for finite-time partial stability of nonlinear dynamical systems using Lyapunov functions that are positive definite with respect to part of the system's state and additionally satisfy a differential inequality involving fractional powers. These results are then specialized to provide sufficient conditions for finite-time stability of nonlinear time-varying systems. In Section 4, we consider a nonlinear system with a performance functional evaluated over the infinite horizon. The performance functional is then evaluated in terms of a Lyapunov function that guarantees finite-time partial stability. We then state an optimal control problem and provide sufficient conditions for characterizing an optimal nonlinear feedback controller guaranteeing finite-time partial stability of the closed-loop system. These results are then used to construct optimal finite-time controllers for nonlinear time-varying dynamical systems. In Section 5, we specialize the results developed in Section 4 to affine in the control dynamical systems as well as develop optimal feedback controllers for affine nonlinear systems using an inverse optimality framework tailored to the finite-time, partial-state stabilization problem. In Section 6, we provide two illustrative numerical examples that highlight the optimal partial-state stabilization framework. Finally, in Section 7, we present conclusions and highlight some future research directions.

2. Notation, Definitions, and Mathematical Preliminaries

In this section, we establish notation, definitions, and introduce the notion of finite-time partial stability. Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}_+$ denote the set of positive real numbers, $\overline{\mathbb{R}}_+$ denote the set of nonnegative numbers, $\mathbb{R}^n$ denote the set of $n \times 1$ real column vectors, $\mathbb{R}^{n\times m}$ denote the set of $n \times m$ real matrices, and $\mathcal{B}_\varepsilon(x)$ denote the *open ball centered* at x with *radius* ε. We write $V'(x) \triangleq \frac{\partial V(x)}{\partial x}$ for the gradient of V at x, $\|\cdot\|$ for the Euclidean vector norm, and I_n or I for the $n \times n$ identity matrix.

In this paper, we consider nonlinear dynamical systems of the form

$$\dot{x}_1(t) = f_1(x_1(t), x_2(t)), \quad x_1(0) = x_{10}, \quad t \in \mathcal{I}_{x_0}, \tag{1}$$

$$\dot{x}_2(t) = f_2(x_1(t), x_2(t)), \quad x_2(0) = x_{20}, \tag{2}$$

where, for every $t \in \mathcal{I}_{x_0}$, $x_1(t) \in \mathcal{D} \subseteq \mathbb{R}^{n_1}$ and $x_2(t) \in \mathbb{R}^{n_2}$, $\mathcal{I}_{x_0} \subset \mathbb{R}$ is the maximal interval of existence of a solution $x(t) \triangleq [x_1^{\mathrm{T}}(t), x_2^{\mathrm{T}}(t)]^{\mathrm{T}}$ of (1) and (2) with initial condition $x_0 \triangleq [x_{10}^{\mathrm{T}}, x_{20}^{\mathrm{T}}]^{\mathrm{T}}$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $f_1 : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1}$ is such that, for every $(x_1, x_2) \in \mathcal{D} \times \mathbb{R}^{n_2}$, $f_1(0, x_2) = 0$ and $f_1(\cdot,\cdot)$ is jointly continuous in x_1 and x_2, and $f_2 : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$ is such that, for every $(x_1, x_2) \in \mathcal{D} \times \mathbb{R}^{n_2}$, $f_2(\cdot,\cdot)$ is jointly continuous in x_1 and x_2. A continuously differentiable function $x : \mathcal{I}_{x_0} \to \mathcal{D} \times \mathbb{R}^{n_2}$ is said to be a *solution* of (1) and (2) on the interval $\mathcal{I}_{x_0} \subset \mathbb{R}$ if $x(\cdot) = [x_1^{\mathrm{T}}(\cdot), x_2^{\mathrm{T}}(\cdot)]^{\mathrm{T}}$ satisfies (1) and (2) for all $t \in \mathcal{I}_{x_0}$. If $x(\cdot) = [x_1^{\mathrm{T}}(\cdot), x_2^{\mathrm{T}}(\cdot)]^{\mathrm{T}}$ is a solution of (1) and (2) on the interval $\mathcal{I}_{x_0} \subset \mathbb{R}$, then $x_1(\cdot)$ is the solution of (1) and $x_2(\cdot)$ is the solution of (2).

The joint continuity of $f(\cdot,\cdot) = [f_1^{\mathrm{T}}(\cdot,\cdot), f_2^{\mathrm{T}}(\cdot,\cdot)]^{\mathrm{T}}$ implies that, for every $(x_1, x_2) \in \mathcal{D} \times \mathbb{R}^{n_2}$, there exists $\tau_0 < 0 < \tau_1$ and a solution $[x_1^{\mathrm{T}}(\cdot), x_2^{\mathrm{T}}(\cdot)]^{\mathrm{T}}$ of (1) and (2) defined on the open interval (τ_0, τ_1) such that $[x_1^{\mathrm{T}}(0), x_2^{\mathrm{T}}(0)]^{\mathrm{T}} = [x_1^{\mathrm{T}}, x_2^{\mathrm{T}}]^{\mathrm{T}}$ [2, Th. 2.24]. A solution $t \mapsto [x_1^{\mathrm{T}}(t), x_2^{\mathrm{T}}(t)]^{\mathrm{T}}$ is said to be *right maximally* defined if $[x_1^{\mathrm{T}}, x_2^{\mathrm{T}}]^{\mathrm{T}}$ cannot be extended (either uniquely or nonuniquely) forward in time. We assume that all right maximal solutions to (1) and (2) exist on $[0, \infty)$, and hence, we assume that (1) and (2) is *forward complete*. Recall that every bounded solution to (1) and (2) can be extended on a semi-infinite interval $[0,\infty)$ [2]. That is, if $x:[0,\tau_{x_0})\to\mathcal{D}\times\mathbb{R}^{n_2}$ is the right maximally defined solution of (1) and (2) such that $x(t)=[x_1^{\rm T}(t),x_2^{\rm T}(t)]^{\rm T}\in\mathcal{D}_{\rm c}\times\mathcal{Q}_{\rm c}$ for all $t\in[0,\tau_{x_0})$, where $\mathcal{D}_{\rm c}\subset\mathcal{D}$ and $\mathcal{Q}_{\rm c}\subset\mathbb{R}^{n_2}$ are compact, then $\tau_{x_0}=\infty$ [2, Cor. 2.5].

We assume that the nonlinear dynamical system given by (1) and (2) possesses unique solutions in forward time for all initial conditions except possibly at $x_1=0$ in the following sense. For every $(x_1,x_2)\in\mathcal{D}\backslash\{0\}\times\mathbb{R}^{n_2}$ there exists $\tau_x>0$, where $x=[x_1^{\rm T},x_2^{\rm T}]^{\rm T}$, such that, if $y_{\rm I}:[0,\tau_1)\to\mathcal{D}\times\mathbb{R}^{n_2}$ and $y_{\rm II}:[0,\tau_2)\to\mathcal{D}\times\mathbb{R}^{n_2}$ are two solutions of (1) and (2) with $y_{\rm I}(0)=y_{\rm II}(0)=x$, then $\tau_x\le\min\{\tau_1,\tau_2\}$ and $y_{\rm I}(t)=y_{\rm II}(t)$ for all $t\in[0,\tau_x)$. Without loss of generality, we assume that, for every (x_1,x_2), τ_x is chosen to be the largest such number in $\overline{\mathbb{R}}_+$. In this case, given $x=[x_1^{\rm T},x_2^{\rm T}]^{\rm T}\in\mathcal{D}\times\mathbb{R}^{n_2}$, we denote by the continuously differentiable map $s^x(\cdot)\triangleq s(\cdot,x_1,x_2)$ the *trajectory* or the unique *solution curve* of (1) and (2) on $[0,\tau_x)$ satisfying $s(0,x_1,x_2)=[x_1^{\rm T},x_2^{\rm T}]^{\rm T}$ and we denote by $s_1^x(\cdot)$ the *partial trajectory* or the unique *solution curve* of (1) on $[0,\tau_x)$. Sufficient conditions for forward uniqueness in the absence of Lipschitz continuity can be found in [26], [27, Section 10], [28], and [29, Section 1]. Finally, we assume that given a continuously differentiable function $x_1:[0,\infty)\to\mathbb{R}^{n_1}$, the solution $x_2(t)$, $t\ge0$, to (2) is unique.

The following definitions introduce the notion of finite-time partial stability.

Definition 2.1. The nonlinear dynamical system (1) and (2) is *finite-time stable with respect to* x_1 if there exist an open neighborhood $\mathcal{D}_0\subseteq\mathcal{D}$ of $x_1=0$ and a function $T:\mathcal{D}_0\backslash\{0\}\times\mathbb{R}^{n_2}\to(0,\infty)$, called the *settling-time function*, such that the following statements hold:

(i) *Finite-time partial convergence*. For every $(x_{10},x_{20})\in\mathcal{D}_0\backslash\{0\}\times\mathbb{R}^{n_2}$, $s^{x_0}(t)$ is defined on $[0,T(x_{10},x_{20}))$, where $x_0=[x_{10}^{\rm T},x_{20}^{\rm T}]^{\rm T}$, $s_1^{x_0}(t)\in\mathcal{D}_0\backslash\{0\}$ for all $t\in[0,T(x_{10},x_{20}))$, and $s_1^{x_0}(t)\to0$ as $t\to T(x_{10},x_{20})$.

(ii) *Partial Lyapunov stability*. For every $\varepsilon>0$ and $x_{20}\in\mathbb{R}^{n_2}$ there exists $\delta=\delta(\varepsilon,x_{20})>0$ such that $\mathcal{B}_\delta(0)\subset\mathcal{D}_0$ and, for every $x_{10}\in\mathcal{B}_\delta(0)\backslash\{0\}$, $s_1^{x_0}(t)\in\mathcal{B}_\varepsilon(0)$ for all $t\in[0,T(x_{10},x_{20}))$.

The nonlinear dynamical system (1) and (2) is *finite-time stable with respect to* x_1 *uniformly in* x_{20} if (1) and (2) is finite-time stable with respect to x_1 and the following statment holds:

(iii) *Partial uniform Lyapunov stability*. For every $\varepsilon>0$ there exists $\delta=\delta(\varepsilon)>0$ such that $\mathcal{B}_\delta(0)\subset\mathcal{D}_0$ and, for every $x_{10}\in\mathcal{B}_\delta(0)\backslash\{0\}$, $s_1^{x_0}(t)\in\mathcal{B}_\varepsilon(0)$ for all $t\in[0,T(x_{10},x_{20}))$ and for all $x_{20}\in\mathbb{R}^{n_2}$.

The nonlinear dynamical system (1) and (2) is *strongly finite-time stable with respect to* x_1 *uniformly in* x_{20} if (1) and (2) is uniformly finite-time stable with respect to x_1 and the following statment holds:

(iv) *Finite-time partial uniform convergence.* For every $(x_{10},x_{20})\in\mathcal{D}_0\backslash\{0\}\times\mathbb{R}^{n_2}$, $s^{x_0}(t)$ is defined on $[0,T(x_{10},x_{20}))$, $s_1^{x_0}(t)\in\mathcal{D}_0\backslash\{0\}$ for all $t\in[0,T(x_{10},x_{20}))$, and $s_1^{x_0}(t)\to0$ as $t\to T(x_{10},x_{20})$ uniformly in x_{20} for all $x_{20}\in\mathbb{R}^{n_2}$.

The nonlinear dynamical system (1) and (2) is *globally finite-time stable with respect to* x_1 (respectively, *globally finite-time stable with respect to* x_1 *uniformly in* x_{20} or *globally strongly finite-time stable with respect to* x_1 *uniformly in* x_{20}) if it is finite-time stable with respect to x_1 (respectively, finite-time stable with respect to x_1 uniformly in x_{20} or strongly finite-time stable with respect to x_1 uniformly in x_{20}) with $\mathcal{D}_0=\mathbb{R}^{n_1}$.

Remark 2.1. It is important to note that there is a key difference between the partial stability definitions given in Definition 2.1 and the definitions of partial stability given in [15]. In particular, the partial stability definitions given in [15] require that both initial conditions x_{10} and x_{20} lie in a neighborhood of the origin, whereas in Definition 2.1, x_{20} can be arbitrary. Furthermore, in the definition of partial stability given in [15], the state $x_1(t)$, $t \geq 0$, converges to zero and the state $x_2(t)$, $t \geq 0$, is bounded and converges to a constant that possibly depends on the system initial conditions. In contrast, in Definition 2.1 the state $x_2(t)$ can diverge as $t \to \infty$. Similar distinctions hold for our partial stabilization definition (see Definition 4.1 below) and the partial stabilization definition given in [14]. As will be seen below, this difference allows us to unify autonomous partial stability theory with time-varying stability theory.

As shown in [2,25], an important application of partial stability theory is the unification it provides between time-invariant stability theory and stability theory for time-varying systems. Specifically, consider the nonlinear time-varying dynamical system given by

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0, \quad t \in \mathcal{I}_{t_0,x_0}, \tag{3}$$

where, for every $t \in \mathcal{I}_{t_0,x_0}$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $\mathcal{I}_{t_0,x_0} \subseteq [t_0, \infty)$ is the maximal interval of existence of a solution $x(t)$ of (3), $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, and $f : \mathcal{I}_{t_0,x_0} \times \mathcal{D} \to \mathbb{R}^n$ is such that, for every $(t, x) \in \mathcal{I}_{t_0,x_0} \times \mathcal{D}$, $f(t, 0) = 0$ and $f(\cdot, \cdot)$ is jointly continuous in t and x. In this paper, we assume that the nonlinear time-varying dynamical system (3) possesses unique solutions in forward time for all initial conditions except possibly $x = 0$ and, given $x_0 \in \mathcal{D}$, we denote by the continuously differentiable map $s^{t_0,x_0}(\cdot) \triangleq s(\cdot, t_0, x_0)$ the *trajectory* or the unique *solution curve* of (3) on $\mathcal{I}_{t_0,x_0}$ satisfying $s(0, t_0, x_0) = x_0$. Now, defining $x_1(\tau) \triangleq x(t)$ and $x_2(\tau) \triangleq t$, where $\tau \triangleq t - t_0$, it follows that the solution $x(t)$, $t \in \mathcal{I}_{t_0,x_0}$, to the nonlinear time-varying dynamical system (3) can be equivalently characterized by the solution $x_1(\tau)$, $\tau \in \mathcal{T}_{t_0,x_0}$, to the nonlinear autonomous dynamical system

$$\dot{x}_1(\tau) = f(x_2(\tau), x_1(\tau)), \quad x_1(0) = x_0, \quad \tau \in \mathcal{T}_{t_0,x_0}, \tag{4}$$

$$\dot{x}_2(\tau) = 1, \quad x_2(0) = t_0, \tag{5}$$

where $\mathcal{T}_{t_0,x_0} \triangleq \{\tau \in \overline{\mathbb{R}}_+ : \tau = t - t_0, t \in \mathcal{I}_{t_0,x_0}\}$. Note that (4) and (5) are in the same form as the system given by (1) and (2), and hence, Definition 2.1 applied to (4) and (5) specializes to the following definition.

