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Abnormal Optimal Trajectory Planning of Multi-Body Systems in the Presence of Holonomic and Nonholonomic Constraints

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Abstract In optimal control problems, the Hamiltonian function is given by the weighted sum of the integrand of the cost function and the dynamic equation. The coefficient multiplying the integrand of the cost function is either zero or one; and if this coefficient is zero, then the optimal control problem is known as *abnormal*; otherwise it is *normal*. This paper provides a characterization of the abnormal optimal control problem for multi-body mechanical systems, subject to external forces and moments, and holonomic and nonholonomic constraints. This study does not only account for first-order necessary conditions, such as Pontryagin's principle, but also for higher-order conditions, which allow the analysis of singular optimal controls.

Keywords Optimal trajectory planning · Singular controls · Pontryagin's principle · Normal and abnormal optimal control

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1 Introduction

Industrial and medical applications require extensive use of robotic manipulators [43]. Usually, the manipulators' links are not fully actuated and are constrained to operate within certain boundaries imposed by their workplace environment. Motivated by these applications, in this paper we address the optimal trajectory planning problem for underactuated systems of N rigid bodies, subject to holonomic and nonholonomic constraints; that is, constraints on the system's configuration and the translational and rotational velocities.

Consider a nonlinear differential equation capturing the controlled equations of motion of a system of N rigid bodies and a performance measure in integral form. Necessary conditions for the controls to minimize the system's performance measure are that the variations with respect to the control of the Hamiltonian function are equal to zero along the optimal trajectory; well-known formulations of this first-order variational principle are Pontryagin's principle [44] and the Euler-Lagrange necessary conditions of optimality [21].

The Hamiltonian function is the weighted sum of the integrand of the cost function and the dynamic equation, and it can be shown that the coefficient multiplying the integrand of the cost function is either equal to zero or one [13]. If the coefficient is zero, then the optimal control problem is known as *abnormal*;

otherwise it is *normal*. Notable studies on the abnormal optimal control problem are given in [1, 26, 27, 37, 41].

In several cases, first-order variational principles are satisfied by any admissible control, and hence, do not provide any useful information about the optimal controls. These optimal control problems are known as *singular* and are typical, for example, of affine in the control systems; the notions of normality and singularity are not necessarily related. Time-optimal control problems involving mechanical systems, such as multi-link robotic manipulators, which are affine in the control dynamical systems, are characterized by singular controls [50, 51].

In the first part of the paper, we provide a set of differential equations, which capture the dynamics of a system of N rigid bodies, subject to external forces and moments, and holonomic and nonholonomic constraints. One of the key contributions of this paper is that we provide a *minimal* set of equations of motion for a system of rigid bodies, that is, the system dynamics cannot be uniquely expressed by a smaller set of differential equations, and hence our approach is advantageous for numerical applications. Moreover, we derive the equations of motion using the kinetic energy of the *constrained* dynamical system, that is, we account for the kinetic energy of the system subject to the nonholonomic constraints; this approach is simpler than other approaches in the literature, such as the one by Boltzmann-Hamel [24, Ch.5], wherein the kinetic energy of the unconstrained system is considered. Furthermore, our formulation accounts for the case where the nonholonomic constraints result in the dynamical system to be underactuated. In the second part of the paper, we provide a solution to the abnormal optimal control problem for a mechanical system by accounting for both singular and nonsingular controls along optimal trajectories.

None of the work currently available in the literature addresses the singular control problem for systems of N rigid bodies subject to external forces and moments, and holonomic and nonholonomic constraints. The control effort minimization problem for formations of N vehicles has been addressed in [33], where the authors do not consider the singular control problem and provide necessary conditions for the existence of a maximum under the simplifying assumptions that the system constraints are holonomic. Moreover, the authors addressed the fuel-optimal control problem in [34], without providing any of the details, proofs, and technical discussions presented herein.

Optimal trajectory planning problems for mechanical systems usually do not allow analytical solutions and one needs to resort to numerical optimization tools. However, numerical approaches to the optimal control problem are based on first-order necessary conditions and are not specifically designed to find candidate optimal singular controls [5, 45, 46, 48]. To overcome this limitation, the theoretical framework developed in this paper can be applied as follows. Firstly, a numerical solution of the control effort minimization problem is attempted without prior knowledge of the existence of singular controls. Then, the necessary conditions proven in this paper are applied to verify the validity of numerical results and identify singular controls and identify singular arcs. Finally, as recommended in [45], candidate optimal controls are computed along singular arcs using only a lower error tolerance. The application of the necessary conditions proven in this paper is straightforward, since one needs to solve three scalar nonlinear differential equations and check the positive definiteness of a three-by-three matrix function.

Kelley [29] and Kopp and Moyer [31] were the first to systematically address the singular control problem and obtained second-order necessary conditions for the existence of weak local minima that were verified using new control variations along singular arcs, namely, the first and the second derivatives of the Dirac δ-function. A more general result that combines both Kelley's and Kopp and Moyer's necessary conditions is known as the generalized Legendre-Clebsch condition or Kelley-Contensou test [30] and can be obtained either by approximating higher derivatives of the δ-function or by applying the method introduced in [20], which avoids using an explicit form of the control variation. It is also important to mention the work of Robbins [47], Goh [22], and Speyer [52], who further extended Kelley's condition to multi-input controls. Third-order necessary conditions for optimality of singular controls have been addressed in [32] and, more recently, in [36].

Among the classical studies on singular arcs that are not based on Kelley's approach, Jacobson's necessary condition [25] and the method of state transformation [53] are worth noting. If we interpret the path of a dynamical system as a flow on a differentiable manifold, then the governing differential equation is a vector field on that manifold and the directions in which the system moves on the manifold are given by the Lie brackets of the vector field [4, 14, 38]. This geometric approach has been particularly successful to address the singular control problem [11, 16, 54]. For example, the authors in [17] characterized singular controls in the time optimal control problem for affine systems applying Lie algebra, Delgado-Tellez [18] proved the existence and uniqueness of solutions for singular optimal control problems using the geometric recursive constraints algorithm, Bonnard [12] related singular trajectories for autonomous nonlinear systems to the solutions of Pontryagin's principle for the time optimal control problem by exploiting the properties of sub-Riemannian geometry, and the authors in [15] prove that given a system, which is affine in the control and satisfies the Lie algebra rank condition, is characterized by strictly abnormal singular trajectories. Finally, it is important to mention that McDannel and Powers [40] provide necessary conditions for characterizing the junction between singular and non-singular arcs. A further discussion on these topics can be found in [6, 19, 23].

2 Notation and Definitions

The mathematical notation used in this paper is fairly standard. In particular, we write $\mathbb{N}$ for the set of positive integers, $\mathbb{R}$ for the set of real numbers, $\overline{\mathbb{R}}_+$ for the set of nonnegative real numbers, $\mathbb{R}^n$ for the set of $n \times 1$ column vectors on the field of real numbers, $\mathbb{R}^{n\times m}$ for the set of real $n \times m$ matrices, and $\mathrm{int}(\mathcal{A})$ for the interior of the set $\mathcal{A} \subset \mathbb{R}^{n\times m}$. Given $t \in \mathbb{R}$, $\mathcal{I}(t)$ denotes a neighborhood of t. If $f : (t_1, t_2) \to \mathbb{R}^n$ is continuous with its first p derivatives, then $f(\cdot) \in C^p(t_1, t_2)$. The zero vector in $\mathbb{R}^n$ is denoted by 0_n or 0, the zero matrix in $\mathbb{R}^{n\times m}$ is denoted by $0_{n\times m}$ or 0, and the identity matrix in $\mathbb{R}^{n\times n}$ is denoted by I_n or I. Given $x \in \mathbb{R}^3$ with $x \triangleq [x_1,\ x_2,\ x_3]^{\mathrm{T}}$, define

$$x^{\times} \triangleq \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}.$$

The nullspace of $A \in \mathbb{R}^{n\times m}$ is denoted by $\mathcal{N}(A)$, the transpose of A is denoted by A^{T}, and the inverse transpose of $B \in \mathbb{R}^{n\times n}$ is denoted by $B^{-\mathrm{T}}$. The matrix B is nonnegative (respectively, positive) definite, that is, $B \geq 0_{n\times n}$ (respectively, $B > 0_{n\times n}$), if $B = B^{\mathrm{T}}$ and the eigenvalues of B are nonnegative (respectively, positive.) Given $N \triangleq \begin{bmatrix} 0_{m\times m} & 0_{m\times(n-m)} \\ 0_{(n-m)\times m} & I_{n-m} \end{bmatrix}$ and $C \in \mathbb{R}^{l\times n}$, the map $\mathcal{O} : \mathbb{R}^{n\times n} \times \mathbb{R}^{l\times n} \to \mathbb{R}^{l\times(n-m)}$ is such that if $D = \mathcal{O}(N, C)$, then $CN = [0_{l\times m} D]$. For brevity, we write $D = \mathcal{O}N(C)$ for $D = \mathcal{O}(N, C)$. Moreover, if $l = n$, then we denote $(\mathcal{O}N(C^{\mathrm{T}}))^{\mathrm{T}}$ by $\mathcal{O}^{\mathrm{T}}N(C)$ and $\mathcal{O}N((\mathcal{O}N(C))^{\mathrm{T}})$ by $\mathcal{O}N(C)\mathcal{O}$.

Time is the only independent variable used in this paper and is denoted by t, and we let $t \in [t_1, t_2] \subset \mathbb{R}$, where t_1 is fixed and assigned a priori, and t_2 is found by solving the optimization problem. Given a system of N rigid bodies, the components of $q : [t_1, t_2] \to \mathbb{R}^{\gamma}$ are the γ *independent generalized coordinates*, which uniquely identify the system configuration at every $t \in [t_1, t_2]$ and account for holonomic constraints [24, Ch. 1], [33]. Specifically, the *position vector* of the center of mass of the αth rigid body, $\alpha = 1, \ldots, N$, in a given inertial reference frame is denoted by $r_\alpha : \mathbb{R}^{\gamma} \to \mathbb{R}^3$, the *attitude vector* of the αth rigid body in the modified Rodrigues parameters [49] is denoted by $\sigma_\alpha : \mathbb{R}^{\gamma} \to \mathbb{R}^3$, the *state vector* of the αth rigid body is denoted by $x_\alpha \triangleq [r_\alpha^{\mathrm{T}}, \sigma_\alpha^{\mathrm{T}}]^{\mathrm{T}}$, and the *system's configuration* at time t is given by $[x_1^{\mathrm{T}}(q(t)), \ldots, x_N^{\mathrm{T}}(q(t))]^{\mathrm{T}}$. The mapping $x_\alpha(q(t))$, $t \in [t_1, t_2]$, is the *trajectory* of the αth rigid body and the mapping $x_\alpha(q(t))$, $t \in [\hat{t}_1, \hat{t}_2] \subseteq [t_1, t_2]$, is the *arc* of the αth body between $\hat{t}_1$ and $\hat{t}_2$.