Definition 2.2. The nonlinear dynamical system (3) is *finite-time stable* if there exist an open neighborhood $\mathcal{D}_0 \subseteq \mathcal{D}$ of the origin and a function $T : [0, \infty) \times \mathcal{D}_0 \backslash \{0\} \to (t_0, \infty)$, called the *settling-time function*, such that the following statements hold:

(i) *Finite-time convergence.* For every $(t_0, x_0) \in [0, \infty) \times \mathcal{D}_0 \backslash \{0\}$, $s^{t_0,x_0}(t)$ is defined on $[t_0, T(t_0, x_0))$, $s^{t_0,x_0}(t) \in \mathcal{D}_0 \backslash \{0\}$ for all $t \in [t_0, T(t_0, x_0))$, and $s^{t_0,x_0}(t) \to 0$ as $t \to T(t_0, x_0)$.
(ii) *Lyapunov stability.* For every $\varepsilon > 0$ and $t_0 \in [0, \infty)$ there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that $\mathcal{B}_\delta(0) \subset \mathcal{D}_0$ and, for every $x_0 \in \mathcal{B}_\delta(0) \backslash \{0\}$, $s^{t_0,x_0}(t) \in \mathcal{B}_\varepsilon(0)$ for all $t \in [t_0, T(t_0, x_0))$.

The nonlinear dynamical system (3) is *uniformly finite-time stable* if (3) is finite-time stable and the following statement holds:

(iii) *Uniform Lyapunov stability.* For every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\mathcal{B}_\delta(0) \subset \mathcal{D}_0$ and, for every $x_0 \in \mathcal{B}_\delta(0) \backslash \{0\}$, $s^{t_0,x_0}(t) \in \mathcal{B}_\varepsilon(0)$ for all $t \in [t_0, T(t_0, x_0))$ and for all $t_0 \in [0, \infty)$.

The nonlinear dynamical system (3) is *strongly uniformly finite-time stable* if (3) is uniformly finite-time stable and the following statement holds:

(iv) *Uniform finite-time convergence.* For every $(t_0, x_0) \in [0, \infty) \times \mathcal{D}_0 \backslash \{0\}$, $s^{t_0, x_0}(t)$ is defined on $[t_0, T(t_0, x_0))$, $s^{t_0, x_0}(t) \in \mathcal{D}_0 \backslash \{0\}$ for all $t \in [t_0, T(t_0, x_0))$, and $s^{t_0, x_0}(t) \to 0$ as $t \to T(t_0, x_0)$ uniformly in t_0 for all $t_0 \in [0, \infty)$.

The nonlinear dynamical system (3) is *globally finite-time stable* (respectively, *globally uniformly finite-time stable* or *globally strongly uniformly finite-time stable*) if it is finite-time stable (respectively, uniformly finite-time stable or strongly uniformly finite-time stable) with $\mathcal{D}_0 = \mathbb{R}^n$.

3. Finite-time partial stability theory

In this section, we present sufficient conditions for finite-time partial stability using a Lyapunov function satisfying a differential inequality involving fractional powers. The following proposition shows that if the nonlinear dynamical system (1) and (2) is finite-time stable with respect to x_1, then it possesses a unique solution $s(\cdot, x_{10}, x_{20})$ defined on $\overline{\mathbb{R}}_+ \times \mathcal{D}_0 \times \mathbb{R}^{n_2}$ for every x_{10} in a neighborhood of $x_1 = 0$, including $x_1 = 0$, and, for every $x_{20} \in \mathbb{R}^{n_2}$, $s_1(t, x_{10}, x_{20}) = 0$ for all $t \geq T(x_{10}, x_{20})$, where $T(0, x_{20}) \triangleq 0$.

Proposition 3.1. *Consider the nonlinear dynamical system $\mathcal{G}$ given by (1) and (2). Assume $\mathcal{G}$ is finite-time stable with respect to x_1 and let $\mathcal{D}_0 \subseteq \mathcal{D}$ and $T : \mathcal{D}_0 \backslash \{0\} \times \mathbb{R}^{n_2} \to (0, \infty)$ be defined as in Definition 2.1. Then, for every $(x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}$, there exists a unique solution $s(t, x_{10}, x_{20}) = [s_1^{\mathrm{T}}(t, x_{10}, x_{20}), s_2^{\mathrm{T}}(t, x_{10}, x_{20})]^{\mathrm{T}}$, $t \geq 0$, to (1) and (2) defined on $\overline{\mathbb{R}}_+ \times \mathcal{D}_0 \times \mathbb{R}^{n_2}$ such that $s_1(t, x_{10}, x_{20}) \in \mathcal{D}_0$, $t \in [0, T(x_{10}, x_{20}))$, and such that $s_1(t, x_{10}, x_{20}) = 0$, $t \geq T(x_{10}, x_{20})$, where $T(0, x_{20}) \triangleq 0$.*

Proof. It follows from the partial Lyapunov stability of (1) and (2) with respect to x_1 that $x_1(t) \equiv 0$, $t \geq 0$, is the unique solution of (1) satisfying $x_1(0) = 0$ for all $x_{20} \in \mathbb{R}^{n_2}$. Thus, $s_1(t, 0, x_{20}) = 0$ for all $t \geq 0$ and $x_{20} \in \mathbb{R}^{n_2}$. Next, let $(x_{10}, x_{20}) \in \mathcal{D}_0 \backslash \{0\} \times \mathbb{R}^{n_2}$, and define

$$x_1(t) \triangleq \begin{cases} s_1(t, x_{10}, x_{20}), & 0 \leq t < T(x_{10}, x_{20}), \\ 0, & t \geq T(x_{10}, x_{20}). \end{cases} \tag{6}$$

Note that by construction, $x_1(\cdot)$ is continuously differentiable on $\overline{\mathbb{R}}_+ \backslash \{T(x_{10}, x_{20})\}$ and satisfies (1) on $\overline{\mathbb{R}}_+ \backslash \{T(x_{10}, x_{20})\}$. Furthermore, since $f_1(\cdot, \cdot)$ is jointly continuous,

$$\lim_{t \to T^-(x_{10}, x_{20})} \dot{x}_1(t) = \lim_{t \to T^-(x_{10}, x_{20})} f_1(x_1(t), x_2(t)) = \lim_{t \to T^+(x_{10}, x_{20})} \dot{x}_1(t), \tag{7}$$

and hence, $x_1(\cdot)$ is continuously differentiable at $T(x_{10}, x_{20})$ and $x_1(t)$ satisfies (1). Hence, it follows from the assumptions on $f_2(\cdot, \cdot)$ that, given $x_1(t)$, $t \geq 0$, there exists $x_2(t)$ such that $x(t) = [x_1^{\mathrm{T}}(t), x_2^{\mathrm{T}}(t)]^{\mathrm{T}}$ is solution of (1) and (2) for all $(x_{10}, x_{20}) \in \mathcal{D}_0 \backslash \{0\} \times \mathbb{R}^{n_2}$ and for all $t \geq 0$.

Given $(x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}$, to show uniqueness, assume $y_1(\cdot)$ satisfies (1) for all $t \geq 0$. In this case, $x_1(t) = y_1(t)$ for all $t \in [0, T(x_{10}, x_{20}))$ by the uniqueness assumption in Section 2. In addition, by continuity, $x_1(t) = y_1(t)$ at $t = T(x_{10}, x_{20})$, and hence, $x_1(t) = y_1(t)$ for all $t \in [0, T(x_{10}, x_{20})]$, which implies that $y_1(T(x_{10}, x_{20})) = 0$. Now, partial Lyapunov stability with respect to x_1 implies that $y_1(t) = 0$ for $t > T(x_{10}, x_{20})$, which proves uniqueness of $x_1(\cdot)$. Hence uniqueness of $x(\cdot) = [x_1^{\mathrm{T}}(\cdot), x_2^{\mathrm{T}}(\cdot)]^{\mathrm{T}}$ immediately follows from the assumptions in Section 2.This proves the result. □

It follows from Proposition 3.1 and the assumptions on $f_2(\cdot,\cdot)$ that if the nonlinear dynamical system (1) and (2) is finite-time stable with respect to x_1, then it defines a *global semiflow* on $\mathcal{D}_0 \times \mathbb{R}^{n_2}$; that is, the solution curve $s(\cdot,\cdot,\cdot)$ of (1) and (2) satisfies the consistency property $s(0,x_1,x_2)=[x_1^{\mathrm{T}},x_2^{\mathrm{T}}]^{\mathrm{T}}$ and the semigroup property $s(t,s_1(\tau,x_1,x_2),s_2(\tau,x_1,x_2))=s(t+\tau,x_1,x_2)$ for every $(x_1,x_2)\in\mathcal{D}_0\times\mathbb{R}^{n_2}$ and $t,\tau\in\overline{\mathbb{R}}_+$. Furthermore, $s(\cdot,\cdot,\cdot)$ satisfies

$$s_1(T(x_{10},x_{20})+t_1,x_{10},x_{20})=0 \tag{8}$$

for all $(x_{10},x_{20})\in\mathcal{D}_0\times\mathbb{R}^{n_2}$ and $t_1\ge 0$.

In general, finite-time partial stability does not imply that the settling-time function $T(\cdot,\cdot)$ is continuous [30]. The following proposition generalizes Proposition 2.4 of [30] to show that the settling-time function $T(\cdot,\cdot)$ of a finite-time partially stable system is jointly continuous on $\mathcal{D}_0\times\mathbb{R}^{n_2}$ if and only if it is continuous at $(0,\cdot)$.

Proposition 3.2. *Consider the nonlinear dynamical system $\mathcal{G}$ given by (1) and (2). Assume $\mathcal{G}$ is finite-time stable with respect to x_1, let $\mathcal{D}_0\subseteq\mathcal{D}$ be as defined in Definition 2.1, and let $T:\mathcal{D}_0\backslash\{0\}\times\mathbb{R}^{n_2}\to[0,\infty)$ be the settling-time function of $\mathcal{G}$. Then the following statements hold:*

(i) *If $t_1\ge 0$ and $(x_{10},x_{20})\in\mathcal{D}_0\times\mathbb{R}^{n_2}$, then*

$$T(s_1(t_1,x_{10},x_{20}),s_2(t_1,x_{10},x_{20}))=\max\{T(x_{10},x_{20}),t_1\}. \tag{9}$$

(ii) *$T(\cdot,\cdot)$ is jointly continuous on $\mathcal{D}_0\times\mathbb{R}^{n_2}$ if and only if $T(\cdot,\cdot)$ is jointly continuous at $(0,x_2)$, $x_2\in\mathbb{R}^{n_2}$.*

Proof. (i) It follows from Definition 2.1 that

$$T(x_{10},x_{20})=\inf\{t\in\mathbb{R}_+ : s_1(t,x_{10},x_{20})=0\} \tag{10}$$

for all $(x_{10},x_{20})\in\mathcal{D}_0\backslash\{0\}\times\mathbb{R}^{n_2}$. Hence, $T(s_1(t_1,x_{10},x_{20}),s_2(t_1,x_{10},x_{20}))=\inf\{t_2\in\mathbb{R}_+ : s_1(t_2, s_1(t_1,x_{10},x_{20}),s_2(t_1,x_{10},x_{20}))=0\}$. Now, for $0\le t_1\le T(x_{10},x_{20})$, the semigroup property and (10) imply that $T(s_1(t_1,x_{10},x_{20}),s_2(t_1,x_{10},x_{20}))=\inf\{t_2\in\mathbb{R}_+ : s_1(t_2,x_{10},x_{20})=0\}=T(x_{10},x_{20})$. Alternatively, for $0\le T(x_{10},x_{20})\le t_1$, $T(s_1(t_1,x_{10},x_{20}),s_2(t_1,x_{10},x_{20}))=t_1$, which proves (9).

(ii) Necessity is immediate. To prove sufficiency, suppose that $T(\cdot,\cdot)$ is jointly continuous at $(0,x_2)$, $x_2\in\mathbb{R}^{n_2}$. Let $(x_1,x_2)\in\mathcal{D}_0\times\mathbb{R}^{n_2}$ and consider the sequences $\{x_{1n}\}_{n=1}^{\infty}\subset\mathcal{D}_0$ converging to x_1 and $\{x_{2n}\}_{n=1}^{\infty}\subset\mathbb{R}^{n_2}$ converging to x_2. Let $\tau^-=\liminf_{n\to\infty}T(x_{1n},x_{2n})$ and $\tau^+=\limsup_{n\to\infty}T(x_{1n},x_{2n})$. Note that τ^-, $\tau^+\in\mathbb{R}_+$ and

$$\tau^-\le\tau^+. \tag{11}$$

Next, let $\{x_{1n_m}\}_{m=0}^{\infty}\subset\mathcal{D}_0$ be a subsequence of $\{x_{1n}\}$ and $\{x_{2n_m}\}_{m=0}^{\infty}\subset\mathbb{R}^{n_2}$ be a subsequence of $\{x_{2n}\}$ such that $T(x_{1n_m},x_{2n_m})\to\tau^+$ as $m\to\infty$. The sequence $\{(T(x_1,x_2),x_{1n_m},x_{2n_m})\}_{m=1}^{\infty}$ converges in $\overline{\mathbb{R}}_+\times\mathcal{D}_0\times\mathbb{R}^{n_2}$ to $(T(x_1,x_2),x_1,x_2)$ as $m\to\infty$. Since $s_1(T(x_1,x_2)+t_1,x_1,x_2)=0$ for all $t_1\ge 0$ and since all solutions to (1) and (2) are continuous in their initial conditions, it follows that $s_1(T(x_1,x_2),x_{1n_m},x_{2n_m})\to s_1(T(x_1,x_2),x_1,x_2)=0$ as $m\to\infty$. Thus, since $T(0,x_2)$ is continuous for all $x_2\in\mathbb{R}^{n_2}$, it follows that

$$\lim_{m\to\infty}T(s_1(T(x_1,x_2),x_{1n_m},x_{2n_m}),s_2(T(x_1,x_2),x_{1n_m},x_{2n_m}))=T(x_1,x_2). \tag{12}$$

Now, with $t_1=T(x_1,x_2)$, $x_{10}=x_{1n_m}$, and $x_{20}=x_{2n_m}$, it follows from (9) and (12) that $T(s_1(T(x_1,x_2),x_{1n_m},x_{2n_m}),s_2(T(x_1,x_2),x_{1n_m},x_{2n_m}))=\max\{T(x_{1n_m},x_{2n_m}),T(x_1,x_2)\}$ and $\max\{T(x_{1n_m},x_{2n_m}),T(x_1,x_2)\}\to T(x_1,x_2)$ as $m\to\infty$. Thus, $\max\{\tau^+,T(x_1,x_2)\}=T(x_1,x_2)$, which implies that

$$\tau^+ \le T(x_1, x_2). \tag{13}$$

Finally, let $\{x_{1n_k}\}_{k=0}^{\infty} \subset \mathcal{D}_0$ be a subsequence of $\{x_{1n}\}$ and $\{x_{2n_k}\}_{k=0}^{\infty} \subset \mathbb{R}^{n_2}$ be a subsequence of $\{x_{2n}\}$ such that $T(x_{1n_k}, x_{2n_k}) \to \tau^-$ as $k \to \infty$. It follows from (11) and (13) that $\tau^- \in \mathbb{R}_+$, and hence, the sequence $\{(T(x_{1n_k}, x_{2n_k}), x_{1n_k}, x_{2n_k})\}_{k=1}^{\infty}$ converges to (τ^-, x_1, x_2) as $k \to \infty$. Since $s_1(\cdot,\cdot,\cdot)$ is jointly continuous, it follows that $s_1(T(x_{1n_k}, x_{2n_k}), x_{1n_k}, x_{2n_k}) \to s_1(\tau^-, x_1, x_2)$ as $k \to \infty$. Now, since $s_1(T(x_1,x_2)+t_1, x_1, x_2) = 0$ for all $t_1 \ge 0$, $s_1(T(x_{1n_k}, x_{2n_k}), x_{1n_k}, x_{2n_k}) = 0$ for each k. Hence, $s_1(\tau^-, x_1, x_2) = 0$ and, by the definition of settling-time function,

$$T(x_1, x_2) \le \tau^-. \tag{14}$$

Now, it follows from (11), (13) and (14) that $\tau^- = T(x_1, x_2) = \tau^+$, and hence, $T(x_{1n}, x_{2n}) \to T(x_1, x_2)$ as $n \to \infty$, which proves that $T(\cdot,\cdot)$ is jointly continuous on $\mathcal{D}_0 \times \mathbb{R}^{n_2}$. □

Next, we present sufficient conditions for finite-time partial stability using a Lyapunov function involving a scalar differential inequality. Given the nonlinear dynamical system (1) and (2), for the statement of the following result define

$$\dot{V}(x_1, x_2) \triangleq V'(x_1, x_2) f(x_1, x_2),$$

where $f(x_1, x_2) \triangleq [f_1^{\mathrm{T}}(x_1, x_2), f_2^{\mathrm{T}}(x_1, x_2)]^{\mathrm{T}}$ and $V : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$ is a continuously differentiable function, and recall the definitions of class $\mathcal{K}$ and $\mathcal{K}_\infty$ functions given in [2, Def. 3.3].

Theorem 3.1. *Consider the nonlinear dynamical system $\mathcal{G}$ given by* (1) *and* (2). *Then the following statements hold*:

(i) *If there exist a continuously differentiable function $V : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$, a class $\mathcal{K}$ function $\alpha(\cdot)$, a continuous function $k : [0, \infty) \to \mathbb{R}_+$, a real number $\theta \in (0, 1)$, and an open neighborhood $\mathcal{M} \subseteq \mathcal{D}$ of $x_1 = 0$ such that*

$$V(0, x_2) = 0, \quad x_2 \in \mathbb{R}^{n_2}, \tag{15}$$

$$\alpha(\|x_1\|) \le V(x_1, x_2), \quad (x_1, x_2) \in \mathcal{M} \times \mathbb{R}^{n_2}, \tag{16}$$

$$\dot{V}(x_1, x_2) \le -k(\|x_2\|)(V(x_1, x_2))^{\theta}, \quad (x_1, x_2) \in \mathcal{M} \times \mathbb{R}^{n_2}, \tag{17}$$

then $\mathcal{G}$ is finite-time stable with respect to x_1. Moreover, there exist a neighborhood $\mathcal{D}_0$ of $x_1 = 0$ and a settling-time function $T : \mathcal{D}_0 \times \mathbb{R}^{n_2} \to [0, \infty)$ such that

$$T(x_{10}, x_{20}) \le q^{-1}\left(\frac{(V(x_{10}, x_{20}))^{1-\theta}}{1-\theta}\right), \quad (x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}, \tag{18}$$

where $q : [0, \infty) \to \mathbb{R}$ is continuously differentiable and satisfies

$$\dot{q}(t) = k(\|x_2(t)\|), \quad q(0) = 0, \quad t \ge 0, \tag{19}$$

and $T(\cdot,\cdot)$ is jointly continuous on $\mathcal{D}_0 \times \mathbb{R}^{n_2}$.

(ii) If $\mathcal{M} = \mathcal{D} = \mathbb{R}^{n_1}$ and there exist a continuously differentiable function $V : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$, a class $\mathcal{K}_\infty$ function $\alpha(\cdot)$, a continuous function $k : [0, \infty) \to \mathbb{R}_+$, and a real number $\theta \in (0, 1)$ such that (15)–(17) hold, then $\mathcal{G}$ is globally finite-time stable with respect to x_1. Moreover, there exists a settling-time function $T : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to [0, \infty)$ such that (18) holds with $\mathcal{D}_0 = \mathbb{R}^{n_1}$ and $T(\cdot,\cdot)$ is jointly continuous on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$.

(iii) If there exist a continuously differentiable function $V:\mathcal{D}\times\mathbb{R}^{n_2}\to\mathbb{R}$, class $\mathcal{K}$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, a continuous function $k:[0,\infty)\to\mathbb{R}_+$, a real number $\theta\in(0,1)$, and an open neighborhood $\mathcal{M}\subseteq\mathcal{D}$ of $x_1=0$ such that (16) and (17) hold, and

$$V(x_1,x_2)\le\beta(\|x_1\|),\quad (x_1,x_2)\in\mathcal{M}\times\mathbb{R}^{n_2},\tag{20}$$

then $\mathcal{G}$ is finite-time stable with respect to x_1 uniformly in x_{20}. Moreover, there exist a neighborhood $\mathcal{D}_0$ of $x_1=0$ and a settling-time function $T:\mathcal{D}_0\times\mathbb{R}^{n_2}\to[0,\infty)$ such that (18) holds and $T(\cdot,\cdot)$ is jointly continuous on $\mathcal{D}_0\times\mathbb{R}^{n_2}$.

(iv) If $\mathcal{M}=\mathcal{D}=\mathbb{R}^{n_1}$ and there exist a continuously differentiable function $V:\mathcal{D}\times\mathbb{R}^{n_2}\to\mathbb{R}$, class $\mathcal{K}_\infty$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, a continuous function $k:[0,\infty)\to\mathbb{R}_+$, and a real number $\theta\in(0,1)$ such that (16), (17) and (20) hold, then $\mathcal{G}$ is globally finite-time stable with respect to x_1 uniformly in x_{20}. Moreover, there exists a settling-time function $T:\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to[0,\infty)$ such that (18) holds with $\mathcal{D}_0=\mathbb{R}^{n_1}$ and $T(\cdot,\cdot)$ is jointly continuous on $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$.

(v) If there exist a continuously differentiable function $V:\mathcal{D}\times\mathbb{R}^{n_2}\to\mathbb{R}$, class $\mathcal{K}$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, a real number $\theta\in(0,1)$, and an open neighborhood $\mathcal{M}\subseteq\mathcal{D}$ of $x_1=0$ such that (16), (17) and (20) hold with $k(\|x_2\|)=k\in\mathbb{R}_+$, $x_2\in\mathbb{R}^{n_2}$, then $\mathcal{G}$ is strongly finite-time stable with respect to x_1 uniformly in x_{20}. Moreover, there exist a neighborhood $\mathcal{D}_0$ of $x_1=0$ and a settling-time function $T:\mathcal{D}_0\times\mathbb{R}^{n_2}\to[0,\infty)$ such that

$$T(x_{10},x_{20})\le\frac{(V(x_{10},x_{20}))^{1-\theta}}{k(1-\theta)},\quad (x_{10},x_{20})\in\mathcal{D}_0\times\mathbb{R}^{n_2},\tag{21}$$

and $T(\cdot,\cdot)$ is jointly continuous on $\mathcal{D}_0\times\mathbb{R}^{n_2}$.

(vi) If $\mathcal{M}=\mathcal{D}=\mathbb{R}^{n_1}$ and there exist a continuously differentiable function $V:\mathcal{D}\times\mathbb{R}^{n_2}\to\mathbb{R}$, class $\mathcal{K}_\infty$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, and a real number $\theta\in(0,1)$ such that (16), (17) and (20) hold with $k(\|x_2\|)=k\in\mathbb{R}_+$, $x_2\in\mathbb{R}^{n_2}$, then $\mathcal{G}$ is globally strongly finite-time stable with respect to x_1 uniformly in x_{20}. Moreover, there exists a settling-time function $T:\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to[0,\infty)$ such that (21) holds with $\mathcal{D}_0=\mathbb{R}^{n_1}$ and $T(\cdot,\cdot)$ is jointly continuous on $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$.

Proof. (i) Let $x_{20}\in\mathbb{R}^{n_2}$, let $\varepsilon>0$ be such that $\mathcal{B}_\varepsilon(0)\subseteq\mathcal{M}$, define $\eta\triangleq\alpha(\varepsilon)$, and define $\mathcal{D}_\eta\triangleq\{x_1\in\mathcal{B}_\varepsilon(0):V(x_1,x_{20})<\eta\}$. Since $V(\cdot,\cdot)$ is continuous and $V(0,x_2)=0$, it follows that $\mathcal{D}_\eta$ is nonempty and there exists $\delta=\delta(\varepsilon,x_{20})>0$ such that $V(x_1,x_{20})<\eta$, $x_1\in\mathcal{B}_\delta(0)$. Hence, $\mathcal{B}_\delta(0)\subseteq\mathcal{D}_\eta$. Next, it follows from (17) that $V(x_1(t),x_2(t))$ is a nonincreasing function of time and, hence, for every $x_{10}\in\mathcal{B}_\delta(0)\subseteq\mathcal{D}_\eta$, it follows that

$$\alpha(\|x_1(t)\|)\le V(x_1(t),x_2(t))\le V(x_{10},x_{20})<\eta=\alpha(\varepsilon),\quad t\ge0.\tag{22}$$

Thus, for every $x_{10}\in\mathcal{B}_\delta(0)$, $x_1(t)\in\mathcal{B}_\varepsilon(0)$, $t\ge0$, which proves partial Lyapunov stability with respect to x_1.

Next, let $z:[0,\infty)\to\mathbb{R}_+$ be a continuous function defined on $[0,\infty)$ and note that the solution to

$$\dot v(t)=-z(t)(v(t))^\theta,\quad v(0)=v_0=V(x_{10},x_{20}),\quad t\ge0,\tag{23}$$

is given by

$$v(t)=\begin{cases} V(x_{10},x_{20})\left[(V(x_{10},x_{20}))^{1-\theta}-(1-\theta)\int_0^t z(\tau)\,\mathrm{d}\tau\right]^{\frac{1}{1-\theta}}, & 0\le t<t_1,\ v_0\neq 0,\\ 0, & t\ge t_1,\ v_0\neq 0,\\ 0, & t\ge 0,\ v_0=0,\end{cases} \tag{24}$$

where $t_1>0$ is such that

$$\int_0^{t_1} z(\tau)\mathrm{d}\tau=\frac{(V(x_{10},x_{20}))^{1-\theta}}{1-\theta}. \tag{25}$$

Hence,

$$t_1=q^{-1}\left(\frac{(V(x_{10},x_{20}))^{1-\theta}}{1-\theta}\right), \tag{26}$$

where $q:[0,\infty)\to\mathbb{R}$ is continuously differentiable and satisfies

$$\dot{q}(t)=k(\|x_2(t)\|),\quad q(0)=q_0,\quad t\ge 0, \tag{27}$$

for some $q_0\in\overline{\mathbb{R}}_+$. Now, let $w:[0,\infty)\to\mathbb{R}$ be a continuously differentiable function such that

$$\dot{w}(t)\le -z(t)(v(t))^{\theta},\quad w(0)=V(x_{10},x_{20}),\quad t\ge 0, \tag{28}$$

where $v(t)$ is given by (24). Then, it follows from (23) and (28), and the comparison lemma [2, p. 126] that

$$w(t)\le v(t),\quad t\ge 0. \tag{29}$$

Thus, it follows from (17), (23), (24), (28), and (29), with $z(t)=k(\|x_2(t)\|)$ and $w(t)=(V(x_1(t),x_2(t)))^{1-\theta}$, $t\ge 0$, that

$$V(x_1(t),x_2(t))\le v(t),\quad t\ge 0, \tag{30}$$

and hence, using (15), (16), (24), and (30),

$$x_1(t)=0,\quad t\ge t_1, \tag{31}$$

where t_1 is given in (26), which proves finite-time convergence of the trajectory of (1) for all $(x_{10},x_{20})\in\mathcal{B}_\delta(0)\times\mathbb{R}^{n_2}$. Hence, the nonlinear dynamical system $\mathcal{G}$ is finite-time stable with respect to x_1.

Finally, since $s_1(0,x_1,x_2)=x_1$ and $s_1(\cdot,\cdot,\cdot)$ is continuous, $\inf\{t\in\mathbb{R}_+ : s_1(t,x_1,x_2)=0\}>0$, $x_{10}\in\mathcal{B}_\delta(0)\backslash\{0\}$. Furthermore, it follows from (31) that $\inf\{t\in\mathbb{R}_+ : s_1(t,x_1,x_2)=0\}<\infty$, $x_{10}\in\mathcal{B}_\delta(0)$. Now, defining $\mathcal{D}_0\triangleq\mathcal{B}_\delta(0)$ and $T:\mathcal{D}_0\times\mathbb{R}^{n_2}\to\mathbb{R}_+$ by (24) and (26), (18) is immediate. Moreover, it follows from the finite-time stability of $\mathcal{G}$ with respect to x_1 and Proposition 3.1 that $T(\cdot,\cdot)$ can be extended to $\overline{\mathbb{R}}_+$ and $T(0,x_{20})=0$, which implies that $q_0=0$ in (27). Thus, (19) immediately follows from (27). Finally, the right-hand side of (18) is jointly continuous at $(0,x_2)$, $x_2\in\mathbb{R}^{n_2}$, and hence, by Proposition 3.2, it is jointly continuous on $\mathcal{D}_0\times\mathbb{R}^{n_2}$.

(ii) Let $\delta>0$, $x_{10}\in\mathbb{R}^{n_1}$, and $x_{20}\in\mathbb{R}^{n_2}$ be such that $\|x_{10}\|<\delta$. Since $\alpha(\cdot)$ is a $\mathcal{K}_\infty$ function, it follows that there exists $\varepsilon>0$ such that $V(x_{10},x_{20})\le\alpha(\varepsilon)$. Now, (17) implies that $V(x_1(t),x_2(t))$ is a nonincreasing function of time, and hence, it follows from (16) that

$$\alpha(\|x_1(t)\|)\le V(x_1(t),x_2(t))\le V(x_{10},x_{20})=\alpha(\varepsilon),\quad t\ge 0. \tag{32}$$

Hence, for every $x_{10}\in\mathcal{B}_\delta(0)$, $x_1(t)\in\mathcal{B}_\varepsilon(0)$, $t\ge 0$, which proves Lyapunov stability with respect to x_1. Finite-time partial convergence follows as in the proof of (i), implying global finite-time stability of $\mathcal{G}$ with respect to x_1. In addition, the existence of a settling-time function $T:\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to[0,\infty)$ satisfying (18) and is jointly continuous on $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ follows as in the proof of (i).

(iii) Let $\varepsilon>0$ and $\mathcal{B}_\varepsilon(0)$ be given as in the proof of (i). Let $\delta=\delta(\varepsilon)>0$ be such that $\beta(\delta)=\alpha(\varepsilon)$. Hence, it follows from (16) and (20) that, for all $(x_{10},x_{20})\in\mathcal{B}_\delta(0)\times\mathbb{R}^{n_2}$,

$$\alpha(\|x_1(t)\|)\le V(x_1(t),x_2(t))\le V(x_{10},x_{20})<\beta(\delta)=\alpha(\varepsilon),\quad t\ge 0. \tag{33}$$

Thus, for every $x_{10}\in\mathcal{B}_\delta(0)$, $x_1(t)\in\mathcal{B}_\varepsilon(0)$, $t\ge 0$, which proves partial uniform Lyapunov stability with respect to x_1. Finite-time partial convergence follows as in the proof of (i), implying finite-time stability of $\mathcal{G}$ with respect to x_1 uniformly in x_{20}. In addition, the existence of a settling-time function $T:\mathcal{D}_0\times\mathbb{R}^{n_2}\to[0,\infty)$ such that (18) holds and is jointly continuous on $\mathcal{D}_0\times\mathbb{R}^{n_2}$ follows as in the proof of (i).