The advantage of using independent generalized coordinates is that $x_1(q), \ldots, x_N(q)$ always account for algebraic equality and inequality constraints on the system's configuration, which are known as *holonomic constraints* [24, 33]. Given the scope of the present paper, any set of parameters for the attitude representation, such as quaternions or Euler angles, are also suitable. However, modified Rodrigues parameters are commonly employed in multi-agent formations path planning because they guarantee the fastest computational time in numerical optimization algorithms [2].

The components of

$$q_{\mathrm{dot}}(t) \triangleq D(q(t))\dot{q}(t) + d(q(t)), \qquad t \in [t_1, t_2], \quad (1)$$

are the *quasi-velocities*, where $D : \mathbb{R}^{\gamma} \to \mathbb{R}^{\gamma\times\gamma}$ is invertible and continuously differentiable, and $d : \mathbb{R}^{\gamma} \to \mathbb{R}^{\gamma}$ is continuously differentiable. Detailed discussions on quasi-velocities can be found in [10, 24]. The vector $v_\alpha(q, q_{\mathrm{dot}}) \triangleq \dot{r}_\alpha(q)$, $\alpha = 1, \ldots, N$, denotes the *velocity* of the center of mass of the αth rigid body and $\omega_\alpha(q, q_{\rm dot}) \triangleq R_{\rm rod}^{-1}(\sigma_\alpha)\dot{\sigma}_\alpha(q)$ denotes the *angular velocity* of the αth rigid body in a principal body reference frame, where $R_{\rm rod}(\sigma_\alpha) \triangleq \frac{1}{4}(1 - \sigma_\alpha^{\rm T}\sigma_\alpha)I_3 + \frac{1}{2}\sigma_\alpha^\times + \frac{1}{2}\sigma_\alpha\sigma_\alpha^{\rm T}$ [49]. Finally, the *augmented state vector* of the αth rigid body is denoted by $\tilde{x}_\alpha \triangleq [r_\alpha^{\rm T}, v_\alpha^{\rm T}, \sigma_\alpha^{\rm T}, \omega_\alpha^{\rm T}]^{\rm T}, \alpha = 1, \ldots, N$.

For a given set of real constants $\rho_{\alpha,1}, \rho_{\alpha,2}, \rho_{\alpha,3}$, and $\rho_{\alpha,4}$, $\alpha = 1, \ldots, N$, such that $0 \le \rho_{\alpha,1} < \rho_{\alpha,2}$ and $0 \le \rho_{\alpha,3} < \rho_{\alpha,4}$, define

$$\mathcal{G}_{\alpha,\rm tran} \triangleq \left\{z \in \mathbb{R}^3 : \rho_{\alpha,1} \le \|z\|_2 \le \rho_{\alpha,2}\right\} \cup \{0_3\},$$
$$\mathcal{G}_{\alpha,\rm rot} \triangleq \left\{z \in \mathbb{R}^3 : \rho_{\alpha,3} \le \|z\|_2 \le \rho_{\alpha,4}\right\} \cup \{0_3\},$$

and let $u_{\alpha,\rm tran} : [t_1, t_2] \to \mathcal{G}_{\alpha,\rm tran}$ (respectively, $u_{\alpha,\rm rot} : [t_1, t_2] \to \mathcal{G}_{\alpha,\rm rot}$) be the *force* (respectively, *moment*) applied by the control system of the αth rigid body. The vector $u_{\alpha,\rm tran}$ (respectively, $u_{\alpha,\rm rot}$) is also referred to as the *αth translational control vector* (respectively, the *rotational control vector*.) The following definition is needed.

Definition 1 ([44]) If $u_{\alpha,\rm tran} : [t_1, t_2] \to \mathcal{G}_{\alpha,\rm tran}$ (respectively, $u_{\alpha,\rm rot} : [t_1, t_2] \to \mathcal{G}_{\alpha,\rm rot}$), $\alpha = 1, \ldots, N$, is such that *i)* $u_{\alpha,\rm tran}(\cdot)$ (respectively, $u_{\alpha,\rm rot}(\cdot)$) is continuous at the endpoints of $[t_1, t_2]$, *ii)* $u_{\alpha,\rm tran}(\cdot)$ (respectively, $u_{\alpha,\rm rot}(\cdot)$) is continuous for all $t \in (t_1, t_2)$ with the exception of a finite number of times t at which $u_{\alpha,\rm tran}(t)$ may have discontinuities of the first kind, and *iii)* $u_{\alpha,\rm tran}(\tau) = \lim_{t\to\tau^-} u_{\alpha,\rm tran}(t)$ (respectively, $u_{\alpha,\rm rot}(\tau) = \lim_{t\to\tau^-} u_{\alpha,\rm rot}(t)$), where $\tau \in [t_1, t_2]$ is a point of discontinuity of first kind for $u_{\alpha,\rm tran}(t)$ (respectively, $u_{\alpha,\rm rot}(\cdot)$), then $u_{\alpha,\rm tran}(\cdot)$ (respectively, $u_{\alpha,\rm rot}(\cdot)$) is an *admissible control* in $\mathcal{G}_{\alpha,\rm tran}$ (respectively, $\mathcal{G}_{\alpha,\rm rot}$.)

Finally, define $\tilde{u} \triangleq [u_1^{\rm T}, \ldots, u_N^{\rm T}]^{\rm T}$, where $u_\alpha \triangleq [{\rm c}u_{\alpha,\rm tran}^{\rm T}, u_{\alpha,\rm rot}^{\rm T}]^{\rm T}$, $\alpha = 1, \ldots, N$, and c $= 1$ with units of distance.

3 A Minimal Set of Equations of Motion for a Constrained System of Rigid Bodies

In this section, we provide a minimal set of differential equations that uniquely captures the dynamics of a system of N six-degrees-of-freedom rigid bodies subject to external forces and moments, and holonomic and nonholonomic constraints. The kinetic energy of a system of N rigid bodies, whose reference points are centered in the bodies' centers of mass, is given by *König's theorem* [24] and for our problem takes the form

$$T(q, q_{\rm dot}) = \frac{1}{2}\sum_{\alpha=1}^{N} m_\alpha v_\alpha^{\rm T}(q, q_{\rm dot})v_\alpha(q, q_{\rm dot}) + \frac{1}{2}\sum_{\alpha=1}^{N} \omega_\alpha^{\rm T}(q, q_{\rm dot})I_{{\rm in},\alpha}\omega_\alpha(q, q_{\rm dot}), \quad (2)$$

where $m_\alpha \in \mathbb{R}$ and $I_{{\rm in},\alpha} \in \mathbb{R}^3$ are the mass and inertia matrix of the αth rigid body, respectively, which are assumed constant. The system's dynamic equations can be written as [24]

$$\frac{\rm d}{{\rm d}t}\frac{\partial T(q, q_{\rm dot})}{\partial q_{\rm dot}} = \sum_{\alpha=1}^{N} m_\alpha v_\alpha^{\rm T}(q, q_{\rm dot})\frac{\rm d}{{\rm d}t}\frac{\partial v_\alpha(q, q_{\rm dot})}{\partial q_{\rm dot}} + \sum_{\alpha=1}^{N} \omega_\alpha^{\rm T}(q, q_{\rm dot})I_{{\rm in},\alpha}\frac{\rm d}{{\rm d}t}\frac{\partial \omega_\alpha(q, q_{\rm dot})}{\partial q_{\rm dot}} + \sum_{\alpha=1}^{N}(a(\tilde{x}_\alpha) + u_{\alpha,\rm tran})^{\rm T}\frac{\partial v_\alpha(q, q_{\rm dot})}{\partial q_{\rm dot}} + \sum_{\alpha=1}^{N}(m(\tilde{x}_\alpha) + u_{\alpha,rot})^{\rm T}\frac{\partial \omega_\alpha(q, q_{\rm dot})}{\partial q_{\rm dot}}, \quad (3)$$

where $a : \mathbb{R}^{12} \to \mathbb{R}^3$ and $m : \mathbb{R}^{12} \to \mathbb{R}^3$ are continuously differentiable and denote the *external forces* and *moments* acting on the rigid body, respectively. Since $a(\cdot)$ and $m(\cdot)$ are functions of the augmented state vector, the external forces and moments acting on the αth vehicle, $\alpha = 1, \ldots, N$, are functions of the body's translational and angular positions and velocities. The boundary conditions for Eq. 3 are given by Eqs. 13 and 14 below.

In most applications, *nonholonomic constraints* can be expressed as

$$\tilde{N}_{\rm nh}q_{\rm dot}(t) = 0_\gamma, \qquad t \in [t_1, t_2], \quad (4)$$

where $\tilde{N}_{\rm nh} \triangleq \begin{bmatrix} I_\zeta & 0_{\zeta\times(\gamma-\zeta)} \\ 0_{(\gamma-\zeta)\times\zeta} & 0_{(\gamma-\zeta)\times(\gamma-\zeta)} \end{bmatrix}$. In this case, the first ζ scalar differential equations in Eq. 3

are identically equal to zero [28, Ch. 2]. Nonlinear nonholonomic constraints are not needed to address mechanical problems [8] and linear nonholonomic constraints can always be reduced to the form Eq. 4 by properly choosing $D(\cdot)$ in Eq. 1 [28, Ch. 2].

The following result gives the system's equations of motion in explicit form.

Theorem 1 *Consider a system of N rigid bodies with dynamic (3). If, for all $q \in \mathbb{R}^{\gamma}$, $Z(q) > 0_{(\gamma-\zeta)\times(\gamma-\zeta)}$, where*

$$Z^{-1}(q) \triangleq \mathcal{O} N_{\rm nh} \left(D^{-\rm T}(q) \left[\sum_{\alpha=1}^{N} \left(\frac{\partial \sigma_\alpha(q)}{\partial q} \right)^{\rm T} R_{\rm rod}^{-\rm T}(\sigma_\alpha) {\rm I}_{{\rm in},\alpha} R_{\rm rod}^{-1}(\sigma_\alpha) \frac{\partial \sigma_\alpha(q)}{\partial q} + \sum_{\alpha=1}^{N} m_\alpha \left(\frac{\partial r_\alpha(q)}{\partial q} \right)^{\rm T} \frac{\partial r_\alpha(q)}{\partial q} \right] D^{-1}(q) \right) \mathcal{O},$$

then the system's equations of motion are given by the set of $2\gamma - \zeta$ scalar differential equations

$$\begin{bmatrix} \dot{q}(t) \\ \mathcal{O}^{\rm T} N_{\rm nh}(\dot{q}_{\rm dot}(t)) \end{bmatrix} = \begin{bmatrix} D^{-1}(q(t)) N_{\rm nh} [q_{dot}(t) - d(q(t))] \\ f_{\rm dyn}(q(t), q_{dot}(t)) \end{bmatrix} + \begin{bmatrix} 0_{\gamma \times 6N} \\ M(q(t)) \end{bmatrix} \tilde{u}(t), \quad t \in [t_1, t_2], \tag{5}$$

where

$$M(q) \triangleq Z(q) \mathcal{O}^{\rm T} N_{\rm nh} \left[\left(\frac{\partial v_1(q, q_{\rm dot})}{\partial q_{\rm dot}} \right)^{\rm T}, \left(\frac{\partial \omega_1(q, q_{\rm dot})}{\partial q_{\rm dot}} \right)^{\rm T}, \ldots, \left(\frac{\partial \omega_N(q, q_{\rm dot})}{\partial q_{\rm dot}} \right)^{\rm T} \right]$$
$$= Z(q) \mathcal{O}^{\rm T} N_{\rm nh} \left(D^{-\rm T}(q) \left[\left(\frac{\partial r_1(q)}{\partial q} \right)^{\rm T}, \left(R_{\rm rod}^{-1}(\sigma_1) \frac{\partial \sigma_1(q)}{\partial q} \right)^{\rm T}, \ldots, \left(\frac{\partial r_N(q)}{\partial q} \right)^{\rm T}, \left(R_{\rm rod}^{-1}(\sigma_N) \frac{\partial \sigma_N(q)}{\partial q} \right)^{\rm T} \right] \right), \tag{6}$$

$$f_{dyn}(q, q_{dot}) \triangleq Z(q) \mathcal{O}^{\rm T} N_{nh} \left[\sum_{\alpha=1}^{N} m_\alpha \left(\frac{\rm d}{{\rm d}t} \frac{\partial v_\alpha(q, q_{\rm dot})}{\partial q_{\rm dot}} \right)^{\rm T} v_\alpha(q, q_{\rm dot}) + \sum_{\alpha=1}^{N} \left(\frac{\partial v_\alpha(q, q_{\rm dot})}{\partial q_{\rm dot}} \right)^{\rm T} a(\tilde{x}_\alpha) + \sum_{\alpha=1}^{N} \left(\frac{\rm d}{{\rm d}t} \frac{\partial \omega_\alpha(q, q_{\rm dot})}{\partial q_{\rm dot}} \right)^{\rm T} I_{{\rm in},\alpha} \omega_\alpha(q, q_{\rm dot}) + \sum_{\alpha=1}^{N} \left(\frac{\partial \omega_\alpha(q, q_{\rm dot})}{\partial q_{\rm dot}} \right)^{\rm T} m(\tilde{x}_\alpha) - \frac{\partial^2 T(q, q_{\rm dot})}{\partial q \, \partial q_{\rm dot}} \dot{q} \right], \tag{7}$$

and

$$N_{\rm nh} \triangleq I_\gamma - \tilde{N}_{\rm nh}. \quad (8)$$

Proof The first γ scalar equations in Eq. 5 are given by Eqs. 1 and 4. Next, since $\frac{\rm d}{{\rm d}t}\frac{\partial T(q,q_{\rm dot})}{\partial q_{\rm dot}} = \dot{q}_{\rm dot}^{\rm T}\frac{\partial^2 T(q,q_{\rm dot})}{\partial q_{\rm dot}^2} + \dot{q}^{\rm T}\left(\frac{\partial^2 T(q,q_{\rm dot})}{\partial q\,\partial q_{\rm dot}}\right)^{\rm T}$, (3) is equivalent to

$$\begin{aligned}\dot{q}_{\rm dot}^{\rm T}\frac{\partial^2 T(q,q_{\rm dot})}{\partial q_{\rm dot}^2} &= \sum_{\alpha=1}^{N} m_\alpha v_\alpha^{\rm T}(q,q_{\rm dot})\frac{\rm d}{{\rm d}t}\frac{\partial v_\alpha(q,q_{\rm dot})}{\partial q_{\rm dot}}\\ &+\sum_{\alpha=1}^{N}\left(a(\tilde{x}_\alpha)+u_{\alpha,\rm tran}\right)^{\rm T}\frac{\partial v_\alpha(q,q_{\rm dot})}{\partial q_{\rm dot}}\\ &+\sum_{\alpha=1}^{N}\omega_\alpha^{\rm T}(q,q_{\rm dot})I_{{\rm in},\alpha}\frac{\rm d}{{\rm d}t}\frac{\partial \omega_\alpha(q,q_{\rm dot})}{\partial q_{\rm dot}}\\ &+\sum_{\alpha=1}^{N}\left(m(\tilde{x}_\alpha)+u_{\alpha,rot}\right)^{\rm T}\frac{\partial \omega_\alpha(q,q_{\rm dot})}{\partial q_{\rm dot}}\\ &-\dot{q}^{\rm T}\left(\frac{\partial^2 T(q,q_{\rm dot})}{\partial q\,\partial q_{\rm dot}}\right)^{\rm T}. \end{aligned} \quad (9)$$

Next, note that $\frac{\partial v_\alpha(q,q_{\rm dot})}{\partial \dot{q}} = \frac{\partial r_\alpha(q)}{\partial q}$ and $\frac{\partial \omega_\alpha(q,q_{\rm dot})}{\partial \dot{q}} = R_{\rm rod}^{-1}(\sigma_\alpha)\frac{\partial \sigma_\alpha(q)}{\partial q}$, $\alpha \in \{1,\ldots,N\}$. Thus, $\frac{\partial v_\alpha(q,q_{\rm dot})}{\partial q_{\rm dot}} = \frac{\partial r_\alpha(q)}{\partial q}D^{-1}(q)N_{\rm nh}$, $\alpha \in \{1,\ldots,N\}$, $\frac{\partial \omega_\alpha(q,q_{\rm dot})}{\partial q_{\rm dot}} = R_{\rm rod}^{-1}(\sigma_\alpha)\frac{\partial \sigma_\alpha(q)}{\partial q}D^{-1}(q)N_{\rm nh}$,

$$\begin{aligned}\frac{\partial T(q,q_{\rm dot})}{\partial q_{\rm dot}} &= \left(\sum_{\alpha=1}^{N} m_\alpha v_\alpha^{\rm T}\frac{\partial r_\alpha(q)}{\partial q}\right)D^{-1}(q)N_{\rm nh}\\ &+\left(\sum_{\alpha=1}^{N}\omega_\alpha^{\rm T}I_{{\rm in},\alpha}R_{\rm rod}^{-1}(\sigma_\alpha)\frac{\partial\sigma_\alpha(q)}{\partial q}\right)D^{-1}(q)N_{\rm nh},\end{aligned} \quad (10)$$

and

$$\begin{aligned}\frac{\partial^2 T(q,q_{\rm dot})}{\partial q_{\rm dot}^2} &= N_{\rm nh}D^{-\rm T}(q)\left[\sum_{\alpha=1}^{N}\left(\frac{\partial\sigma_\alpha(q)}{\partial q}\right)^{\rm T}R_{\rm rod}^{-\rm T}(\sigma_\alpha)I_{{\rm in},\alpha}R_{\rm rod}^{-1}(\sigma_\alpha)\frac{\partial\sigma_\alpha(q)}{\partial q}\right.\\ &\left.+\sum_{\alpha=1}^{N}m_\alpha\left(\frac{\partial r_\alpha(q)}{\partial q}\right)^{\rm T}\frac{\partial r_\alpha(q)}{\partial q}\right]D^{-1}(q)N_{\rm nh},\end{aligned} \quad (11)$$

which is a nonnegative definite block matrix in $\mathbb{R}^{\gamma\times\gamma}$. Since three of the four blocks of $\frac{\partial^2 T(q,q_{\rm dot})}{\partial q_{\rm dot}^2}$ are zero matrices and the nonzero block is $Z^{-1}(q)$, (5) follows from (9), (11), and the definition of $\mathcal{O}^{\rm T}N_{\rm nh}(\dot{q}_{\rm dot})$. □

The boundary conditions of (5) are given by (13) and (14) below. It is important to remark that (3), and hence (5), are deduced from d'Alembert's principle and not using the classical Euler-Lagrange formulation [24, Ch. 2].

It follows from Theorem 1 that if the matrix $M(q)$, $q \in \mathbb{R}^\gamma$, is not full-rank, then the dynamical system (5) is underactuated. Moreover, if $Z(q) \geq 0_{(\gamma-\zeta)\times(\gamma-\zeta)}$, $q \in \mathbb{R}^\gamma$, then $\mathcal{N}(Z(q))$ is not trivial and describes the equilibrium manifold of the systems of rigid bodies. Finally, note that neither $Z(q)$, $q \in \mathbb{R}^\gamma$, nor $M(q)$ explicitly depend on the quasi-velocities vector $q_{\rm dot}$.

Remark 1 Theorem 1 provides sufficient conditions for writing the equations of motion for a system of N rigid bodies as an explicit first-order nonlinear differential equation that is affine in the control $\tilde{u}(\cdot)$. In addition, Theorem 1 allows us to compute the kinetic energy $T(\cdot,\cdot)$ for the constrained dynamical system, that is, accounting for the nonholonomic constraints (4). Finally, (5) represents a systems of $2\gamma-\zeta$ first-order differential equations, and hence, comprises a *minimal* set of equations of motion for a system of N rigid bodies [24, Ch. 4].

The dynamic equations for a system of N rigid bodies can be alternatively derived using the classical Euler-Lagrange equation, the Maggi equation, or the Boltzmann-Hamel equation. In order to apply the Euler-Lagrange equation, the kinetic energy must be computed as a function of q and $\dot{q}$ and a costate vector must be introduced to account for Eq. 1 and the nonholonomic constraints (4). Therefore, in this case, the system's equations of motion are given by a set of $2\gamma + \zeta$ first order differential equations [42, Ch. 3]. The Maggi equation modifies the Euler-Lagrange equation as to not introduce a costate vector, and hence, reduces the equations of motion to 2γ ordinary differential equations. Finally, the Boltzmann-Hamel equation further reduces the number of equations of motion to $2\gamma - \zeta$ by writing the kinetic energy $T(\cdot,\cdot)$ as a function of q and $q_{\rm dot}$. However, the Boltzmann-Hamel equation holds if the kinetic energy $T(\cdot,\cdot)$ is computed for the unconstrained systems, that is, not accounting for the nonholonomic constraints (4). Consequently, (5) is more efficient than the Boltzmann-Hamel equation [24, Ch. 4].

It is worth noting that if nonholonomic constraints are in Pfaffian differential form [24, Ch. 1], then the Euler-Lagrange equation, the Maggi equation, and the Boltzmann-Hamel equation reduce to a set of $2\gamma - \zeta$ differential equations. Although quasi-velocities are generally chosen as simple as possible, a judicious choice of $q_{\rm dot}$, that is, of $D(\cdot)$ in Eq. 1, can simplify the expression of $M(\cdot)$ in Eq. 5 and, consequently, further reduce the number of differential equations needed to impose first-order necessary conditions for optimality [39].