(iv) Let $\delta>0$, $x_{10}\in\mathbb{R}^{n_1}$, and $x_{20}\in\mathbb{R}^{n_2}$ be such that $\|x_{10}\|<\delta$. Since $\alpha(\cdot)$ and $\beta(\cdot)$ are $\mathcal{K}_\infty$ functions, it follows that there exists $\varepsilon>0$ such that $\beta(\varepsilon)\le\alpha(\varepsilon)$. Now, (17) implies that $V(x_1(t),x_2(t))$ is a nonincreasing function of time, and hence, it follows from (16) that

$$\alpha(\|x_1(t)\|)\le V(x_1(t),x_2(t))\le V(x_{10},x_{20})=\alpha(\varepsilon),\quad t\ge 0. \tag{34}$$

Hence, for every $x_{10}\in\mathcal{B}_\delta(0)$, $x_1(t)\in\mathcal{B}_\varepsilon(0)$, $t\ge 0$, which proves uniform Lyapunov stability with respect to x_1. Finite-time partial convergence follows as in the proof of (i), implying global finite-time stability of $\mathcal{G}$ with respect to x_1 uniformly in x_{20}. In addition, the existence of a settling-time function $T:\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to[0,\infty)$ that verifies (18) and is jointly continuous on $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ follows as in the proof of (i).

(v) Uniform finite-time stability of $\mathcal{G}$ with respect to x_1 directly follows from (iii). Now, using similar arguments as in the proof of (i), it follows from (16) and (17) that

$$\alpha_1(\|x_1(t)\|)\le V(x_1(t),x_2(t))\le v(t),\quad t\ge 0, \tag{35}$$

where

$$v(t)=\begin{cases}\left[(V(x_{10},x_{20}))^{1-\theta}-(1-\theta)kt\right]^{\frac{1}{1-\theta}}, & 0\le t<t_1,\ v_0\ne 0,\\ 0, & t\ge t_1,\ v_0\ne 0,\\ 0, & t\ge 0,\ v_0=0,\end{cases} \tag{36}$$

and

$$t_1=\frac{(V(x_{10},x_{20}))^{1-\theta}}{k(1-\theta)}. \tag{37}$$

Now, the existence of a neighborhood $\mathcal{D}_0$ of $x_1=0$ and a settling-time function $T:\mathcal{D}_0\times\mathbb{R}^{n_2}\to[0,\infty)$ such that (21) holds and is jointly continuous on $\mathcal{D}_0\times\mathbb{R}^{n_2}$ follows as in the proof of (i). Hence, for $t\ge T(x_{10},x_{20})$, uniform finite-time convergence of $x_1(t)$ to zero is immediate. Alternatively, for every $t<T(x_{10},x_{20})$ and $\varepsilon>0$, there exists $\delta=\alpha_1^{-1}\left(\frac{\varepsilon^{1-\theta}}{k(1-\theta)}\right)$ such that if $\|x_1(t)\|\le\alpha_1^{-1}(v(t))<\varepsilon$, then $T(x_{10},x_{20})-t\le t_1-t<\delta$, which proves strong finite-time convergence of $\mathcal{G}$ with respect to x_1 uniformly in x_{20}.

(vi) The proof of finite-time stability of $\mathcal{G}$ with respect to x_1 uniformly in x_{20} follows as in the proof of (iv), whereas the proof of uniform finite-time convergence of $\mathcal{G}$ with respect to x_1 follows as in the proof of (v). Hence, the nonlinear dynamical system $\mathcal{G}$ is globally strongly finite-time stable with respect to x_1 uniformly in x_{20}. □

Example 3.1. Consider the nonlinear dynamical system given by

$$\dot{x}_1(t) = -x_2(t)(x_1(t))^{\frac{1}{3}}, \quad x_1(0) = x_{10}, \quad t \geq t_0, \tag{38}$$

$$\dot{x}_2(t) = x_2(t), \quad x_2(0) = x_{20}, \tag{39}$$

where $x_{20} > 0$, and hence, $x_2(t) > 0$, $t \geq 0$. To show that (38) and (39) is globally finite-time stable with respect to x_1, consider the Lyapunov function candidate $V(x_1, x_1) = x_1^{\frac{4}{3}}$ and let $\mathcal{D} = \mathbb{R}$. Clearly, (16) and (20) hold, and

$$\dot{V}(x_1, x_2) = \frac{4}{3} x_1^{\frac{1}{3}} \left(-x_2 x_1^{\frac{1}{3}} \right) = -\frac{4}{3} x_2 x_1^{\frac{2}{3}} \leq -k(x_2)(V(x_1, x_2))^{\frac{1}{2}}, \tag{40}$$

where $k(x_2) = \frac{4}{3} x_2 > 0$ and $x_2 > 0$. Hence, it follows from *iv*) of Theorem 3.1 that (38) and (39) is globally finite-time stable with respect to x_1. □

The following results specialize Propositions 3.1 and 3.2, and Theorem 3.1 to nonlinear time-varying dynamical systems.

Proposition 3.3. *Consider the nonlinear dynamical system $\mathcal{G}$ given by (3). Assume $\mathcal{G}$ is finite-time stable and let $\mathcal{D}_0 \subseteq \mathcal{D}$ and $T : [0, \infty) \times \mathcal{D}_0 \backslash \{0\} \to (t_0, \infty)$ be defined as in Definition 2.2. Then, for every $(t_0, x_0) \in [0, \infty) \times \mathcal{D}_0$, there exists a unique solution $s(t, t_0, x_0)$, $t \geq t_0$, to (3) such that $s(t, t_0, x_0) \in \mathcal{D}_0$, $t \in [t_0, T(t_0, x_0))$, and such that $s(t, t_0, x_0) = 0$, $t \geq T(t_0, x_0)$, where $T(t_0, 0) \triangleq t_0$.*

Proof. The result is a direct consequence of Proposition 3.1 with $n_1 = n$, $n_2 = 1$, $x_1(t - t_0) = x(t)$, $x_2(t - t_0) = t$, $f_1(x_1, x_2) = f_1(x_2, x_1) = f(t, x)$, $f_2(x_1, x_2) = 1$, and $T(x_{10}, x_{20}) = T(x_{20}, x_{10}) = T(t_0, x_0)$. □

Proposition 3.4. *Consider the nonlinear dynamical system $\mathcal{G}$ given by (3). Assume $\mathcal{G}$ is finite-time stable, let $\mathcal{D}_0 \subseteq \mathcal{D}$ be as defined in Definition 2.2, and let $T : [0, \infty) \times \mathcal{D}_0 \backslash \{0\} \to [t_0, \infty)$ be the settling-time function of $\mathcal{G}$. Then the following statements hold:*

(i) *If $t_1 \geq t_0$ and $(t_0, x_0) \in [0, \infty) \times \mathcal{D}_0$, then*
$$T(t_1, s(t_1, t_0, x_0)) = \max\{T(t_0, x_0), t_1\}. \tag{41}$$
(ii) *$T(\cdot, \cdot)$ is jointly continuous on $\mathbb{R}_+ \times \mathcal{D}_0$ if and only if $T(\cdot, \cdot)$ is jointly continuous at $(t, 0)$, $t \in [t_0, \infty)$.*

Proof. The result is a direct consequence of Proposition 3.2 with $n_1 = n$, $n_2 = 1$, $x_1(t - t_0) = x(t)$, $x_2(t - t_0) = t$, $f_1(x_1, x_2) = f_1(x_2, x_1) = f(t, x)$, $f_2(x_1, x_2) = 1$, and $T(x_{10}, x_{20}) = T(x_{20}, x_{10}) = T(t_0, x_0)$. □

Given the nonlinear time-varying dynamical system (3), for the statement of the following result define

$$\dot{V}(t, x) \triangleq \frac{\partial V(t, x)}{\partial t} + \frac{\partial V(t, x)}{\partial x} f(t, x),$$

where $V : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function.

Theorem 3.2. *Consider the nonlinear dynamical system* $\mathcal{G}$ *given by* (3). *Then the following statements hold*:

(i) *If there exist a continuously differentiable function* $V:[t_0,\infty)\times\mathcal{D}\to\mathbb{R}$, *a class* $\mathcal{K}$ *function* $\alpha(\cdot)$, *a continuous function* $k:[t_0,\infty)\to\mathbb{R}_+$, *a real number* $\theta\in(0,1)$, *and an open neighborhood* $\mathcal{M}\subseteq\mathcal{D}$ *of the origin such that*

$$V(t,0)=0,\quad t\in[t_0,\infty), \tag{42}$$

$$\alpha(\|x\|)\le V(t,x),\quad (t,x)\in[t_0,\infty)\times\mathcal{M}, \tag{43}$$

$$\dot V(t,x)\le -k(t)(V(t,x))^{\theta},\quad (t,x)\in[t_0,\infty)\times\mathcal{M}, \tag{44}$$

then $\mathcal{G}$ *is finite-time stable. Moreover, there exist a neighborhood* $\mathcal{D}_0$ *of the origin and a settling-time function* $T:[0,\infty)\times\mathcal{D}_0\to[t_0,\infty)$ *such that*

$$T(t_0,x_0)\le q^{-1}\left(\frac{(V(t_0,x_0))^{1-\theta}}{1-\theta}\right),\quad (t_0,x_0)\in[0,\infty)\times\mathcal{D}_0, \tag{45}$$

where $q:[t_0,\infty)\to\mathbb{R}$ *is continuously differentiable and*

$$\dot q(t)=k(t),\quad q(t_0)=0,\quad t\ge t_0, \tag{46}$$

and $T(\cdot,\cdot)$ *is jointly continuous on* $[0,\infty)\times\mathcal{D}_0$.

(ii) *If* $\mathcal{M}=\mathcal{D}=\mathbb{R}^n$ *and there exist a continuously differentiable function* $V:[t_0,\infty)\times\mathcal{D}\to\mathbb{R}$, *a class* $\mathcal{K}_\infty$ *function* $\alpha(\cdot)$, *a continuous function* $k:[t_0,\infty)\to\mathbb{R}_+$, *and a real number* $\theta\in(0,1)$ *such that* (42)–(44) *hold, then* $\mathcal{G}$ *is globally finite-time stable. Moreover, there exists a settling-time function* $T:[0,\infty)\times\mathbb{R}^n\to[t_0,\infty)$ *such that* (45) *holds with* $\mathcal{D}_0=\mathbb{R}^n$ *and* $T(\cdot,\cdot)$ *is jointly continuous on* $[0,\infty)\times\mathbb{R}^n$.

(iii) *If there exist a continuously differentiable function* $V:[t_0,\infty)\times\mathcal{D}\to\mathbb{R}$, *class* $\mathcal{K}$ *functions* $\alpha(\cdot)$ *and* $\beta(\cdot)$, *a continuous function* $k:[t_0,\infty)\to\mathbb{R}_+$, *a real number* $\theta\in(0,1)$, *and an open neighborhood* $\mathcal{M}\subseteq\mathcal{D}$ *of the origin such that* (43) *and* (44) *hold and*

$$V(t,x)\le\beta(\|x\|),\quad (t,x)\in[t_0,\infty)\times\mathcal{M}, \tag{47}$$

then $\mathcal{G}$ *is uniformly finite-time stable. Moreover, there exist a neighborhood* $\mathcal{D}_0$ *of the origin and a settling-time function* $T:[0,\infty)\times\mathcal{D}_0\to[t_0,\infty)$ *such that* (45) *holds and* $T(\cdot,\cdot)$ *is jointly continuous on* $[0,\infty)\times\mathcal{D}_0$.

(iv) *If* $\mathcal{M}=\mathcal{D}=\mathbb{R}^n$ *and there exist a continuously differentiable function* $V:[t_0,\infty)\times\mathcal{D}\to\mathbb{R}$, *class* $\mathcal{K}_\infty$ *functions* $\alpha(\cdot)$ *and* $\beta(\cdot)$, *a continuous function* $k:[t_0,\infty)\to\mathbb{R}_+$, *and a real number* $\theta\in(0,1)$ *such that* (43), (44), *and* (47) *hold, then* $\mathcal{G}$ *is globally uniformly finite-time stable. Moreover, there exists a settling-time function* $T:[0,\infty)\times\mathbb{R}^n\to[t_0,\infty)$ *such that* (45) *holds with* $\mathcal{D}_0=\mathbb{R}^n$ *and* $T(\cdot,\cdot)$ *is jointly continuous on* $[0,\infty)\times\mathbb{R}^{n_2}$.

(v) *If there exist a continuously differentiable function* $V:[t_0,\infty)\times\mathbb{R}^n\to\mathbb{R}$, *class* $\mathcal{K}$ *functions* $\alpha(\cdot)$ *and* $\beta(\cdot)$, *a real number* $\theta\in(0,1)$, *and an open neighborhood* $\mathcal{M}\subseteq\mathcal{D}$ *of the origin such that* (43), (44), *and* (47) *hold with* $k(t)=k\in\mathbb{R}_+$, $t\ge t_0$, *then* $\mathcal{G}$ *is strongly uniformly finite-time stable. Moreover, there exist a neighborhood* $\mathcal{D}_0$ *of the origin and a settling-time function* $T:[0,\infty)\times\mathcal{D}_0\to[t_0,\infty)$ *such that*

$$T(t_0,x_0)\le\frac{(V(t_0,x_0))^{1-\theta}}{k(1-\theta)},\quad (t_0,x_0)\in[0,\infty)\times\mathcal{D}_0, \tag{48}$$

and $T(\cdot,\cdot)$ *is jointly continuous on* $[0,\infty)\times\mathcal{D}_0$.

(vi) *If $\mathcal{M}=\mathcal{D}=\mathbb{R}^n$ and there exist a continuously differentiable function $V:\mathcal{D}\times\mathbb{R}^n\to\mathbb{R}$, class $\mathcal{K}_\infty$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, and a real number $\theta\in(0,1)$ such that (43), (44), and (47) hold with $k(t)=k\in\mathbb{R}_+$, $t\geq t_0$, then $\mathcal{G}$ is globally strongly uniformly finite-time stable. Moreover, there exists a settling-time function $T:[0,\infty)\times\mathbb{R}^n\to[t_0,\infty)$ such that (48) holds with $\mathcal{D}_0=\mathbb{R}^n$ and $T(\cdot,\cdot)$ is jointly continuous on $[0,\infty)\times\mathbb{R}^{n_1}$.*

Proof. The result is a direct consequence of Theorem 3.1 with $n_1=n$, $n_2=1$, $x_1(t-t_0)=x(t)$, $x_2(t-t_0)=t$, $f_1(x_1,x_2)=f_1(x_2,x_1)=f(t,x)$, $f_2(x_1,x_2)=1$, and $T(x_{10},x_{20})=T(x_{20},x_{10})=T(t_0,x_0)$. □

Remark 3.1. Propositions 3.3 and 3.4 along with Statements (i)–(iv) of Theorem 3.2 appear in [31]. See also [32].