4 Statement of the Optimal Control Problem

The optimal control problem considered in this paper can be stated as follows. Let $\tilde{x}_{\alpha,1} \in \mathbb{R}^{12}$, $\alpha = 1,\dots,N$, be given and consider the continuously differentiable map $s_2 : \mathbb{R}^{12N} \to \mathbb{R}^{\zeta_2}$. For all $\alpha = 1,\dots,N$, find the translational controls $u_{\alpha,\rm tran}(\cdot)$ and the rotational controls $u_{\alpha,\rm rot}(\cdot)$ among all admissible controls in $\mathcal{G}_{\alpha,\rm tran}$ and $\mathcal{G}_{\alpha,\rm rot}$, respectively, such that the performance measure

$$J[\tilde{u}(\cdot)] \triangleq \int_{t_1}^{t_2} L(q(t), q_{\rm dot}(t), \tilde{u}(t))\,{\rm d}t, \tag{12}$$

where $L : \mathbb{R}^{\gamma} \times \mathbb{R}^{\gamma-\zeta} \times \mathbb{R}^{6N} \to \mathbb{R}$, is minimized and the system's equations of motion (5) hold with boundary conditions

$$[\tilde{x}_1^{\rm T}(q(t_1), q_{\rm dot}(t_1)),\ \dots,\ \tilde{x}_N^{\rm T}(q(t_1), q_{\rm dot}(t_1))]^{\rm T} = [\tilde{x}_{1,1}^{\rm T},\ \dots,\ \tilde{x}_{N,1}^{\rm T}]^{\rm T}, \tag{13}$$

$$s_2([\tilde{x}_1^{\rm T}(q(t_2), q_{\rm dot}(t_2)),\ \dots,\ \tilde{x}_N^{\rm T}(q(t_2), q_{\rm dot}(t_2))]^{\rm T}) = 0_{\zeta_2}. \tag{14}$$

In this paper, we assume that there exists at least one set of $2N$ admissible controls $\{u_{1,\rm tran}(\cdot), \dots, u_{N,\rm tran}(\cdot),\ u_{1,\rm rot}(\cdot), \dots, u_{N,\rm rot}(\cdot)\}$ such that Eqs. 5, 13, and 14 are satisfied. (13) and (14) are referred to as *endpoint constraints*. The constant $\tilde{x}_{\alpha,1} \in \mathbb{R}^{12}$, $\alpha = 1,\dots,N$, in (13) is assigned a priori and $s_2 : D_{\rm abs} \to \mathbb{R}^{\zeta_2}$ in (14) is continuously differentiable. (13) implies that the system's configuration is known at time t_1, which is usually the case for most applications, and (14) partly imposes the system's configuration at t_2. For example, (14) can be used to impose that the N rigid bodies reach a surface described by s_2 at time t_2 with a prescribed velocity.

5 Mathematical Background

In this section, we review some of the mathematical background needed to address the optimal control problem posed in Section 4.

5.1 Pontryagin's Principle

In the following, we state Pontryagin's principle, which is a first-order necessary condition for the existence of admissible controllers $u_{\alpha,\rm tran} : [t_1, t_2] \to \mathcal{G}_{\alpha,\rm tran}$ and $u_{\alpha,\rm rot} : [t_1, t_2] \to \mathcal{G}_{\alpha,\rm rot}$, $\alpha = 1,\dots,N$, that solve the trajectory optimization problem stated in Section 4. Let $s : \mathbb{R}^n \to \mathbb{R}^m$ be a continuously differentiable manifold and let the manifold tangent to s at y_0 be given by

$$\mathcal{T}(s(\cdot), y_0) \triangleq \left\{ y \in \mathbb{R}^n : \left.\frac{\partial s(y)}{\partial y}\right|_{y=y_0} (y - y_0) = 0_m \right\}. \tag{15}$$

Every vector $v \in \mathbb{R}^n$ that is normal to the manifold tangent to $s(\cdot)$ at y_0, that is, $v^{\rm T} y = 0$, $y \in \mathcal{T}(s(\cdot), y_0)$, is said to satisfy the *transversality condition* for $s(\cdot)$ at y_0.

Given the dynamical system (5) with performance measure (12), the *Hamiltonian function* is defined as

$$\mathfrak{h}(q, q_{\rm dot}, \tilde{u}, p) \triangleq p_0 L(q, q_{\rm dot}, \tilde{u}) + p_{\rm dot}^{\rm T}[D^{-1}(q) N_{\rm nh}(q_{\rm dot} - d(q))] + p_{\rm dyn}^{\rm T}(f_{\rm dyn}(q, q_{\rm dot}) + M(q)\tilde{u}), \quad (16)$$

where $p_0 \in \overline{\mathbb{R}}_+$, $p(t) \triangleq \left[p_0, p_{\rm dot}^{\rm T}(t), p_{\rm dyn}^{\rm T}(t)\right]^{\rm T}$, $t \in [t_1, t_2]$, the *costate vectors* $p_{\rm dot} : [t_1, t_2] \to \mathbb{R}^\gamma$ and $p_{\rm dyn} : [t_1, t_2] \to \mathbb{R}^{\gamma-\zeta}$ are the solutions of the *costate equation*

$$\frac{\rm d}{{\rm d}t}\begin{bmatrix} p_{\rm dot}(t) \\ p_{\rm dyn}(t) \end{bmatrix} = -\begin{bmatrix} \frac{\partial}{\partial q}[D^{-1}(q(t))N_{\rm nh}(q_{\rm dot}(t) - d(q(t)))]\mathcal{O}N_{\rm nh}(D^{-1}(q(t))) \\ \frac{\partial}{\partial q}(f_{\rm dyn}(q, q_{\rm dot}(t)) + M(q)\tilde{u}(t))\mathcal{O}N_{\rm nh}\left(\frac{\partial f_{\rm dyn}(q(t), q_{\rm dot})}{\partial q_{\rm dot}}\right) \end{bmatrix}^{\rm T} \cdot \begin{bmatrix} p_{\rm dot}(t) \\ p_{\rm dyn}(t) \end{bmatrix}, \qquad t \in [t_1, t_2], \quad (17)$$

and the boundary conditions for (17) are given in Theorem 2 below. Finally, let

$$\mathfrak{m}(q(t), q_{\rm dot}(t), p(t)) \triangleq \min_{\tilde{u} \in \prod_{\alpha=1}^{N}(\mathcal{G}_{\alpha,\rm tran} \times \mathcal{G}_{\alpha,\rm rot})} \mathfrak{h}(q(t), q_{\rm dot}(t), \tilde{u}, p(t)), \qquad t \in [t_1, t_2]. \quad (18)$$

In this paper, we refer to the following theorem as *Pontryagin's principle* [44].

Theorem 2 *For all* $\alpha = 1, \ldots, N$, *let* $u_{\alpha,\rm tran}^*(t)$ *and* $u_{\alpha,\rm rot}^*(t)$, $t \in [t_1, t_2]$, *be admissible controls in* $\mathcal{G}_{\alpha,\rm tran}$ *and* $\mathcal{G}_{\alpha,\rm rot}$, *respectively, that minimize the performance measure* (12) *subject to the dynamic* Eq. 5 *and the constraints* (13) and (14). *Then there exist* $p_0^* \in \overline{\mathbb{R}}_+$, $p_{\rm dyn}^*(t)$, *and* $p_{\rm dot}^*(t)$ *such that i)* $|p_0^*| + \|p_{\rm dyn}^*(t)\|_2 + \|p_{\rm dot}^*(t)\|_2 \neq 0$, $t \in [t_1, t_2]$, *ii)* (17) *is satisfied, iii)* $p_{\rm dyn}^*(t_1)$ *and* $p_{\rm dot}^*(t_1)$ *are arbitrary, iv)* $p_{\rm dyn}^*(t_2)$ *and* $p_{\rm dot}^*(t_2)$) *satisfy the transversality condition for* s_2 *at* $q^*(t_2)$, *and v)* $\mathfrak{h}(q^*(t), q_{\rm dot}^*(t), \tilde{u}^*(t), p^*(t))$ *attains its minimum* (18) *almost everywhere on* $[t_1, t_2]$, *which is equal to zero, except on a finite number of points.*

Pontryagin's principle is a necessary condition for the existence of strong local minima, and hence, it provides *candidate* optimal control vectors. Sufficient conditions for optimality that are currently available in the literature do not apply to the optimization problem discussed in this paper. There exist some versions of Pontryagin's principle that allow accounting for singular controls [32]. However, given the scope of the present paper, these formulations are equivalent to the approach discussed herein. We say that the optimization problem is *normal* if $p_0^* \neq 0$; otherwise the optimization problem is *abnormal*. For normal problems, we assume, without loss of generality, that $p_0^* = 1$. Recall that a common technique for verifing normality is to compute $\frac{\partial}{\partial \tilde{u}}\mathfrak{h}(q, q_{\rm dot}, \tilde{u}, p) = 0$ for $p_0 = 0$ and find in what cases condition *i*) of Theorem 2 is satisfied [33].

5.2 Necessary Conditions for Singular Controls

In this paper, we define singular controls and singular arcs as follows.

Definition 2 Consider the performance measure (12) subject to (5), (17), (13), (14), and conditions iii) and iv) of Theorem 2, and let $u_{\alpha,\rm tran}(\cdot)$ (respectively, $u_{\alpha,\rm rot}(\cdot)$) be an admissible control in $\mathcal{G}_{\alpha,\rm tran}$ (respectively, $\mathcal{G}_{\alpha,\rm rot}$), $\alpha = 1, \ldots, N$. If

$$\frac{\partial}{\partial u_{\alpha,\rm tran}}\mathfrak{h}(q(t), q_{\rm dot}(t), \tilde{u}, p(t)) = 0_3^{\rm T}, \quad t \in [\hat{t}_{1,\alpha,\rm tran}, \hat{t}_{2,\alpha,\rm tran}],$$

(respectively,

$$\frac{\partial}{\partial u_{\alpha,\rm rot}}\mathfrak{h}(q(t), q_{\rm dot}(t), \tilde{u}, p(t)) = 0_3^{\rm T} \quad t \in [\hat{t}_{1,\alpha,\rm rot}, \hat{t}_{2,\alpha,\rm rot}],)$$

where $[\hat{t}_{1,\alpha,\text{tran}}, \hat{t}_{2,\alpha,\text{tran}}] \subset [t_1, t_2]$ (respectively, $[\hat{t}_{1,\alpha,\text{rot}}, \hat{t}_{2,\alpha,\text{rot}}] \subset [t_1, t_2]$), then $u_{\alpha,\text{tran}}(t)$ (respectively, $u_{\alpha,\text{rot}}(t)$), is a *singular translational* (respectively, *rotational*) *control*. Furthermore, if $u_{\alpha,\text{tran}}(t)$ is a singular translational control and $u_{\alpha,\text{rot}}(t)$ is a singular rotational control, $t \in [\hat{t}_{1,\alpha}, \hat{t}_{2,\alpha}] \subset [t_1, t_2]$, then $x_\alpha(t)$, $t \in [\hat{t}_{1,\alpha}, \hat{t}_{2,\alpha}]$, is a *singular arc*.