Example 3.2. Consider the nonlinear time-varying dynamical system given by

$$\dot{x}(t)=-t(x(t))^{\frac{1}{3}}-t(x(t))^{\frac{1}{5}},\quad x(0)=x_0,\quad t\geq t_0. \tag{49}$$

To show that the zero solution $x(t)\equiv 0$ to (49) is globally uniformly finite-time stable, consider the Lyapunov function candidate $V(t,x)=x^{\frac{4}{3}}$ and let $\mathcal{D}=\mathbb{R}$. Clearly, (43) and (47) hold, and

$$\dot{V}(t,x)=\frac{4}{3}x^{\frac{1}{3}}\left(-tx^{\frac{1}{3}}-tx^{\frac{1}{5}}\right)=-\frac{4}{3}t\left(x^{\frac{2}{3}}+x^{\frac{8}{15}}\right)\leq -k(t)(V(t,x))^{\frac{1}{2}}, \tag{50}$$

where $k(t)=2t>0$, $t\geq t_0$. Hence, it follows from *iv*) of Theorem 3.2 that the zero solution $x(t)\equiv 0$ to (49) is globally uniformly finite-time stable. □

4. Optimal Finite-Time, Partial-State Stabilization

In the first part of this section, we provide connections between Lyapunov functions and nonquatratic cost evaluation. Specifically, we consider the problem of evaluating a nonlinear-nonquadratic performance measure that depends on the solution of the nonlinear dynamical system given by (1) and (2). In particular, we prove finite-time partial stability of (1) and (2), and show that the nonlinear-nonquadratic performance measure

$$J(x_{10},x_{20})\triangleq\int_0^\infty L(x_1(t),x_2(t))\mathrm{d}t, \tag{51}$$

where $L:\mathcal{D}\times\mathbb{R}^{n_2}\to\mathbb{R}$ is jointly continuous in x_1 and x_2, and $x_1(t)$ and $x_2(t)$, $t\geq 0$, satisfy (1) and (2), can be evaluated in a convenient form so long as (1) and (2) are related to an underlying Lyapunov function that is positive definite and decrescent with respect to x_1 and is related to an underlying Lyapunov function satisfying a differential inequality involving fractional powers.

Theorem 4.1. *Consider the nonlinear dynamical system $\mathcal{G}$ given by (1) and (2) with performance measure (51). Assume that there exists a continuously differentiable function $V:\mathcal{D}\times\mathbb{R}^{n_2}\to\mathbb{R}$, class $\mathcal{K}$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, a real number $\theta\in(0,1)$, and an open neighborhood $\mathcal{M}\subseteq\mathcal{D}$ of $x_1=0$ such that*

$$\alpha(\|x_1\|)\leq V(x_1,x_2)\leq\beta(\|x_1\|),\quad (x_1,x_2)\in\mathcal{M}\times\mathbb{R}^{n_2}, \tag{52}$$

$$\dot{V}(x_1,x_2)\leq -k(V(x_1,x_2))^{\theta},\quad (x_1,x_2)\in\mathcal{M}\times\mathbb{R}^{n_2}, \tag{53}$$

$$L(x_1,x_2)+V'(x_1,x_2)f(x_1,x_2)=0,\quad (x_1,x_2)\in\mathcal{M}\times\mathbb{R}^{n_2}. \tag{54}$$

Then the nonlinear dynamical system $\mathcal{G}$ is strongly finite-time stable with respect to x_1 uniformly in x_{20} and there exist a neighborhood $\mathcal{D}_0 \subseteq \mathcal{M}$ of $x_1 = 0$ and a settling-time function $T : \mathcal{D}_0 \times \mathbb{R}^{n_2} \to [0, \infty)$, jointly continuous on $\mathcal{D}_0 \times \mathbb{R}^{n_2}$, such that

$$T(x_{10}, x_{20}) \le \frac{(V(x_{10}, x_{20}))^{1-\theta}}{k(1-\theta)}, \quad (x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}. \tag{55}$$

In addition, for all $(x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}$,

$$J(x_{10}, x_{20}) = V(x_{10}, x_{20}). \tag{56}$$

Finally, if $\mathcal{M} = \mathcal{D} = \mathbb{R}^{n_1}$ and the functions $\alpha(\cdot)$ and $\beta(\cdot)$ satisfying (52) are class $\mathcal{K}_\infty$, then $\mathcal{G}$ is globally strongly finite-time stable with respect to x_1 uniformly in x_{20}.

Proof. Let $x_1(t)$ and $x_2(t)$, $t \ge 0$, satisfy (1) *and* (2). Then it follows from (53) that

$$\dot{V}(x_1(t), x_2(t)) = V'(x_1(t), x_2(t))f(x_1(t), x_2(t)) \le -k(V(x_1(t), x_2(t)))^\theta, \quad t \ge 0. \tag{57}$$

Thus, it follows from (52) and (53), and (v) of Theorem 3.1 that $\mathcal{G}$ is strongly finite-time stable with respect to x_1 uniformly in x_{20}. In addition, it follows from Theorem 3.1 that there exist an open neighborhood $\mathcal{D}_0$ of $x_1 = 0$ and a jointly continuous settling-time function $T : \mathcal{D}_0 \times \mathbb{R}^{n_2} \to [0, \infty)$ such that (55) holds and $x_1(t) \to 0$ as $t \to T(x_{10}, x_{20})$ for all initial condition $(x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}$. Now, since

$$0 = -\dot{V}(x_1(t), x_2(t)) + V'(x_1(t), x_2(t))f(x_1(t), x_2(t)), \quad t \ge 0, \tag{58}$$

it follows from (54) that

$$\begin{aligned} L(x_1(t), x_2(t)) &= -\dot{V}(x_1(t), x_2(t)) + L(x_1(t), x_2(t)) + V'(x_1(t), x_2(t))f(x_1(t), x_2(t)) \\ &= -\dot{V}(x_1(t), x_2(t)), \quad t \ge 0. \end{aligned} \tag{59}$$

Next, integrating (59) over $[0, t]$ yields

$$\int_0^t L(x_1(s), x_2(s))\mathrm{d}s = V(x_{10}, x_{20}) - V(x_1(t), x_2(t)), \quad t \ge 0. \tag{60}$$

Now, using (52) and letting $t \to \infty$ it follows from (60) that

$$V(x_{10}, x_{20}) - \beta\Big(\lim_{t\to\infty} \|x_1(t)\|\Big) \le \int_0^\infty L(x_1(s), x_2(s))\mathrm{d}s \le V(x_{10}, x_{20}) - \alpha\Big(\lim_{t\to\infty} \|x_1(t)\|\Big), \tag{61}$$

and hence, (56) is a direct consequence of (61) using the fact that $\lim_{t\to T(x_{10},x_{20})} x_1(t) = \lim_{t\to\infty} x_1(t) = 0$ and $\alpha(\cdot)$ and $\beta(\cdot)$ are class $\mathcal{K}$ functions. Finally, if $\mathcal{M} = \mathcal{D} = \mathbb{R}^{n_1}$ and $\alpha(\cdot)$ and $\beta(\cdot)$ are class $\mathcal{K}_\infty$ functions, then global strong finite-time stability with respect to x_1 uniformly in x_{20} is a direct consequence of (vi) of Theorem 3.1. □

The following corollary to Theorem 4.1 considers the nonautonomous dynamical system (3) with performance measure

$$J(t_0, x_0) \triangleq \int_{t_0}^\infty L(t, x(t))\mathrm{d}t, \tag{62}$$

where $L : [t_0, \infty) \times \mathcal{D} \to \mathbb{R}$ is jointly continuous in t and x, and $x(t)$, $t \ge t_0$, satisfies (3).

Corollary 4.1. *Consider the nonlinear time-varying dynamical system* (3) *with performance measure* (62). *Assume that there exist a continuously differentiable function* $V : [t_0, \infty) \times \mathcal{D} \to \mathbb{R}$, *class* $\mathcal{K}$ *functions* $\alpha(\cdot)$ *and* $\beta(\cdot)$, *a real number* $\theta \in (0,1)$, *and an open neighborhood* $\mathcal{M} \subseteq \mathcal{D}$ *of the origin such that*

$$\alpha(\|x\|) \le V(t,x) \le \beta(\|x\|), \quad (t,x) \in [t_0, \infty) \times \mathcal{M}, \tag{63}$$

$$\dot{V}(t,x) \le -k(V(t,x))^{\theta}, \quad (t,x) \in [t_0, \infty) \times \mathcal{M}, \tag{64}$$

$$0 = \frac{\partial V(t,x)}{\partial t} + L(t,x) + \frac{\partial V(t,x)}{\partial x} f(t,x), \quad (t,x) \in [t_0, \infty) \times \mathcal{M}. \tag{65}$$

Then the nonlinear time-varying dynamical system (3) *is strongly uniformly finite-time stable and there exist a neighborhood of the origin* $\mathcal{D}_0 \subseteq \mathcal{M}$ *and a settling-time function* $T : [0, \infty) \times \mathcal{D}_0 \to [t_0, \infty)$, *jointly continuous on* $[0, \infty) \times \mathcal{D}_0$, *such that*

$$T(t_0, x_0) \le \frac{(V(t_0, x_0))^{1-\theta}}{k(1-\theta)}, \quad (t_0, x_0) \in [0, \infty) \times \mathcal{D}_0. \tag{66}$$

In addition, for all $(t_0, x_0) \in [0, \infty) \times \mathcal{D}_0$,

$$J(t_0, x_0) = V(t_0, x_0). \tag{67}$$

Finally, if $\mathcal{D} = \mathbb{R}^n$ *and the functions* $\alpha(\cdot)$ *and* $\beta(\cdot)$ *satisfying* (52) *are class* $\mathcal{K}_\infty$, *then* $\mathcal{G}$ *is globally strongly finite-time stable.*

Proof. The result is a direct consequence of Theorem 4.1 with $n_1 = n$, $n_2 = 1$, $x_1(t - t_0) = x(t)$, $x_2(t - t_0) = t$, $f_1(x_1, x_2) = f_1(x_2, x_1) = f(t,x)$, $f_2(x_1, x_2) = 1$, $T(x_{10}, x_{20}) = T(x_{20}, x_{10}) = T(t_0, x_0)$, and $V(x_1, x_2) = V(x_2, x_1) = V(t,x)$. □

Next, we use the framework developed in Theorem 4.1 to obtain a characterization of optimal feedback controllers that guarantee closed-loop finite-time partial stabilization. Specifically, sufficient conditions for optimality are given in a form that corresponds to a steady-state version of the Hamilton-Jacobi-Bellman equation. To address the problem of characterizing finite-time partially stabilizing feedback controllers, consider the controlled nonlinear dynamical system

$$\dot{x}_1(t) = F_1(x_1(t), x_2(t), u(t)), \quad x_1(0) = x_{10}, \quad t \ge 0, \tag{68}$$

$$\dot{x}_2(t) = F_2(x_1(t), x_2(t), u(t)), \quad x_2(0) = x_{20}, \tag{69}$$

where, for every $t \ge 0$, $x_1(t) \in \mathcal{D} \subseteq \mathbb{R}^{n_1}$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $x_2(t) \in \mathbb{R}^{n_2}$, $u(t) \in U \subseteq \mathbb{R}^m$ with $0 \in U$, $F_1 : \mathcal{D} \times \mathbb{R}^{n_2} \times U \to \mathbb{R}^{n_1}$ and $F_2 : \mathcal{D} \times \mathbb{R}^{n_2} \times U \to \mathbb{R}^{n_2}$ are jointly continuous in x_1, x_2, and u, and $F_1(0, x_2, 0) = 0$ for every $x_2 \in \mathbb{R}^{n_2}$. The control $u(\cdot)$ in (68) and (69) is restricted to the class of *admissible* controls consisting of measurable functions $u(\cdot)$ such that $u(t) \in U$, $t \ge 0$.

A measurable function $\phi : \mathcal{D} \times \mathbb{R}^{n_2} \to U$ satisfying $\phi(0, x_2) = 0$, $x_2 \in \mathbb{R}^{n_2}$, is called a *control law*. If $u(t) = \phi(x_1(t), x_2(t))$, $t \ge 0$, where $\phi(\cdot,\cdot)$ is a control law and $x_1(t)$ and $x_2(t)$ satisfy (68) and (69), then we call $u(\cdot)$ a *feedback control law*. Note that the feedback control law is an admissible control since $\phi(\cdot,\cdot)$ has values in U. Given a control law $\phi(\cdot,\cdot)$ and a feedback control law $u(t) = \phi(x_1(t), x_2(t))$, $t \ge 0$, the *closed-loop system* (68) and (69) is given by

$$\dot{x}_1(t) = F_1(x_1(t), x_2(t), \phi(x_1(t), x_2(t))), \quad x_1(0) = x_{10}, \quad t \ge 0, \tag{70}$$

$$\dot{x}_2(t) = F_2(x_1(t), x_2(t), \phi(x_1(t), x_2(t))), \quad x_2(0) = x_{20}. \tag{71}$$

We now consider the problem of finite-time partial-state stabilization.

Definition 4.1. Consider the controlled nonlinear dynamical system given by (68) and (69). The feedback control law $u=\phi(x_1,x_2)$ is *strongly finite-time stabilizing with respect to* x_1 *uniformly in* x_{20} if the closed-loop system (70) and (71) is strongly finite-time stable with respect to x_1 uniformly in x_{20}. Furthermore, the feedback control law $u=\phi(x_1,x_2)$ is *globally strongly finite-time stabilizing with respect to* x_1 *uniformly in* x_{20} if the closed-loop system (70) and (71) is globally strongly finite-time stable with respect to x_1 uniformly in x_{20}.

Next, we present a main theorem for strong finite-time, partial-state stabilization characterizing feedback controllers that guarantee closed-loop finite-time partial stability and minimize a nonlinear-nonquadratic performance functional. For the statement of this result, define $F(x_1,x_2,u)\triangleq[F_1^{\mathrm{T}}(x_1,x_2,u),F_2^{\mathrm{T}}(x_1,x_2,u)]^{\mathrm{T}}$, let $L:\mathcal{D}\times\mathbb{R}^{n_2}\times U\to\mathbb{R}$ be jointly continuous in x_1, x_2, and u, and define the set of partial regulation controllers given by

$$\mathcal{S}(x_{10},x_{20})\triangleq\{u(\cdot):u(\cdot)\text{is admissible and } x_1(\cdot)\text{ given by (68)}\\ \text{satisfies } x_1(t)\to 0 \quad \text{as } t\to T(x_{10},x_{20})\},$$

where $T:\mathcal{D}_0\times\mathbb{R}^{n_2}\to(0,\infty)$ is the settling-time function and $\mathcal{D}_0\subseteq\mathcal{D}$ is an open neighborhood of $x_1=0$. Note that restricting our minimization problem to $u(\cdot)\in\mathcal{S}(x_{10},x_{20})$, that is, inputs corresponding to partial-state null convergent solutions, can be interpreted as incorporating a partial-state system detectability condition through the cost. In addition, since finite-time partial convergence is a stronger condition than asymptotic partial-state convergence, $\mathcal{S}(x_{10},x_{20})$ includes the set of all partial-state null asymptotically convergent controllers.

Theorem 4.2. *Consider the controlled nonlinear dynamical system* $\mathcal{G}$ *given by* (68) *and* (69) *with*

$$J(x_{10},x_{20},u(\cdot))\triangleq\int_0^\infty L(x_1(t),x_2(t),u(t))\,\mathrm{d}t, \tag{72}$$

where $u(\cdot)$ *is an admissible control. Assume that there exist a continuously differentiable function* $V:\mathcal{D}\times\mathbb{R}^{n_2}\to\mathbb{R}$, *class* $\mathcal{K}$ *functions* $\alpha(\cdot)$ *and* $\beta(\cdot)$, *a real number* $\theta\in(0,1)$, *an open neighborhood* $\mathcal{M}\subseteq\mathcal{D}$ *of* $x_1=0$, *and a control law* $\phi:\mathcal{M}\times\mathbb{R}^{n_2}\to U$ *such that*

$$\alpha(\|x_1\|)\le V(x_1,x_2)\le\beta(\|x_1\|),\quad (x_1,x_2)\in\mathcal{M}\times\mathbb{R}^{n_2}, \tag{73}$$

$$V'(x_1,x_2)F(x_1,x_2,\phi(x_1,x_2))\le -k(V(x_1,x_2))^{\theta},\quad (x_1,x_2)\in\mathcal{M}\times\mathbb{R}^{n_2}, \tag{74}$$

$$\phi(0,x_2)=0,\quad x_2\in\mathbb{R}^{n_2}, \tag{75}$$

$$L(x_1,x_2,\phi(x_1,x_2))+V'(x_1,x_2)F(x_1,x_2,\phi(x_1,x_2))=0,\quad (x_1,x_2)\in\mathcal{M}\times\mathbb{R}^{n_2}, \tag{76}$$

$$L(x_1,x_2,u)+V'(x_1,x_2)F(x_1,x_2,u)\ge 0,\quad (x_1,x_2,u)\in\mathcal{M}\times\mathbb{R}^{n_2}\times U. \tag{77}$$

Then, with the feedback control $u=\phi(x_1,x_2)$, *the closed-loop system given by* (70) *and* (71) *is strongly finite-time stable with respect to* x_1 *uniformly in* x_{20} *and there exist a neighborhood* $\mathcal{D}_0\subseteq\mathcal{M}$ *of* $x_1=0$ *and a settling-time function* $T:\mathcal{D}_0\times\mathbb{R}^{n_2}\to[0,\infty)$, *jointly continuous on* $\mathcal{D}_0\times\mathbb{R}^{n_2}$, *such that*

$$T(x_{10},x_{20})\le\frac{(V(x_{10},x_{20}))^{1-\theta}}{k(1-\theta)},\quad (x_{10},x_{20})\in\mathcal{D}_0\times\mathbb{R}^{n_2}. \tag{78}$$

In addition, if $(x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}$, *then*

$$J(x_{10}, x_{20}, \phi(x_1(\cdot), x_2(\cdot))) = V(x_{10}, x_{20}), \quad (x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2} \tag{79}$$

and the feedback control $u(\cdot) = \phi(x_1(\cdot), x_2(\cdot))$ *minimizes* $J(x_{10}, x_{20}, u(\cdot))$ *in the sense that*

$$J(x_{10}, x_{20}, \phi(\cdot,\cdot)) = \min_{u(\cdot) \in \mathcal{S}(x_{10}, x_{20})} J(x_{10}, x_{20}, u(\cdot)). \tag{80}$$

Finally, if $\mathcal{M} = \mathcal{D} = \mathbb{R}^{n_1}$, $U = \mathbb{R}^m$, *and the functions* $\alpha(\cdot)$ *and* $\beta(\cdot)$ *satisfying* (73) *are class* $\mathcal{K}_\infty$, *then the closed-loop system* (70) *and* (71) *is globally strongly finite-time stable with respect to* x_1 *uniformly in* x_{20}.

Proof. Local and global strong finite-time stability with respect to x_1 uniformly in x_{20} are a direct consequence of (73) and (74) by applying Theorem 3.1 to the closed-loop system given by (70) and (71). In addition, it follows from Theorem 3.1 that there exist an open neighborhood $\mathcal{D}_0$ of $x_1 = 0$ and a jointly continuous settling-time function $T : \mathcal{D}_0 \times \mathbb{R}^{n_2} \to [0, \infty)$ such that (78) holds and $x_1(t) \to 0$ as $t \to T(x_{10}, x_{20})$ for all initial conditions $(x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}$. Furthermore, using (76), condition (79) is a restatement of (56) as applied to the closed-loop system.

Next, let $(x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}$, let $u(\cdot) \in \mathcal{S}(x_{10}, x_{20})$, and let $x_1(t)$ and $x_2(t)$, $t \geq 0$, be solutions of (68) and (69). Then, it follows that

$$0 = -\dot{V}(x_1(t), x_2(t)) + V'(x_1(t), x_2(t)) F(x_1(t), x_2(t), u(t)), \quad t \geq 0. \tag{81}$$

Hence,

$$\begin{aligned} L(x_1(t), x_2(t), u(t)) = & -\dot{V}(x_1(t), x_2(t)) + L(x_1(t), x_2(t), u(t)) \\ & + V'(x_1(t), x_2(t)) F(x_1(t), x_2(t), u(t)), \quad t \geq 0. \end{aligned} \tag{82}$$

Now, using (73) and the fact that $\mathcal{G}$ is strongly finite-time stable with respect to x_1 uniformly in x_{20}, it follows that

$$0 = \lim_{t \to \infty} \alpha(\|x_1(t)\|) \leq \lim_{t \to \infty} V(x_1(t), x_2(t)) \leq \lim_{t \to \infty} \beta(\|x_1(t)\|) = 0. \tag{83}$$

Thus, it follows from (82), (83), (77), (79), and the strong finite-time stability of $\mathcal{G}$ with respect to x_1 uniformly in x_{20}, that

$$\begin{aligned} \int_0^\infty L(x_1(t), x_2(t), u(t))\mathrm{d}t = & \int_0^\infty -\dot{V}(x_1(t), x_2(t))\mathrm{d}t + \int_0^\infty L(x_1(t), x_2(t), u(t))\mathrm{d}t \\ & + \int_0^\infty \left(\frac{\partial V(x_1, x_2)}{\partial x_1} F_1(x_1(t), x_2(t), u(t)) \right. \\ & \left. + \frac{\partial V(x_1, x_2)}{\partial x_2} F_2(x_1(t), x_2(t), u(t)) \right) \mathrm{d}t \\ \geq & \int_0^\infty -\dot{V}(x_1(t), x_2(t))\mathrm{d}t \\ = & -\lim_{t \to \infty} V(x_1(t), x_2(t)) + V(x_{10}, x_{20}) \\ = & -V\left(\lim_{t \to \infty} (x_1(t), x_2(t)) \right) + V(x_{10}, x_{20}) \\ = & -V\left(\lim_{t \to T(x_{10}, x_{20})} (x_1(t), x_2(t)) \right) + V(x_{10}, x_{20}) \\ = & J(x_{10}, x_{20}, \phi(x_1(\cdot), x_2(\cdot))), \end{aligned} \tag{84}$$

which yields (80). □

Note that (76) is the steady-state, Hamilton-Jacobi-Bellman equation for the nonlinear controlled dynamical system (68) and (69) with performance criterion (72). Furthermore, conditions (76) and (77) guarantee optimality with respect to the set of admissible finite-time partially stabilizing controllers $\mathcal{S}(x_{10}, x_{20})$. However, it is important to note that an explicit characterization of $\mathcal{S}(x_{10}, x_{20})$ is not required. In addition, the optimal strongly finite-time stabilizing with respect to x_1 uniformly in x_{20} *feedback* control law $u = \phi(x_1, x_2)$ is independent of the initial condition (x_{10}, x_{20}) and is given by

$$\phi(x_1, x_2) = \underset{u \in \mathcal{S}(x_{10}, x_{20})}{\arg\min} \left[L(x_1, x_2, u) + \frac{\partial V(x_1, x_2)}{\partial x_1} F_1(x_1, x_2, u) + \frac{\partial V(x_1, x_2)}{\partial x_2} F_2(x_1, x_2, u) \right]. \tag{85}$$

Finally, we use Theorem 4.2 to provide a unification between optimal finite-time, partial-state stabilization and optimal finite-time control for nonlinear time-varying systems. Specifically, consider the controlled nonlinear time-varying dynamical system

$$\dot{x}(t) = F(t, x(t), u(t)), \quad x(t_0) = x_0, \quad t \ge t_0, \tag{86}$$

with performance measure

$$J(t_0, x_0, u(\cdot)) \triangleq \int_{t_0}^{\infty} L(t, x(t), u(t)) \mathrm{d}t, \tag{87}$$

where, for every $t \ge t_0$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $u(t) \in U \subseteq \mathbb{R}^m$ with $0 \in U$, and $L : [t_0, \infty) \times \mathcal{D} \times U \to \mathbb{R}$ and $F : [t_0, \infty) \times \mathcal{D} \times U \to \mathbb{R}^n$ are jointly continuous in t, x, and u on $[t_0, \infty) \times \mathcal{D} \times U$. For the statement of the next result, define the set of regulation controllers

$$\mathcal{S}(t_0, x_0) \triangleq \{u(\cdot) : u(\cdot) \quad \text{is admissible and } x(\cdot) \quad \text{given by (86)} \\ \text{satisfies } x(t) \to 0 \text{ as } t \to T(t_0, x_0)\},$$

where $T : [0, \infty) \times \mathcal{D}_0 \to (t_0, \infty)$ is the settling-time function and $\mathcal{D}_0 \subseteq \mathcal{D}$ is an open neighborhood of the origin.

Corollary 4.2. *Consider the controlled nonlinear time-varying dynamical system (86) with performance measure (87) where $u(\cdot)$ is an admissible control. Assume that there exist a continuously differentiable function $V : [t_0, \infty) \times \mathcal{D} \to \mathbb{R}$, class $\mathcal{K}$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, a real number $\theta \in (0, 1)$, an open neighborhood $\mathcal{M} \subseteq \mathcal{D}$ of the origin, and a control law $\phi : [t_0, \infty) \times \mathcal{M} \to U$ such that*

$$\alpha(\|x\|) \le V(t, x) \le \beta(\|x\|), \quad (t, x) \in [t_0, \infty) \times \mathcal{M}, \tag{88}$$

$$\frac{\partial V(t, x)}{\partial t} + \frac{\partial V(t, x)}{\partial x} F(t, x, \phi(t, x)) \le -k(V(t, x(t))^{\theta}, \quad (t, x) \in [t_0, \infty) \times \mathcal{M}, \tag{89}$$

$$\phi(t, 0) = 0, \quad t \in [t_0, \infty), \tag{90}$$

$$L(t, x, \phi(t, x)) + \frac{\partial V(t, x)}{\partial t} + \frac{\partial V(t, x)}{\partial x} F(t, x, \phi(t, x)) = 0, \quad (t, x) \in [t_0, \infty) \times \mathcal{M}, \tag{91}$$

$$L(t, x, u) + \frac{\partial V(t, x)}{\partial t} + \frac{\partial V(t, x)}{\partial x} F(t, x, u) \ge 0, \quad (t, x, u) \in [t_0, \infty) \times \mathcal{M} \times U. \tag{92}$$

Then, with the feedback control $u=\phi(t,x)$, the closed-loop system given by

$$\dot{x}(t)=F(t,x(t),\phi(x(t))),\quad x(0)=x_0,\quad t\geq t_0, \tag{93}$$

is strongly uniformly finite-time stable and there exists a neighborhood of the origin $\mathcal{D}_0\subseteq\mathcal{M}$ and a settling-time function $T:[0,\infty)\times\mathcal{D}_0\to[t_0,\infty)$, jointly continuous on $[0,\infty)\times\mathcal{D}_0$, such that

$$T(t_0,x_0)\leq\frac{(V(t_0,x_0))^{1-\theta}}{k(1-\theta)},\quad (t_0,x_0)\in[0,\infty)\times\mathcal{D}_0. \tag{94}$$

In addition, if $(t_0,x_0)\in[0,\infty)\times\mathcal{D}_0$, then

$$J(t_0,x_0,\phi(\cdot,\cdot))=V(t_0,x_0),\quad (t_0,x_0)\in[0,\infty)\times\mathcal{D}_0. \tag{95}$$

and the feedback control $u(\cdot)=\phi(\cdot,x(\cdot))$ minimizes $J(t_0,x_0,u(\cdot))$ in the sense that

$$J(t_0,x_0,\phi(\cdot,\cdot))=\min_{u(\cdot)\in\mathcal{S}(t_0,x_0)}J(t_0,x_0,u(\cdot)). \tag{96}$$

Finally, if $\mathcal{D}=\mathbb{R}^n$, $U=\mathbb{R}^m$, and the functions $\alpha(\cdot)$ and $\beta(\cdot)$ satisfying (88) are class $\mathcal{K}_\infty$, then the nonlinear dynamical system $\mathcal{G}$ is globally uniformly asymptotically stable.

Proof. The proof is a direct consequence of Theorem 4.2 with $n_1=n$, $n_2=1$, $x_1(t-t_0)=x(t)$, $x_2(t-t_0)=t$, $F_1(x_1,x_2,u)=F_1(x_2,x_1,u)=F(t,x,u)$, $F_2(x_1,x_2,u)=1$, $\phi(x_1,x_2)=\phi(x_2,x_1)=\phi(t,x)$, $T(x_{10},x_{20})=T(x_{20},x_{10})=T(t_0,x_0)$, and $V(x_1,x_2)=V(x_2,x_1)=V(t,x)$. □

Note that (91) and (92) give the classical Hamilton-Jacobi-Bellman equation

$$-\frac{\partial V(t,x)}{\partial t}=\min_{u\in\mathcal{S}(t_0,x_0)}\left[L(t,x,u)+\frac{\partial V(t,x)}{\partial x}F(t,x,u)\right],\quad (t,x)\in[t_0,\infty)\times\mathcal{D}, \tag{97}$$

which characterizes the optimal control

$$\phi(t,x)=\arg\min_{u\in\mathcal{S}(t_0,x_0)}\left[L(t,x,u)+\frac{\partial V(t,x)}{\partial x}F(t,x,u)\right] \tag{98}$$

for time-varying systems on a finite or infinite interval.