It follows from Definition 2 that if $u_{\alpha,\text{tran}}(\cdot)$ (respectively, $u_{\alpha,\text{rot}}(\cdot)$) is a singular translational (respectively, rotational) control, then the variations of the Hamiltonian function with respect to $u_{\alpha,\text{tran}}(\cdot)$ (respectively, $u_{\alpha,\text{rot}}(\cdot)$) are identically equal to zero. Furthermore, if both $u_{\alpha,\text{tran}}(\cdot)$ and $u_{\alpha,\text{rot}}(\cdot)$ are singular controls, then the corresponding trajectory is a singular arc. It is worth noting that Definition 2 holds irrespectively of the normality of the optimal control problem considered.

Pontryagin's principle and the Legendre-Clebsch necessary condition hold along singular arcs. However, these theorems do not provide any useful information for identifying singular translational and rotational controls. In these cases, one can apply the following theorem, known as the *generalized Legendre-Clebsch necessary condition* [6, 22, 29, 30].

Theorem 3 *For all $\alpha = 1, \ldots, N$, let $u^*_{\alpha,\text{tran}}(\cdot) \in \text{int}(\mathcal{G}_{\alpha,\text{tran}})$ (respectively, $u^*_{\alpha,\text{rot}}(\cdot) \in \text{int}(\mathcal{G}_{\alpha,\text{rot}})$) be an admissible control in $\mathcal{G}_{\alpha,\text{tran}}$ (respectively, $\mathcal{G}_{\alpha,\text{rot}}$) that minimizes the performance measure (12) subject to (5), (17), (13), (14), and conditions iii) and iv) of Theorem 2. Then, there exists $[\hat{t}_{1,\alpha,\text{tran}}, \hat{t}_{2,\alpha,\text{tran}}] \subseteq [t_1, t_2]$ (respectively, $[\hat{t}_{1,\alpha,\text{rot}}, \hat{t}_{2,\alpha,\text{rot}}] \subseteq [t_1, t_2]$) and an integer $\nu_{\alpha,\text{tran}}$ (respectively, $\nu_{\alpha,\text{rot}}$) such that*

$$\frac{\mathrm{d}^\kappa}{\mathrm{d}t^\kappa} \frac{\partial}{\partial u^*_{\alpha,\text{tran}}} \mathfrak{h}(q^*(t), q^*_{dot}(t), \tilde{u}^*, p^*(t)) = 0_3^{\mathrm{T}}, \quad t \in [\hat{t}_{1,\alpha,\text{tran}}, \hat{t}_{2,\alpha,\text{tran}}], \tag{19}$$

(respectively,

$$\frac{\mathrm{d}^\lambda}{\mathrm{d}t^\lambda} \frac{\partial}{\partial u^*_{\alpha,\text{rot}}} \mathfrak{h}(q^*(t), q^*_{dot}(t), \tilde{u}^*, p^*(t)) = 0_3^{\mathrm{T}}, \quad t \in [\hat{t}_{1,\alpha,\text{rot}}, \hat{t}_{2,\alpha,\text{rot}}],) \tag{20}$$

for $\kappa = 0, \ldots, 2\nu_{\alpha,\text{tran}} - 1$ (respectively, $\lambda = 0, \ldots, 2\nu_{\alpha,\text{rot}} - 1$). Furthermore, $u^*_{\alpha,\text{tran}}(t)$ (respectively, $u^*_{\alpha,\text{rot}}(t)$) appears explicitly in the left-hand-side of Eq. 19 *(respectively, (20)) for $\kappa = 2\nu_{\alpha,\text{tran}}$ (respectively, $\lambda = 2\nu_{\alpha,\text{rot}}$.) Finally,*

$$(-1)^{\nu_{\alpha,\text{tran}}} \frac{\partial}{\partial u^*_{\alpha,\text{tran}}} \left(\frac{\mathrm{d}^{2\nu_{\alpha,\text{tran}}}}{\mathrm{d}t^{2\nu_{\alpha,\text{tran}}}} \frac{\partial}{\partial u^*_{\alpha,\text{tran}}} \mathfrak{h}(q^*(t), q^*_{dot}(t), \tilde{u}^*, p^*(t)) \right)^{\mathrm{T}} \geq 0_{3\times3}, \quad t \in [\hat{t}_{1,\alpha,\text{tran}}, \hat{t}_{2,\alpha,\text{tran}}], \tag{21}$$

(respectively,

$$(-1)^{\nu_{\alpha,\text{rot}}} \frac{\partial}{\partial u^*_{\alpha,\text{rot}}} \left(\frac{\mathrm{d}^{2\nu_{\alpha,\text{rot}}}}{\mathrm{d}t^{2\nu_{\alpha,\text{rot}}}} \frac{\partial}{\partial u^*_{\alpha,\text{rot}}} \mathfrak{h}(q^*(t), q^*_{dot}(t), \tilde{u}^*, p^*(t)) \right)^{\mathrm{T}} \geq 0_{3\times3}, \quad t \in [\hat{t}_{1,\alpha,\text{rot}}, \hat{t}_{2,\alpha,\text{rot}}].) \tag{22}$$

The integer $\nu_{\alpha,\text{tran}}$ (respectively, $\nu_{\alpha,\text{rot}}$) is the *order of the translational* (respectively, *rotational*) *singularity*. If $\nu_{\alpha,\text{tran}} = 0$ (respectively, $\nu_{\alpha,\text{rot}} = 0$), then (19) (respectively, (20)) reduces to the Euler-Lagrange necessary condition and Eq. 21 (respectively, (22)) reduces to the Legendre necessary condition. A discussion on the order of singularity is given in reference [35]. Pontryagin's principle applies to strong local minima, whereas Theorem 3 applies to weak local minima. Thus, the generalized Legendre-Clebsch necessary condition holds on a smaller class of candidate optimal controls. Furthermore, Pontryagin's principle holds for $\tilde{u}^*(\cdot) \in \prod_{\alpha=1}^n (\mathcal{G}_{\alpha,\text{tran}} \times \mathcal{G}_{\alpha,\text{rot}})$, which is a closed set, whereas the generalized Legendre-Clebsh necessary condition holds for $\tilde{u}^*(\cdot) \in \prod_{\alpha=1}^n (\text{int}(\mathcal{G}_{\alpha,\text{tran}}) \times \text{int}(\mathcal{G}_{\alpha,\text{rot}}))$. This limitation is due to the fact that Theorem 3 is proven by using two sided variations of the control vectors.

The next result allows us to characterize candidate optimal translational and rotational singular controls.

Corollary 1 *Assume the conditions of Theorem 3 are satisfied. Then, a candidate optimal singular translational (respectively, rotational) control* $u^*_{\alpha,\mathrm{tran}}(t)$*,* $t \in [\hat{t}_{1,\alpha,\mathrm{tran}}, \hat{t}_{2,\alpha,\mathrm{tran}}]$ *(respectively,* $u^*_{\alpha,\mathrm{rot}}(t)$*,* $t \in [\hat{t}_{1,\alpha,\mathrm{rot}}, \hat{t}_{2,\alpha,\mathrm{rot}}]$*) is given by*

$$\frac{\mathrm{d}^{2\nu_{\alpha,\mathrm{rot}}}}{\mathrm{d}t^{2\nu_{\alpha,\mathrm{tran}}}}\frac{\partial}{\partial u^*_{\alpha,\mathrm{tran}}}\mathfrak{h}(q^*(t), q^*_{dot}(t), \tilde{u}^*, p^*(t)) = 0_3^{\mathrm{T}},\quad t \in [\hat{t}_{1,\alpha,\mathrm{tran}}, \hat{t}_{2,\alpha,\mathrm{tran}}], \tag{23}$$

(respectively,

$$\frac{\mathrm{d}^{2\nu_{\alpha,\mathrm{rot}}}}{\mathrm{d}t^{2\nu_{\alpha,\mathrm{rot}}}}\frac{\partial}{\partial u^*_{\alpha,\mathrm{rot}}}\mathfrak{h}(q^*(t), q^*_{dot}(t), \tilde{u}^*, p^*(t)) = 0_3^{\mathrm{T}},\quad t \in [\hat{t}_{1,\alpha,\mathrm{rot}}, \hat{t}_{2,\alpha,\mathrm{rot}}].) \tag{24}$$

for $\alpha = 1, \ldots, N$*.*

Proof The result follows as a direct consequence of Theorem 3. □

The next theorem characterizes the junction between singular and nonsingular controls. For the statement of this result, recall that $f : [t_1, t_2] \to \mathbb{R}^n$ is piecewise analytic in a neighborhood $\mathcal{I}(\hat{t})$ of $\hat{t} \in (t_1, t_2)$ if $f(t)$ possesses derivatives of all orders and $f(t)$, $t \in \mathcal{I}(\hat{t})$, coincides with its Taylor series except at a finite number of points.

Theorem 4 ([40]) *Consider the performance measure (12) subject to* Eqs. 5, 17, 13, 14*, and conditions* $iii)$ *and* $iv)$ *of Theorem 2. Let* $u^*_{\alpha,\mathrm{tran}}(t) \in \mathrm{int}(\mathcal{G}_{\alpha,\mathrm{tran}})$*,* $t \in [\hat{t}_{1,\alpha,\mathrm{tran}}, \hat{t}_{2,\alpha,\mathrm{tran}}] \subset [t_1, t_2]$ *(respectively,* $u^*_{\alpha,\mathrm{rot}}(t) \in \mathrm{int}(\mathcal{G}_{\alpha,\mathrm{rot}})$*,* $t \in [\hat{t}_{1,\alpha,\mathrm{rot}}, \hat{t}_{2,\alpha,\mathrm{rot}}] \subset [t_1, t_2]$*),* $\alpha = 1, \ldots, N$*, be a candidate optimal singular translational (respectively, rotational) control with order of translational (respectively, rotational) singularity* $\nu_{\alpha,\mathrm{tran}}$ *(respectively,* $\nu_{\alpha,\mathrm{rot}}$*.) Assume that* $u^*_{\alpha,\mathrm{tran}}(t)$ *(respectively,* $u^*_{\alpha,\mathrm{rot}}(t)$*), is piecewise analytic in some neighborhood of* $\hat{t}_{1,\alpha,\mathrm{tran}}$ *and* $\hat{t}_{2,\alpha,\mathrm{tran}}$ *(respectively,* $\hat{t}_{1,\alpha,\mathrm{rot}}$ *and* $\hat{t}_{1,\alpha,\mathrm{rot}}$*), where*

$$(-1)^{\nu_{\alpha,\mathrm{tran}}}\frac{\partial}{\partial u^*_{\alpha,\mathrm{tran}}}\left(\frac{\mathrm{d}^{2\nu_{\alpha,\mathrm{tran}}}}{\mathrm{d}t^{2\nu_{\alpha,\mathrm{tran}}}}\frac{\partial}{\partial u^*_{\alpha,\mathrm{tran}}}\mathfrak{h}(q^*(t), q^*_{dot}(t), \tilde{u}^*, p^*(t))\right)^{\mathrm{T}} > 0_{3\times 3} \tag{25}$$