5. Finite-time stabilization for affine dynamical systems and connections to inverse optimal control

In this section, we specialize the results of Section 4 to nonlinear affine dynamical systems of the form

$$\dot{x}_1(t)=f_1(x_1(t),x_2(t))+G_1(x_1(t),x_2(t))u(t),\quad x_1(0)=x_{10},\ t\geq 0, \tag{99}$$

$$\dot{x}_2(t)=f_2(x_1(t),x_2(t))+G_2(x_1(t),x_2(t))u(t),\quad x_2(0)=x_{20}, \tag{100}$$

where, for every $t\geq 0$, $x_1(t)\in\mathbb{R}^{n_1}$, $x_2(t)\in\mathbb{R}^{n_2}$, and $u(t)\in\mathbb{R}^m$, and $f_1:\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to\mathbb{R}^{n_1}$, $f_2:\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to\mathbb{R}^{n_2}$, $G_1:\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to\mathbb{R}^{n_1\times m}$, and $G_2:\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to\mathbb{R}^{n_2\times m}$ are such that $f_1(0,x_2)=0$ for all $x_2\in\mathbb{R}^{n_2}$, and $f_1(\cdot,\cdot)$, $f_2(\cdot,\cdot)$, $G_1(\cdot,\cdot)$, and $G_2(\cdot,\cdot)$ are jointly continuous in x_1 and x_2 on $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$. Furthermore, we consider performance integrands $L(x_1,x_2,u)$ of the form

$$L(x_1,x_2,u)=L_1(x_1,x_2)+L_2(x_1,x_2)u+u^{\mathrm{T}}R_2(x_1,x_2)u,\quad (x_1,x_2,u)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^m, \tag{101}$$

where $L_1:\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to\mathbb{R}$, $L_2:\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to\mathbb{R}^{1\times m}$, and $R_2(x_1,x_2)\geq N(x_1)>0$, $(x_1,x_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$, so that (72) becomes

$$J(x_{10},x_{20},u(\cdot)) = \int_0^\infty \big[L_1(x_1(t),x_2(t)) + L_2(x_1(t),x_2(t))u(t) + u^{\mathrm{T}}(t)R_2(x_1(t),x_2(t))u(t)\big]\,\mathrm{d}t. \tag{102}$$

For the statement of the next result, define

$$f(x_1,x_2) \triangleq [f_1^{\mathrm{T}}(x_1,x_2), f_2^{\mathrm{T}}(x_1,x_2)]^{\mathrm{T}}, \quad G(x_1,x_2) \triangleq [G_1^{\mathrm{T}}(x_1,x_2), G_2^{\mathrm{T}}(x_1,x_2)]^{\mathrm{T}}. \tag{103}$$

Theorem 5.1. *Consider the controlled nonlinear affine dynamical system (99) and (100) with performance measure (102). Assume that there exist a continuously differentiable function* $V:\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to\mathbb{R}$, *class* $\mathcal{K}_\infty$ *functions* $\alpha(\cdot)$ *and* $\beta(\cdot)$, *and a real number* $\theta\in(0,1)$ *such that*

$$\alpha(\|x_1\|) \le V(x_1,x_2) \le \beta(\|x_1\|), \quad (x_1,x_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}, \tag{104}$$

$$V'(x_1,x_2)\Big[f(x_1,x_2) - \frac{1}{2}G(x_1,x_2)R_2^{-1}(x_1,x_2)L_2^{\mathrm{T}}(x_1,x_2) - \frac{1}{2}G(x_1,x_2)R_2^{-1}(x_1,x_2)G^{\mathrm{T}}(x_1,x_2)V'^{\mathrm{T}}(x_1,x_2)\Big] \le -k(V(x_1,x_2))^\theta, \quad (x_1,x_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}, \tag{105}$$

$$L_2(0,x_2) = 0, \quad x_2\in\mathbb{R}^{n_2}, \tag{106}$$

$$0 = L_1(x_1,x_2) + V'(x_1,x_2)f(x_1,x_2) - \frac{1}{4}[V'(x_1,x_2)G(x_1,x_2) + L_2(x_1,x_2)] \times R_2^{-1}(x_1,x_2)[V'(x_1,x_2)G(x_1,x_2) + L_2(x_1,x_2)]^{\mathrm{T}}, \quad (x_1,x_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}, \tag{107}$$

Then, with the feedback control

$$u = \phi(x_1,x_2) = -\frac{1}{2}R_2^{-1}(x_1,x_2)[L_2(x_1,x_2) + V'(x_1,x_2)G(x_1,x_2)]^{\mathrm{T}}, \tag{108}$$

the closed-loop system

$$\dot{x}_1(t) = f_1(x_1(t),x_2(t)) + G_1(x_1(t),x_2(t))\phi(x_1(t),x_2(t)), \quad x_1(0)=x_{10}, \quad t\ge 0, \tag{109}$$

$$\dot{x}_2(t) = f_2(x_1(t),x_2(t)) + G_2(x_1(t),x_2(t))\phi(x_1(t),x_2(t)), \quad x_2(0)=x_{20}, \tag{110}$$

is globally strongly finite-time stable with respect to x_1 *uniformly in* x_{20} *and there exists a settling-time function* $T:\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to[0,\infty)$, *jointly continuous on* $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$, *such that*

$$T(x_{10},x_{20}) \le \frac{(V(x_{10},x_{20}))^{1-\theta}}{k(1-\theta)}, \quad (x_{10},x_{20})\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}. \tag{111}$$

In addition,

$$J(x_{10},x_{20},\phi(x_1(\cdot),x_2(\cdot))) = V(x_{10},x_{20}), \quad (x_{10},x_{20})\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}. \tag{112}$$

and the performance measure (102) is minimized in the sense of (80).

Proof. The result is a consequence of Theorem 4.2 with $\mathcal{M}=\mathcal{D}=\mathbb{R}^{n_1}$, $U=\mathbb{R}^m$, $F(x_1,x_2,u)=f(x_1,x_2)+G(x_1,x_2)u$, and

$$L(x_1,x_2,u) = L_1(x_1,x_2) + L_2(x_1,x_2)u + u^{\mathrm{T}}R_2(x_1,x_2)u.$$

Specifically, the feedback control law (108) follows from (85) by setting

$$\frac{\partial}{\partial u}\left[L_1(x_1,x_2)+L_2(x_1,x_2)u+u^{\mathrm{T}}R_2(x_1,x_2)u+V'(x_1,x_2)(f(x_1,x_2)+G(x_1,x_2)u)\right]=0. \tag{113}$$

Now, with $u=\phi(x_1,x_2)$ given by (108), conditions (104), (105) and (107) imply (73), (74) and (77), respectively.

Next, since $V(\cdot,\cdot)$ is continuously differentiable and, by (104), $V(0,x_2)$, $x_2\in\mathbb{R}^{n_2}$, is a local minimum of $V(\cdot,\cdot)$, it follows that $V'(0,x_2)=0$, $x_2\in\mathbb{R}^{n_2}$, and hence, it follows from (106) and (108) that $\phi(0,x_2)=0$, $x_2\in\mathbb{R}^{n_2}$, which implies (75). Finally, since

$$\begin{aligned}
&L(x_1,x_2,u)+V'(x_1,x_2)[f(x_1,x_2)+G(x_1,x_2)u]\\
&\quad=L(x_1,x_2,u)+V'(x_1,x_2)[f(x_1,x_2)+G(x_1,x_2)u]-L(x_1,x_2,\phi(x_1,x_2))\\
&\qquad-V'(x_1,x_2)[f(x_1,x_2)+G(x_1,x_2)\phi(x_1,x_2)]\\
&\quad=[u-\phi(x_1,x_2)]^{\mathrm{T}}R_2(x_1,x_2)[u-\phi(x_1,x_2)]\\
&\quad\geq 0,\quad (x_1,x_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2},
\end{aligned} \tag{114}$$

condition (77) holds. The result now follows as a direct consequence of Theorem 4.2. □

The following corollary to Theorem 5.1 considers the nonautonomous dynamical system

$$\dot{x}(t)=f(t,x(t))+G(t,x(t))u(t),\quad x(t_0)=x_0,\quad t\geq t_0, \tag{115}$$

with performance measure

$$J(t_0,x_0,u(\cdot))=\int_{t_0}^{\infty}\left[L_1(t,x(t))+L_2(t,x(t))u(t)+u^{\mathrm{T}}(t)R_2(t,x(t))u(t)\right]\mathrm{d}t, \tag{116}$$

where, for every $t\geq t_0$, $x(t)\in\mathbb{R}^n$ and $u(t)\in\mathbb{R}^m$, $f:[t_0,\infty)\times\mathbb{R}^n\to\mathbb{R}^n$ and $G:[t_0,\infty)\times\mathbb{R}^n\to\mathbb{R}^{n\times m}$ are such that $f(t,0)=0$ for all $t\in[t_0,\infty)$, $f(\cdot,\cdot)$ and $G(\cdot,\cdot)$ are jointly continuous in x_1 and x_2 on $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$, $L_1:[t_0,\infty)\times\mathbb{R}^n\to\mathbb{R}$, $L_2:[t_0,\infty)\times\mathbb{R}^n\to\mathbb{R}^{1\times m}$, and $R_2(t,x)\geq N(x)>0$, $(t,x)\in[t_0,\infty)\times\mathbb{R}^n$.

Corollary 5.1. *Consider the controlled nonlinear affine dynamical system (115) with performance measure (116). Assume that there exist a continuously differentiable function $V:[t_0,\infty)\times\mathbb{R}^n\to\mathbb{R}$, class $\mathcal{K}_\infty$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, and a real number $\theta\in(0,1)$ such that*

$$\alpha(\|x\|)\leq V(t,x)\leq\beta(\|x\|),\quad (t,x)\in[t_0,\infty)\times\mathbb{R}^n, \tag{117}$$

$$\begin{aligned}
&\frac{\partial V(t,x)}{\partial t}\\
&\quad+\frac{\partial V(t,x)}{\partial x}\left[f(t,x)-\frac{1}{2}G(t,x)R_2^{-1}(t,x)L_2^{\mathrm{T}}(t,x)-\frac{1}{2}G(t,x)R_2^{-1}(t,x)G^{\mathrm{T}}(t,x)\left(\frac{\partial V(t,x)}{\partial x}\right)^{\mathrm{T}}\right]\\
&\quad\leq -k(V(t,x))^{\theta},\quad (t,x)\in[t_0,\infty)\times\mathbb{R}^n,
\end{aligned} \tag{118}$$

$$L_2(t,0)=0,\quad t\in[t_0,\infty), \tag{119}$$

$$0=L_1(t,x)+\frac{\partial V(t,x)}{\partial x}f(t,x)-\frac{1}{4}\left[\frac{\partial V(t,x)}{\partial x}G(t,x)+L_2(t,x)\right]$$

$$\times R_2^{-1}(t,x)\left[\frac{\partial V(t,x)}{\partial x}G(t,x)+L_2(t,x)\right]^{\mathrm{T}}, \quad (t,x)\in[t_0,\infty)\times\mathbb{R}^n, \tag{120}$$

Then, with the feedback control

$$u=\phi(t,x)=-\frac{1}{2}R_2^{-1}(t,x)\left[L_2(t,x)+\frac{\partial V(t,x)}{\partial x}G(t,x)\right]^{\mathrm{T}}, \tag{121}$$

the closed-loop system

$$\dot{x}(t)=f(t,x(t))+G(t,x(t))\phi(t,x(t)), \quad x(0)=x_0, \quad t\geq t_0, \tag{122}$$

is globally strongly uniformly finite-time stable and there exists a settling-time function $T:[0,\infty)\times\mathbb{R}^n\to[t_0,\infty)$, *jointly continuous on* $[0,\infty)\times\mathbb{R}^n$, *such that*

$$T(t_0,x_0)\leq\frac{(V(t_0,x_0))^{1-\theta}}{k(1-\theta)}, \quad (t_0,x_0)\in[0,\infty)\times\mathbb{R}^n. \tag{123}$$

In addition,

$$J(t_0,x_0,\phi(\cdot,x(\cdot)))=V(t_0,x_0), \quad (t_0,x_0)\in[0,\infty)\times\mathbb{R}^n \tag{124}$$

and the performance measure (116) is minimized in the sense of (96).

Proof. The proof is a direct consequence of Theorem 5.1 with $n_1=n$, $n_2=1$, $x_1(t-t_0)=x(t)$, $x_2(t-t_0)=t$, $f(x_1,x_2)=f(x_2,x_1)=f(t,x)$, $G(x_1,x_2)=G(x_2,x_1)=G(t,x)$, $L_1(x_1,x_2)=L_1(x_2,x_1)=L_1(t,x)$, $L_2(x_1,x_2)=L_2(x_2,x_1)=L_2(t,x)$, $R_2(x_1,x_2)=R_2(x_2,x_1)=R_2(t,x)$, $\phi(x_1,x_2)=\phi(x_2,x_1)=\phi(t,x)$, $T(x_{10},x_{20})=T(x_{20},x_{10})=T(t_0,x_0)$, and $V(x_1,x_2)=V(x_2,x_1)=V(t,x)$. □

Next, we construct state feedback controllers for nonlinear affine in the control dynamical systems that are predicated on an *inverse optimal control problem* [19–21,33,22]. In particular, to avoid the complexity in solving the steady-state, Hamilton-Jacobi-Bellman equation (107) we do not attempt to minimize a given cost functional, but rather, we parameterize a family of finite-time stabilizing controllers that minimize some derived cost functional that provides flexibility in specifying the control law. The performance integrand is shown to explicitly depend on the nonlinear system dynamics, the Lyapunov function of the closed-loop system, and the stabilizing feedback control law, wherein the coupling is introduced via the Hamilton-Jacobi-Bellman equation. Hence, by varying the parameters in the Lyapunov function and the performance integrand, the proposed framework can be used to characterize a class of globally finite-time partial-state stabilizing controllers that can meet closed-loop system response constraints.

Theorem 5.2. *Consider the controlled nonlinear affine dynamical system (99) and (100) with performance measure (102). Assume there exist a continuously differentiable function* $V:\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to\mathbb{R}$, *class* $\mathcal{K}_\infty$ *functions* $\alpha(\cdot)$ *and* $\beta(\cdot)$, *and a real number* $\theta\in(0,1)$ *such that (104)–(106) hold. Then, with the feedback control (108), the closed-loop system given by (109) and (110) is globally strongly finite-time stable with respect to* x_1 *uniformly in* x_{20} *and there exists a settling-time function* $T:\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to[0,\infty)$, *jointly continuous on* $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$, *such that (111) holds. In addition, the performance functional (102), with*

$$L_1(x_1,x_2)=\phi^{\mathrm{T}}(x_1,x_2)R_2(x_1,x_2)\phi(x_1,x_2)-V'(x_1,x_2)f(x_1,x_2), \tag{125}$$

is minimized in the sense of (80) and (112) holds.

Proof. The proof is identical to the proof of Theorem 5.1. □

The following corollary to Theorem 5.1 considers the nonautonomous dynamical system (115) with performance measure (116).

Corollary 5.2. *Consider the controlled nonlinear affine dynamical system* (115) *with performance measure* (116). *Assume there exist a continuously differentiable function* $V : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}$, *class* $\mathcal{K}_\infty$ *functions* $\alpha(\cdot)$ *and* $\beta(\cdot)$, *and a real number* $\theta \in (0, 1)$ *such that* (115)–(117) *hold. Then, with the feedback control* (121), *the closed-loop system given by* (115) *is globally strongly uniformly finite-time stable and there exists a settling-time function* $T : [0, \infty) \times \mathbb{R}^n \to [t_0, \infty)$, *jointly continuous on* $[t_0, \infty) \times \mathbb{R}^n$, *such that* (123) *holds. In addition, the performance functional* (102), *with*

$$L_1(t,x) = \phi^{\mathrm{T}}(t,x) R_2(t,x) \phi(t,x) - \frac{\partial V(t,x)}{\partial t} - \frac{\partial V(t,x)}{\partial x} f(t,x), \tag{126}$$

is minimized in the sense of (96) *and* (124) *holds.*

Proof. The proof is identical to the proof of Theorem 5.2. □

6. Illustrative numerical examples

In this section, we provide two numerical examples to highlight the optimal and inverse optimal finite-time, partial-state stabilization framework developed in the paper.

6.1. Optimal control of a symmetric spacecraft

Consider the spacecraft with two axes of symmetry [34, p. 753] given by

$$\dot{\omega}_1(t) = \alpha_1 u_1(t), \quad \omega_1(0) = \omega_{10}, \quad t \geq 0, \tag{127}$$

$$\dot{\omega}_2(t) = \alpha_1 u_2(t), \quad \omega_2(0) = \omega_{20}, \tag{128}$$

$$\dot{\omega}_3(t) = \alpha_3 u_1(t) + \alpha_4 u_2(t), \quad \omega_3(0) = \omega_{30}, \tag{129}$$

where $\omega_1 : [0, \infty) \to \mathbb{R}$, $\omega_2 : [0, \infty) \to \mathbb{R}$, and $\omega_3 : [0, \infty) \to \mathbb{R}$ denote the components of the angular velocity vector with respect to a given inertial reference frame expressed in a central body reference frame, $\alpha_1, \alpha_3, \alpha_4 \in \mathbb{R}$, $\alpha_1 \neq 0$, and u_1 and u_2 are the spacecraft control moments. For this example, we apply Theorem 5.1 to find an optimal globally partial-state stabilizing control law $u = [u_1, \ u_2]^{\mathrm{T}} = \phi(x_1, x_2)$, where $x_1 = [\omega_1, \ \omega_2]^{\mathrm{T}}$ and $x_2 = \omega_3$, such that the performance measure

$$J(x_1(0), x_2(0), u(\cdot)) = \int_0^\infty \left[\frac{4}{9} \alpha_1^2 \| x_1(t) \|^{\frac{2}{3}} + u^{\mathrm{T}}(t) u(t) \right] \mathrm{d}t \tag{130}$$

is minimized in the sense of (80), and the spacecraft is finite-time spin-stabilized about its third principle axis of inertia, that is, the dynamical system (127)–(129) is globally strongly finite-time stable with respect to x_1 uniformly in $x_2(0)$.