(respectively,

$$(-1)^{\nu_{\alpha,\mathrm{rot}}}\frac{\partial}{\partial u^*_{\alpha,\mathrm{rot}}}\left(\frac{\mathrm{d}^{2\nu_{\alpha,\mathrm{rot}}}}{\mathrm{d}t^{2\nu_{\alpha,\mathrm{rot}}}}\frac{\partial}{\partial u^*_{\alpha,\mathrm{rot}}}\mathfrak{h}(q^*(t), q^*_{dot}(t), \tilde{u}^*, p^*(t))\right)^{\mathrm{T}} > 0_{3\times 3}). \tag{26}$$

In addition, let $\kappa_{1,\alpha,\mathrm{tran}} \in \mathbb{N} \cup \{0\}$ *and* $\kappa_{2,\alpha,\mathrm{tran}} \in \mathbb{N} \cup \{0\}$ *(respectively,* $\kappa_{1,\alpha,\mathrm{rot}} \in \mathbb{N} \cup \{0\}$ *and* $\kappa_{2,\alpha,\mathrm{rot}} \in \mathbb{N} \cup \{0\}$*),* $\alpha = 1, \ldots, N$*, be the smallest integers such that*

$$\frac{\mathrm{d}^{\kappa_{1,\alpha,\mathrm{tran}}} u^*_{\alpha,\mathrm{tran}}(t)}{\mathrm{d}t^{\kappa_{1,\alpha,\mathrm{tran}}}}$$

is discontinuous at $t = \hat{t}_{1,\alpha,\mathrm{tran}}$ *and*

$$\frac{\mathrm{d}^{\kappa_{2,\alpha,\mathrm{tran}}} u^*_{\alpha,\mathrm{tran}}(t)}{\mathrm{d}t^{\kappa_{2,\alpha,\mathrm{tran}}}}$$

is discontinuous at $t = \hat{t}_{2,\alpha,\mathrm{tran}}$ *(respectively,*

$$\frac{\mathrm{d}^{\kappa_{1,\alpha,\mathrm{rot}}} u^*_{\alpha,\mathrm{rot}}(t)}{\mathrm{d}t^{\kappa_{1,\alpha,\mathrm{rot}}}}$$

is discontinuous at $t = \hat{t}_{1,\alpha,\mathrm{rot}}$ *and*

$$\frac{\mathrm{d}^{\kappa_{2,\alpha,\mathrm{rot}}} u^*_{\alpha,\mathrm{rot}}(t)}{\mathrm{d}t^{\kappa_{2,\alpha,\mathrm{rot}}}}$$

is discontinuous at $t = \hat{t}_{2,\alpha,\mathrm{rot}}$*.) Then* $\nu_{\alpha,\mathrm{tran}} + \kappa_{1,\alpha,\mathrm{tran}}$ *and* $\nu_{\alpha,\mathrm{tran}} + \kappa_{2,\alpha,\mathrm{tran}}$ *(respectively,* $\nu_{\alpha,\mathrm{rot}} + \kappa_{1,\alpha,\mathrm{rot}}$ *and* $\nu_{\alpha,\mathrm{rot}} + \kappa_{2,\alpha,\mathrm{rot}}$*) are odd integers.*

Remark 2 Recall that if $\nu_{\alpha,\mathrm{tran}} > 1$ (respectively, $\nu_{\alpha,\mathrm{rot}} > 1$), then $u^*_{\alpha,\mathrm{tran}}(t)$ (respectively, $u^*_{\alpha,\mathrm{rot}}(t)$) is measurable but not analytic in the neighborhoods of $\hat{t}_{1,\alpha,\mathrm{tran}}$ and $\hat{t}_{2,\alpha,\mathrm{tran}}$ (respectively, $\hat{t}_{1,\alpha,\mathrm{rot}}$ and $\hat{t}_{2,\alpha,\mathrm{rot}}$) [7]. Thus, if $\nu_{\alpha,\mathrm{tran}} > 1$ and $\nu_{\alpha,\mathrm{rot}} > 1$, then Theorem 4 cannot be applied and the control vector $\tilde{u}^*(\cdot)$ is affected by chattering at the junction between singular and nonsingular arcs [17].

6 The Abnormal Optimization Problem

In this section, we provide necessary conditions to solve the abnormal optimization problem for a system of N rigid bodies, whose dynamics is captured by Eqs. 5, 13, and 14. For conciseness, given $p_{\rm dyn} : [t_1, t_2] \to \mathbb{R}^{\gamma-\zeta}$ that satisfies (17) and conditions iii) and iv) of Theorem 2, we define the *auxiliary costate vector* as

$$\hat{p}_{\rm dyn}(t) \triangleq \mathcal{O}N_{\rm nh}\big(D^{-1}(q(t))\big)Z(q(t))p_{\rm dyn}(t), \quad t \in [t_1, t_2]. \quad (27)$$

We also assume, without loss of generality, that the translational control vectors of the first ν rigid bodies and the rotational control vectors of the first κ rigid bodies are singular.

The following lemma provides necessary conditions for identifying candidate optimal singular controls. Specifically, the following result proves that candidate optimal singular controls exist when the costate vector associated to the system's dynamic equations belongs to the nullspace of the Jacobian matrix, which characterizes the transformation from Cartesian coordinates to independent generalized coordinates. In the following, $u^*_{\alpha,\rm tran}(\cdot)$, $\alpha = 1, \ldots, \nu$, and $u^*_{\beta,\rm rot}(\cdot)$, $\beta = 1, \ldots, \kappa$, denote candidate optimal singular controls, whereas $u^*_{\lambda,\rm tran}(\cdot)$, $\lambda = \nu + 1, \ldots, N$, and $u^*_{\chi,\rm rot}(t)$, $\chi = \kappa + 1, \ldots, N$, denote candidate optimal nonsingular controls.

Lemma 1 *Consider the performance measure* (12) *subject to* Eqs. 5. 17, 13, 14, *and conditions iii) and iv) of Theorem 2. If $p^*_0 = 0$ and*

$$\hat{p}^*_{\rm dyn}(t) \in \mathcal{N}\left(\frac{\partial r_\alpha(q^*)}{\partial q^*}\right), \quad t \in [\hat{t}_{1,\alpha,\rm tran}, \hat{t}_{2,\alpha,\rm tran}] \subset [t_1, t_2], \quad \alpha = 1, \ldots, \nu, \quad (28)$$

then $u^*_{\alpha,\rm tran}(t)$ is a candidate optimal singular translational control. In addition, if

$$\hat{p}^*_{\rm dyn}(t) \in \mathcal{N}\left(\frac{\partial \sigma_\beta(q^*)}{\partial q^*}\right), \quad t \in [\hat{t}_{1,\beta,\rm rot}, \hat{t}_{2,\beta,\rm rot}] \subset [t_1, t_2], \quad \beta = 1, \ldots, \kappa, \quad (29)$$

then $u^*_{\beta,\rm rot}(t)$ is a candidate optimal singular rotational control. Furthermore, the candidate optimal translational control $u^*_{\lambda,\rm tran}(t)$ is parallel to the vector $-\frac{\partial r_\lambda(q^*(t))}{\partial q^*}\hat{p}^*_{\rm dyn}(t)$, $t \in [t_1, t_2]$, $\lambda = \nu + 1, \ldots, N$, the candidate optimal rotational control $u^*_{\chi,\rm rot}(t)$ is parallel to $-R^{-1}_{\rm rod}(\sigma_\chi(q^*(t)))\frac{\partial \sigma_\chi(q^*(t))}{\partial q^*}\hat{p}^*_{\rm dyn}(t)$, $t \in [t_1, t_2]$, $\chi = \kappa + 1, \ldots, N$, $\|u^*_{\lambda,\rm tran}(t)\|_2 = \rho_{\lambda,2}$, and $\|u^*_{\chi,\rm rot}(t)\|_2 = \rho_{\chi,4}$.

Proof The result is a consequence of Definition 2 and Theorem 2. Specifically, if $p^*_0 = 0$, then it follows from Eq. 16 that

$$\begin{aligned}
&\mathfrak{h}\ (q^*, q^*_{\rm dot}, \tilde{u}^*, p^*) - p^{*\rm T}_{\rm dot}\Big[D^{-1}(q^*)N_{\rm nh}\big(q^*_{\rm dot} - d(q^*)\big)\Big] \\
&\quad - p^{*\rm T}_{\rm dyn} f_{\rm dyn}(q^*, q^*_{\rm dot}) \\
&= \sum_{\alpha=1}^{\nu} u^{*\rm T}_{\alpha,\rm tran}\frac{\partial r_\alpha(q^*)}{\partial q^*}\hat{p}^*_{\rm dyn} \\
&\quad + \sum_{\lambda=\nu+1}^{N} u^{*\rm T}_{\lambda,\rm tran}\frac{\partial r_\lambda(q^*)}{\partial q^*}\hat{p}^*_{\rm dyn} \\
&\quad + \sum_{\beta=1}^{\kappa} u^{*\rm T}_{\beta,\rm rot}R^{-1}_{\rm rod}(\sigma_\beta(q^*))\frac{\partial \sigma_\beta(q^*)}{\partial q^*}\hat{p}^*_{\rm dyn} \\
&\quad + \sum_{\chi=\kappa+1}^{N} u^{*\rm T}_{\chi,\rm rot}R^{-1}_{\rm rod}(\sigma_\chi(q^*))\frac{\partial \sigma_\chi(q^*)}{\partial q^*}\hat{p}^*_{\rm dyn}. \quad (30)
\end{aligned}$$

Now, if (28) and (29) are satisfied, then it follows from Definition 2 that $u^*_{\alpha,\rm tran}(t)$, $t \in [\hat{t}_{1,\alpha,\rm tran}, \hat{t}_{2,\alpha,\rm tran}]$, $\alpha = 1, \ldots, \nu$, is a singular translational control and $u^*_{\beta,\rm rot}(t)$, $\beta = 1, \ldots, \kappa$, is a singular rotational control.

Next, it follows from Theorem 2 that a necessary condition to minimize the performance measure (12) subject to the dynamic Eq. 5 and the boundary conditions (13) and (14) is that the Hamiltonian function (30) is minimized with respect to $\tilde{u}^*$. Now, if $-u^*_{\lambda,\rm tran}(t)$ and $\frac{\partial r_\lambda(q^*)}{\partial q^*}\hat{p}^*_{\rm dyn}(t)$ are parallel, $\lambda = \nu + 1, \ldots, N$, $-u^*_{\chi,\rm rot}(t)$ and $R^{-1}_{\rm rod}(\sigma_\chi(q^*))\frac{\partial \sigma_\chi(q^*)}{\partial q^*}\hat{p}^*_{\rm dyn}(t)$ are parallel, $\chi = \kappa + 1, \ldots, N$, and $\|u^*_{\lambda,\rm tran}(t)\|_2$ and $\|u^*_{\chi,\rm rot}(t)\|_2$ have maximum magnitudes, then Eq. 30 is minimized, which completes the proof. □

Necessary and sufficient conditions for the existence of singular controls for abnormal optimal control problems can be found in [26] and, more recently, [27], where the existence of the candidate optimal trajectories is related to the rank of the minimial distribution along the optimal trajctory and the dimension of the configuration space.

The next result shows that the order of singularity is one for abnormal optimal control problems involving mechanical systems. Specifically, the control vector does not explicitly appear in the partial derivative of the Hamiltonian function with respect to the control vector along singular arcs, but explicitly appears in its *second* time derivative.

Theorem 5 *Consider the performance measure* (12) *subject to* Eqs. 5, 17, 13, 14, *and conditions iii) and iv) of Theorem 2. If* $p_0^* = 0$ *and* $\hat{p}^*_{\rm dyn}(t) \in \mathcal{N}\left(\frac{\partial r_\alpha(q^*)}{\partial q^*}\right)$, $\alpha = 1, \ldots, \nu$, $t \in [\hat{t}_{1,\alpha,{\rm tran}}, \hat{t}_{2,\alpha,{\rm tran}}] \subset [t_1, t_2]$, *then the order of singularity of the candidate optimal singular translational control* $u^*_{\alpha,{\rm tran}}(t)$, $t \in [\hat{t}_{1,\alpha,{\rm tran}}, \hat{t}_{2,\alpha,{\rm tran}}]$, *is one, that is,* $\nu_{\alpha,{\rm tran}} = 1$. *Moreover, if* $p_0^* = 0$ *and* $\hat{p}^*_{\rm dyn}(t) \in \mathcal{N}\left(\frac{\partial \sigma_\beta(q^*)}{\partial q^*}\right)$, $\beta = 1, \ldots, \kappa$, $t \in [\hat{t}_{1,\beta,{\rm rot}}, \hat{t}_{2,\beta,{\rm rot}}] \subset [t_1, t_2]$, *then the order of the singularity of the candidate optimal singular rotational control* $u^*_{\beta,{\rm rot}}(t)$, $t \in [\hat{t}_{1,\beta,{\rm rot}}, \hat{t}_{2,\beta,{\rm rot}}]$, *is one, that is,* $\nu_{\beta,{\rm rot}} = 1$.

Proof The result is a consequence of Theorem 3. Specifically, it follows from Lemma 1 that $u^*_{\alpha,{\rm tran}}(t)$, $\alpha = 1, \ldots, \nu$, $t \in [\hat{t}_{1,\alpha,{\rm tran}}, \hat{t}_{2,\alpha,{\rm tran}}]$, and $u^*_{\beta,{\rm rot}}(t)$, $\beta = 1, \ldots, \kappa$, $t \in [\hat{t}_{1,\beta,{\rm rot}}, \hat{t}_{2,\beta,{\rm rot}}]$, are candidate optimal singular translational and rotational controls, respectively. Now, it follows from Eq. 30 that Eqs. 19 and 20 specialize, respectively, to

$$\frac{{\rm d}^\kappa}{{\rm d}t^\kappa}\left(\frac{\partial r_\alpha(q^*)}{\partial q^*}\hat{p}^*_{\rm dyn}(t)\right) = 0_3^{\rm T}, \quad t \in [\hat{t}_{1,\alpha,{\rm tran}}, \hat{t}_{2,\alpha,{\rm tran}}], \quad (31)$$

$$\frac{{\rm d}^\lambda}{{\rm d}t^\lambda}\left(R^{-1}_{\rm rod}(\sigma_\beta(q^*(t)))\frac{\partial \sigma_\beta(q^*)}{\partial q^*}\hat{p}^*_{\rm dyn}(t)\right) = 0_3^{\rm T}, \quad t \in [\hat{t}_{1,\beta,{\rm rot}}, \hat{t}_{2,\beta,{\rm rot}}]. \quad (32)$$

Since $\dot{q}^*_{\rm dot}$, and hence, $\tilde{u}^*$ explicitly appears in $\frac{{\rm d}^2}{{\rm d}t^2}\frac{\partial r_\alpha(q^*)}{\partial q^*}$, $\frac{{\rm d}^2}{{\rm d}t^2}\frac{\partial \sigma_\beta(q^*)}{\partial q^*}$, $\frac{{\rm d}^2}{{\rm d}t^2}\hat{p}^*_{\rm dyn}(t)$, and $\frac{{\rm d}^2}{{\rm d}t^2}R^{-1}_{\rm rod}(\sigma_\beta(q^*(t)))$, the result directly follows from Definition 2 and Theorem 3. □

Theorems 6 and 7 below provide necessary conditions for optimality of singular controls. Specifically, Theorems 6 and 7 show that a second-order differential equation and a matrix inequality must be satisfied in the presence of singular controls. For the statement of the next two results, let $a = [a_1, \ldots, a_n]^{\rm T}$ and $a_i \in \mathbb{R}$, $i = 1, \ldots, n$, be the ith component of a, $r_\alpha = [r_{\alpha,1}, r_{\alpha,2}, r_{\alpha,3}]^{\rm T}$ and $r_{\alpha,i} \in \mathbb{R}$ be the ith component of r_α, $\sigma_\beta = [\sigma_{\beta,1}, \sigma_{\beta,2}, \sigma_{\beta,3}]^{\rm T}$ and $\sigma_{\beta,i} \in \mathbb{R}$ be the ith component of σ_β, $p_{\rm dyn} = [p_{{\rm dyn},1}, \ldots, p_{{\rm dyn},\gamma-\zeta}]^{\rm T}$ and $p_{{\rm dyn},i} \in \mathbb{R}$ be the ith component of $p_{\rm dyn}$, $\hat{p}_{\rm dyn} = [\hat{p}_{{\rm dyn},1}, \ldots, \hat{p}_{{\rm dyn},\gamma}]^{\rm T}$ and $\hat{p}_{{\rm dyn},i} \in \mathbb{R}$ be the ith component of $\hat{p}_{\rm dyn}$, and $A \in \mathbb{R}^{n\times m}$ and $A_{(i,j)}$ be the (i,j)th entry of A.

Theorem 6 *Consider the performance measure* (12) *subject to* Eqs. 5, 17, 13, 14, *and conditions iii) and iv) of Theorem 2. If* $p_0^* = 0$ *and* $\hat{p}^*_{\rm dyn}(t) \in \mathcal{N}\left(\frac{\partial r_\alpha(q^*)}{\partial q^*}\right)$, $\alpha = 1, \ldots, \nu$, $t \in [\hat{t}_{1,\alpha,{\rm tran}}, \hat{t}_{2,\alpha,{\rm tran}}] \subset [t_1, t_2]$, *then the candidate optimal singular translational control* $u^*_{\alpha,{\rm tran}}(\cdot)$ *satisfies*

$$\frac{{\rm d}^2}{{\rm d}t^2}\left(\frac{\partial r_\alpha(q^*)}{\partial q^*}\hat{p}^*_{\rm dyn}(t)\right) = 0_3 \quad t \in [\hat{t}_{1,\alpha,{\rm tran}}, \hat{t}_{2,\alpha,{\rm tran}}], \quad (33)$$

and

$$\begin{bmatrix} \hat{p}^{*{\rm T}}_{\rm dyn}(t)\frac{\partial^2 r_{\alpha,1}(q^*)}{\partial q^{*2}} \\ \hat{p}^{*{\rm T}}_{\rm dyn}(t)\frac{\partial^2 r_{\alpha,2}(q^*)}{\partial q^{*2}} \\ \hat{p}^{*{\rm T}}_{\rm dyn}(t)\frac{\partial^2 r_{\alpha,3}(q^*)}{\partial q^{*2}} \end{bmatrix} F_\alpha(q^*(t)) + \frac{\partial r_\alpha(q^*)}{\partial q^*}\begin{bmatrix} \sum_{\lambda=1}^{\gamma-\zeta} p^*_{{\rm dyn},\lambda}(t)\frac{\partial}{\partial q^*}(\mathcal{O}N_{\rm nh}(D^{-1}(q^*))Z(q^*))_{(1,\lambda)} \\ \cdots \\ \sum_{\lambda=1}^{\gamma-\zeta} p^*_{{\rm dyn},\lambda}(t)\frac{\partial}{\partial q^*}(\mathcal{O}N_{\rm nh}(D^{-1}(q^*))Z(q^*))_{(\gamma,\lambda)} \end{bmatrix} F_\alpha(q^*(t)) \le 0_{3\times 3}, \quad t \in [\hat{t}_{1,\alpha,{\rm tran}}, \hat{t}_{2,\alpha,{\rm tran}}], \quad (34)$$

where

$$F_\alpha(q) \triangleq D^{-1}(q)\begin{bmatrix} 0_{\zeta\times\zeta} & 0_{\zeta\times(\gamma-\zeta)} \\ 0_{(\gamma-\zeta)\times\zeta} & Z(q) \end{bmatrix} D^{-{\rm T}}(q)\left(\frac{\partial r_\alpha(q)}{\partial q}\right)^{\rm T}. \quad (35)$$

Furthermore, if $u^*_{\alpha,{\rm tran}}(t)$ *is piecewise analytic and* Eq. 34 *is satisfied as a strict inequality for* $t \in \mathcal{I}(\hat{t}_{1,\alpha,{\rm tran}})$ *and* $t \in \mathcal{I}(\hat{t}_{2,\alpha,{\rm tran}})$, *where the sets* $\mathcal{I}(\hat{t}_{1,\alpha,{\rm tran}})$ *and* $\mathcal{I}(\hat{t}_{2,\alpha,{\rm tran}})$ *denote some neighborhoods of* $\hat{t}_{1,\alpha,{\rm tran}}$ *and* $\hat{t}_{1,\alpha,{\rm tran}}$, *respectively, then* $u^*_{\alpha,{\rm tran}}(\cdot) \in C^{\phi_{1,\alpha,{\rm tran}}}(\mathcal{I}(\hat{t}_{1,\alpha,{\rm tran}}))$ *and* $u^*_{\alpha,{\rm tran}}(\cdot) \in C^{\phi_{2,\alpha,{\rm tran}}}(\mathcal{I}(\hat{t}_{2,\alpha,{\rm tran}}))$, *where* $\phi_{1,\alpha,{\rm tran}}, \phi_{2,\alpha,{\rm tran}} \in \mathbb{N}$ *are odd numbers.*