Note that (127)–(129) with the subquadratic performance measure (130) can be cast in the form of (99) and (100) with performance measure (102). In this case, Theorem 5.1 can be applied with $n_1=2$, $n_2=1$, $m=2$, $f(x_1,x_2)=[0,\ 0,\ 0]^{\mathrm{T}}$, $G(x_1,x_2)=\begin{bmatrix}\alpha_1 & 0 & \alpha_3\\ 0 & \alpha_1 & \alpha_4\end{bmatrix}^{\mathrm{T}}$ $L_1(x_1,x_2)=\frac{4}{9}\alpha_1^2\|x_1(t)\|^{\frac{2}{3}}$, $L_2(x_1,x_2)=0$, and $R_2(x_1,x_2)=I_m$. Specifically, in this case, (107) reduces to

$$0=L_1(x_1,x_2)-\frac{1}{4}V'(x_1,x_2)G(x_1,x_2)G^{\mathrm{T}}(x_1,x_2)V'^{\mathrm{T}}(x_1,x_2),\quad (x_1,x_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2},\tag{131}$$

which is satisfied with $V'(x_1,x_2)=\frac{4}{3}\|x_1\|^{-\frac{2}{3}}[\omega_1,\omega_2,0]^{\mathrm{T}}$. Hence, it follows from (104) that $V(x_1,x_2)=\frac{4}{9}\alpha_1^2\|x_1\|^{2/3}$. Finally, (105) reduces to

$$-\frac{1}{2}V'(x_1,x_2)G(x_1,x_2)G^{\mathrm{T}}(x_1,x_2)V'^{\mathrm{T}}(x_1,x_2)\le -k(V(x_1,x_2))^{\theta},\quad (x_1,x_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2},\tag{132}$$

which is satisfied with $k=\frac{8}{9}\alpha_1^2$ and $\theta=\frac{1}{2}$.

Since all of the conditions of Theorem 5.1 hold, it follows from (108) that the feedback control law

$$\phi(x_1,x_2)=-\tfrac{2}{3}\alpha_1\|x_1\|^{-\frac{2}{3}}x_1,\quad (x_1,x_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2},\tag{133}$$

guarantees that the dynamical system (127)–(129) is globally strongly finite-time stable with respect to x_1 uniformly in $x_2(0)$ and, for all $(x_1(0),x_2(0))\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$,

$$J(x_1(0),x_2(0),\phi(x_1(\cdot),x_2(\cdot)))=\frac{4}{9}\alpha_1^2\|x_1(0)\|^{\frac{2}{3}}.\tag{134}$$

Moreover, there exists a settling-time function $T:\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to[0,\infty)$ such that

$$T(x_1(0),x_2(0))\le\frac{9}{4\alpha_1^2}\|x_1\|^{\frac{2}{3}}.\tag{135}$$

Let $\omega_{10}=2$ Hz, $\omega_{20}=3$ Hz, $\omega_3=1$ Hz, $\alpha_1=1$, $\alpha_3=\frac{\sqrt{2}}{2}$, and $\alpha_4=-\frac{\sqrt{2}}{2}$, Fig. 1 shows the state trajectories of the controlled system versus time. Note that $x_1(t)=0$ for $t=5.2836\ \mathrm{s}<T(x_0)=5.2905$ s. Fig. 2 shows the control signal versus time. Finally, $J(x_{10},x_{20},\phi(x_1(\cdot),x_2(\cdot)))=5.5287\ \mathrm{Hz}^2$.

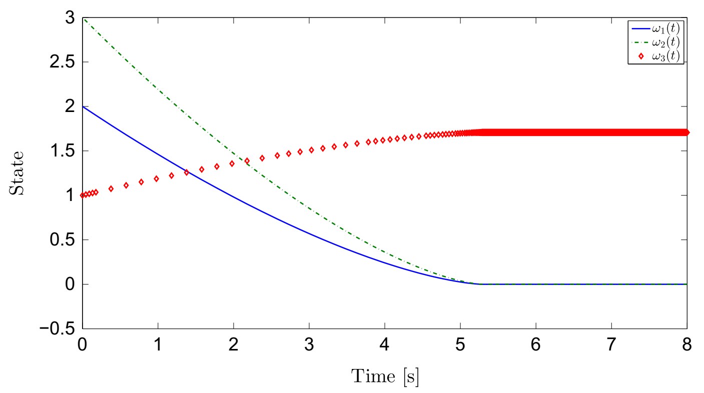

Fig. 1. Closed-loop system trajectories versus time.

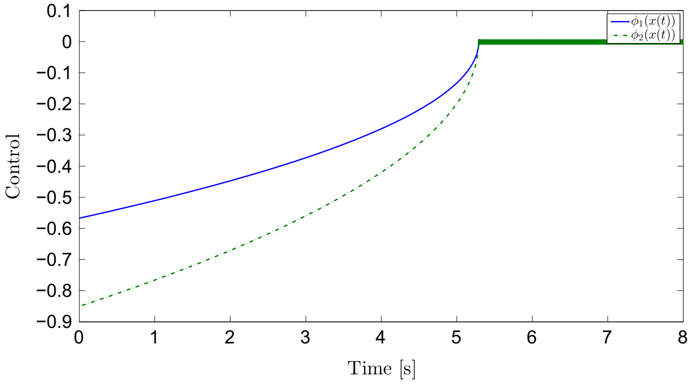

Fig. 2. Control signal versus time.

6.2. Inverse optimal control of an axisymmetric spacecraft

Consider the spacecraft with one axis of symmetry [34, p. 753] given by

$$\dot{\omega}_1(t) = I_{23}\omega_2(t)\omega_3(t) + u_1(t), \quad \omega_1(0) = \omega_{10}, \quad t \geq 0, \tag{136}$$

$$\dot{\omega}_2(t) = -I_{23}\omega_3(t)\omega_1(t) + u_2(t), \quad \omega_2(0) = \omega_{20}, \tag{137}$$

$$\dot{\omega}_3(t) = \alpha_3 u_1(t) + \alpha_4 u_2(t), \quad \omega_3(0) = \omega_{30}, \tag{138}$$

where $I_{23} \triangleq (I_2 - I_3)/I_1$, I_1, I_2, and I_3 are the principal moments of inertia of the spacecraft such that $0 < I_1 = I_2 < I_3$, $\omega_1 : [0, \infty) \to \mathbb{R}$, $\omega_2 : [0, \infty) \to \mathbb{R}$, and $\omega_3 : [0, \infty) \to \mathbb{R}$ denote the components of the angular velocity vector with respect to a given inertial reference frame expressed in a central body reference frame, α_3 and $\alpha_4 \in \mathbb{R}$, and u_1 and u_2 are the spacecraft control moments. For this example, we apply Theorem 5.2 to find an *inverse optimal* globally partial-state stabilizing control law $u = [u_1, \ u_2]^{\mathrm{T}} = \phi(x_1, x_2)$, where $x_1 = [\omega_1, \ \omega_2]^{\mathrm{T}}$ and $x_2 = \omega_3$, such that the spacecraft is finite-time spin-stabilized about its third principle axis of inertia, that is, the dynamical system (136)–(138) is globally strongly finite-time stable with respect to x_1 uniformly in $x_2(0)$. Note that (136)–(138) can be cast in the form of (99) and (100), with $n_1 = 2$, $n_2 = 1$, $m = 2$, $f(x_1, x_2) = [I_{23}\omega_2\omega_3, \ -I_{23}\omega_3\omega_1, \ 0]^{\mathrm{T}}$, and $G(x_1, x_2) = \begin{bmatrix} 1 & 0 & \alpha_3 \\ 0 & 1 & \alpha_4 \end{bmatrix}^{\mathrm{T}}$.

To construct an inverse optimal controller for (136) and (137), let $V(x_1, x_2) = p^{\frac{2}{3}}\left(x_1^{\mathrm{T}} x_1\right)^{\frac{2}{3}}$, where $p > 0$, $L(x_1, x_2, u) = L_1(x_1, x_2) + L_2(x_1, x_2)u + u^{\mathrm{T}}u$, and let $L_2(x_1, x_2) = 2[-I_{23}\omega_3\omega_2, \ I_{23}\omega_3\omega_1]$. Now, the inverse optimal control law (108) is given by

$$u = \phi(x_1, x_2) = \left[-\frac{2}{3} p^{\frac{2}{3}} \omega_1 \| x_1 \|^{-\frac{2}{3}} - I_{23}\omega_3\omega_2, \ -\frac{2}{3} p^{\frac{2}{3}} \omega_2 \| x_1 \|^{-\frac{2}{3}} + I_{23}\omega_3\omega_1 \right]^{\mathrm{T}} \tag{139}$$

and the performance functional (102), with

$$L_1(x_1, x_2) = \left(-\frac{2}{3} p^{\frac{2}{3}} \omega_1 \| x_1 \|^{-\frac{2}{3}} - I_{23}\omega_3\omega_2 \right)^2 + \left(-\frac{2}{3} p^{\frac{2}{3}} \omega_2 \| x_1 \|^{-\frac{2}{3}} + I_{23}\omega_3\omega_1 \right)^2, \tag{140}$$

is minimized in the sense of (80). Furthermore, since (104) holds with $\alpha(\|x_1\|) = \beta(\|x_1\|) = V(x_1, x_2)$ and, since

$$
\begin{aligned}
V'(x_1,x_2)\left[f(x_1,x_2)-\frac{1}{2}G(x_1,x_2)L_2^{\mathrm{T}}(x_1,x_2)-\frac{1}{2}G(x_1,x_2)G^{\mathrm{T}}(x_1,x_2)V'^{\mathrm{T}}(x_1,x_2)\right]\\
= -\frac{8}{9}p^{\frac{4}{3}}(\omega_1^2+\omega_2^2)^{\frac{1}{3}}\\
= -\frac{8}{9}p(V(x_1,x_2))^{\frac{1}{2}}, \quad (x_1,x_2)\in\mathbb{R}^2\times\mathbb{R},
\end{aligned}
\tag{141}
$$

(105) holds with $k = \frac{8}{9}p$ and $\theta = \frac{1}{2}$. Hence, with the feedback control law $\phi(x_1, x_2)$ given by (139), the closed-loop system (136) and (137) is globally finite-time stable with respect to x_1 uniformly in x_{20}. Moreover, there exists a settling-time function $T : \mathbb{R}^2 \times \mathbb{R} \to [0, \infty)$ such that

$$
T(x_{10},x_{20}) \le \frac{9}{4}p^{-\frac{2}{3}}(\omega_{10}^2+\omega_{20}^2)^{\frac{1}{3}}, \quad (x_{10},x_{20})\in\mathbb{R}^2\times\mathbb{R},
\tag{142}
$$

where $x_{10} = [\omega_{10},\ \omega_{20}]^{\mathrm{T}}$ and $x_{20} = \omega_{30}$, and

$$
J(x_{10},x_{20},\phi(x_1(\cdot),x_2(\cdot))) = p^{\frac{2}{3}}(\omega_{10}^2+\omega_{20}^2)^{\frac{2}{3}}, \quad (x_{10},x_{20})\in\mathbb{R}^2\times\mathbb{R}.
\tag{143}
$$

Let $I_1 = I_2 = 4\ \mathrm{kg\cdot m^2}$, $I_3 = 20\ \mathrm{kg\cdot m^2}$, $\omega_{10} = -2$ Hz, $\omega_{20} = 2$ Hz, $\omega_3 = 1$ Hz, $\alpha_3 = \frac{\sqrt{2}}{2}$, $\alpha_4 = -\frac{\sqrt{2}}{2}$, and $p = 1$, Fig. 3 shows the state trajectories of the controlled system versus time. Note that $x_1(t) = 0$ for $t = 4.4943\ \mathrm{s} < T(x_0) = \frac{9}{2}$ s. Fig. 4 shows the control signal versus time. Finally, $J(x_{10}, x_{20}, \phi(x_1(\cdot), x_2(\cdot))) = 4\ \mathrm{Hz}^2$.

7. Conclusion

In this paper, an optimal control problem for finite-time, partial-state stabilization is stated and sufficient conditions are derived to characterize an optimal nonlinear feedback controller that guarantees finite-time stability of part of the closed-loop system state. Specifically, we utilized a steady-state Hamilton-Jacobi-Bellman framework to characterize optimal nonlinear feedback controllers with a notion of optimality that is directly related to a given Lyapunov function that is positive definite and decrescent with respect to part of the system state, and satisfies a differential inequality involving fractional powers. This result was then used to develop optimal finite-time

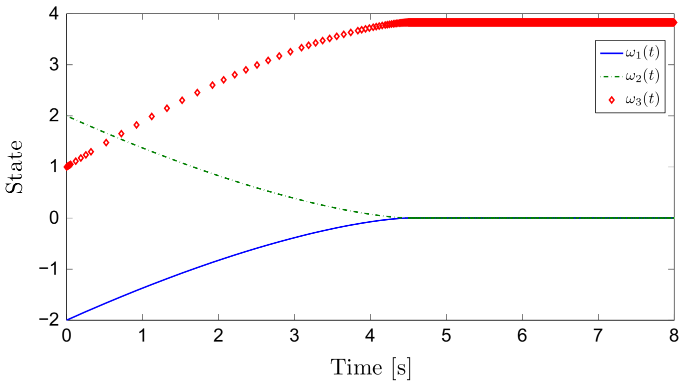

Fig. 3. Closed-loop system trajectories versus time.

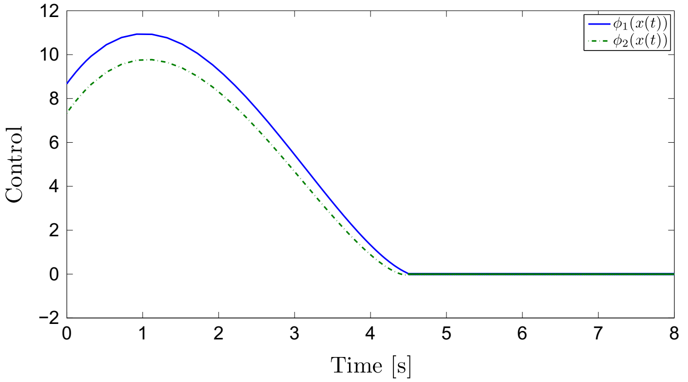

Fig. 4. Control signal versus time.

stabilizing controllers for nonlinear time-varying systems. In addition, we developed inverse optimal feedback controllers for affine nonlinear systems and time-varying systems.

Further extensions of this framework will focus on partial-state semistabilization involving controlled nonlinear systems with a continuum of equilibria for addressing finite-time optimal consensus protocols for multiagent systems. Furthermore, since there exist finite-time stable dynamical systems that do not admit a continuously differentiable Lyapunov function that satisfies the hypothesis of Theorem 3.1 (see, [4,30,35,32]), and hence, Theorems 4.2 and 5.1, a particularly important extension is the consideration of continuous Lyapunov functions leading to viscosity solutions [36] or, equivalently, a proximal analysis formalism [37], of the resulting Hamilton-Jacobi-Bellman equations arising in Theorems 4.2 and 5.1.

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