Proof It follows from Lemma 1 and Theorem 5 that $u^*_{\alpha,\text{tran}}(t)$, $\alpha = 1, \ldots, \nu$, $t \in [\hat{t}_{1,\alpha,\text{tran}}, \hat{t}_{2,\alpha,\text{tran}}]$, is a candidate optimal singular translational control and the order of singularity is $\nu_{\alpha,\text{tran}} = 1$. Now, it follows from Eq. 30 that Eq. 23 is equivalent to

$$\frac{d^2}{\mathrm{d}t^2}\frac{\partial}{\partial u^*_{\alpha,\text{tran}}}\mathfrak{h}(q^*(t), q^*_{\text{dot}}(t), \tilde{u}^*, p^*(t)) = \frac{d^2}{\mathrm{d}t^2}\left[\hat{p}^{*\mathrm{T}}_{\text{dyn}}(t)\left(\frac{\partial r_\alpha(q^*)}{\partial q^*}\right)^{\mathrm{T}}\right] = 0_3^{\mathrm{T}} \tag{36}$$

for all $t \in [\hat{t}_{1,\alpha,\text{tran}}, \hat{t}_{2,\alpha,\text{tran}}]$, which implies (33). In addition, it follows from (33) and (21) with $\nu_{\alpha,\text{tran}} = 1$ that

$$\frac{\partial}{\partial u_{\alpha,\text{tran}}}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial r_\alpha(q^*)}{\partial q^*}\hat{p}^*_{\text{dyn}}(t)\right) = \frac{\partial}{\partial u_{\alpha,\text{tran}}}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial r_\alpha(q^*)}{\partial q^*} + \frac{\partial}{\partial u_{\alpha,\text{tran}}}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\hat{p}^*_{\text{dyn}}(t) \leq 0, \quad t \in [\hat{t}_{1,\alpha,\text{tran}}, \hat{t}_{2,\alpha,\text{tran}}], \tag{37}$$

which is equivalent to (34). Next, if (34) is satisfied with a strict inequality for $t \in \mathcal{I}(\hat{t}_{1,\alpha,\text{tran}})$ and $t \in \mathcal{I}(\hat{t}_{2,\alpha,\text{tran}})$, then it follows from Theorem 4 that the time derivative of the translational control vector is differentiable an odd number of times. □

It follows from Theorem 5 that the order of singularity is one for abnormal optimal control problems involving mechanical systems, and hence, it follows from Remark 2 that for these problems the junction between singular and non-singular arcs is not affected by chattering. Moreover, it follows from Theorem 6 that the candidate optimal control $u^*_{\alpha,tran}(\cdot)$ is continuously differentiable with an odd number of its higher derivatives in some neighborhood of the junction between singular and non-singular arcs.

Theorem 7 *Consider the performance measure (12) subject to (5), (17), (13), (14), and conditions iii) and iv) of Theorem 2. If* $p_0^* = 0$ *and* $\hat{p}^*_{\text{dyn}}(t) \in \mathcal{N}\big(R^{-1}_{\text{rod}}(\sigma_\beta(q^*(t)))\frac{\partial\sigma_\beta(q^*)}{\partial q^*}\big)$, $\beta = 1, \ldots, \kappa$, $t \in [\hat{t}_{1,\beta,\text{rot}}, \hat{t}_{2,\beta,\text{rot}}] \subset [t_1, t_2]$, *then the candidate optimal singular rotational control* $u^*_{\beta,\text{rot}}(\cdot)$ *satisfies*

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(R^{-1}_{\text{rod}}(\sigma_\beta(q^*(t)))\frac{\partial\sigma_\beta(q^*)}{\partial q^*}\hat{p}^*_{\text{dyn}}(t)\right) = 0_3, \quad t \in [\hat{t}_{1,\beta,\text{rot}}, \hat{t}_{2,\beta,\text{rot}}], \tag{38}$$

and

$$\left[\left[\begin{array}{c}\sum_{\lambda=1}^{3}\left(\frac{\partial\sigma_\beta(q^*)}{\partial q^*}\hat{p}^*_{\text{dyn}}(t)\right)_\lambda\frac{\partial(R^{-1}_{\text{rod}}(\sigma^*_\beta))_{(1,\lambda)}}{\partial\sigma^*_\beta}\\ \sum_{\lambda=1}^{3}\left(\frac{\partial\sigma_\lambda(q^*)}{\partial q^*}\hat{p}^*_{\text{dyn}}(t)\right)_\lambda\frac{\partial(R^{-1}_{\text{rod}}(\sigma^*_\beta))_{(2,\lambda)}}{\partial\sigma^*_\beta}\\ \sum_{\lambda=1}^{3}\left(\frac{\partial\sigma_\lambda(q^*)}{\partial q^*}\hat{p}^*_{\text{dyn}}(t)\right)_\lambda\frac{\partial(R^{-1}_{\text{rod}}(\sigma^*_\beta))_{(3,\lambda)}}{\partial\sigma^*_\beta}\end{array}\right]\frac{\partial\sigma_\beta(q^*)}{\partial q^*} + R^{-1}_{\text{rod}}(\sigma_\beta(q^*(t)))\left(\left[\begin{array}{c}\hat{p}^{*\mathrm{T}}_{\text{dyn}}(t)\frac{\partial^2\sigma_{\beta,1}(q^*)}{\partial^2 q^*}\\ \hat{p}^{*\mathrm{T}}_{\text{dyn}}(t)\frac{\partial^2\sigma_{\beta,2}(q^*)}{\partial^2 q^*}\\ \hat{p}^{*\mathrm{T}}_{\text{dyn}}(t)\frac{\partial^2\sigma_{\beta,3}(q^*)}{\partial^2 q^*}\end{array}\right] + \frac{\partial\sigma_\beta(q^*)}{\partial q^*}\left[\begin{array}{c}\sum_{\lambda=1}^{\gamma-\zeta}p^*_{\text{dyn},\lambda}(t)\frac{\partial}{\partial q^*}\big(\mathcal{O}N_{\text{nh}}\big(D^{-1}(q^*)\big)Z(q^*)\big)_{(1,\lambda)}\\ \ldots\\ \sum_{\lambda=1}^{\gamma-\zeta}p^*_{\text{dyn},\lambda}(t)\frac{\partial}{\partial q^*}\big(\mathcal{O}N_{\text{nh}}\big(D^{-1}(q^*)\big)Z(q^*)\big)_{(\gamma,\lambda)}\end{array}\right]\right)\right]H_\beta(q^*(t)) \leq 0_{3\times3}, \quad t \in [\hat{t}_{1,\beta,\text{rot}}, \hat{t}_{2,\beta,\text{rot}}], \tag{39}$$

where

$$H_\beta(q) \triangleq D^{-1}(q)\begin{bmatrix}0_{\zeta\times\zeta} & 0_{\zeta\times(\gamma-\zeta)}\\ 0_{(\gamma-\zeta)\times\zeta} & Z(q)\end{bmatrix}\cdot D^{-\mathrm{T}}(q)\left(R^{-1}_{\text{rod}}(\sigma_\beta)\frac{\partial\sigma_\beta(q)}{\partial q}\right)^{\mathrm{T}}. \tag{40}$$

Furthermore, if $u^*_{\beta,\text{rot}}(t)$ *is piecewise analytic and (39) is satisfied with a strict inequality for* $t \in \mathcal{I}(\hat{t}_{1,\beta,\text{rot}})$ *and* $t \in \mathcal{I}(\hat{t}_{2,\beta,\text{rot}})$, *where* $\mathcal{I}(\hat{t}_{1,\beta,\text{rot}})$ *and* $\mathcal{I}(\hat{t}_{2,\beta,\text{rot}})$ *are some neighborhoods of* $\hat{t}_{1,\beta,\text{rot}}$ *and* $\hat{t}_{2,\beta,\text{rot}}$, *respectively, then* $u^*_{\beta,\text{rot}}(\cdot) \in C^{\phi_{1,\beta,\text{rot}}}(\mathcal{I}(\hat{t}_{1,\beta,\text{rot}}))$ *and*

$u^*_{\beta,\mathrm{rot}}(\cdot) \in C^{\phi_{2,\beta,\mathrm{rot}}}(\mathcal{I}(\hat{t}_{2,\beta,\mathrm{rot}}))$, where $\phi_{1,\beta,\mathrm{rot}} \in \mathbb{N}$ and $\phi_{2,\beta,\mathrm{rot}} \in \mathbb{N}$ are odd numbers.

Proof The result follows as in the proof of Theorem 6 by deducing (38) from (24) for $\lambda = 2\nu_{\beta,\mathrm{rot}} = 2$ and (39) from (22) by noting that $u^*_{\beta,\mathrm{rot}}(t)$ explicitly appears in $\frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial\sigma_\beta(q^*)}{\partial q^*}$, $\frac{\mathrm{d}^2}{\mathrm{d}t^2}R^{-1}_{\mathrm{rod}}(\sigma_\beta(q^*(t)))$, and $\frac{\mathrm{d}^2}{\mathrm{d}t^2}\hat{p}^*_{\mathrm{dyn}}(t)$. □

It follows from Theorem 5 that the order of singularity is one for abnormal optimal control problems involving mechanical systems, and hence, it follows from Remark 2 that for these problems the junction between singular and non-singular arcs is not affected by chattering. Moreover, it follows from Theorem 7 that the candidate optimal control $u^*_{\beta,rot}(\cdot)$ is continuously differentiable with an odd number of its higher derivatives in some neighborhood of the junction between singular and non-singular arcs.

Remark 3 The theoretical results developed in this section can be applied as follows. Singular and non-singular optimal control arcs in the abnormal optimization problem involving systems of N rigid bodies can be identified by applying Lemma 1. Then, Theorems 6 and 7 can be applied to find candidate optimal translational and rotational singular controls by verifying the differential equations (33) and (38), and the nonpositive definiteness of the $\mathbb{R}^{3\times3}$ matrices (34) and (39).

7 Illustrative Numerical Example

In this section, we apply our theoretical framework to the optimal trajectory planning problem for a two-link robotic manipulator. Specifically, consider the planar manipulator shown in Fig. 1, whose first hinge has a knife-edge constraint [24, Ch. 1] orthogonal to the first link. The position of the first hinge is given by the pair $(x, y) \in \mathbb{R}\times\mathbb{R}$ and its velocity is denoted by the pair $(v, w) \in \mathbb{R}\times\mathbb{R}$. The two links are identical, their masses are denoted by m, their length by l, and their angular positions by $\theta_1 \in \mathbb{R}$ and $\theta_2 \in \mathbb{R}$, respectively. The system's configuration is uniquely defined by the vector of independent generalized coordinates $q = [x, y, \theta_1, \theta_2]^{\mathrm{T}}$ and the system's quasi-velocities are given by $q_{\mathrm{dot}} = [v, w, \dot\theta_1, \dot\theta_2]^{\mathrm{T}}$. In this problem, $N = 2$ and the knife-edge constraint is a nonholonomic constraint with $w = 0$.

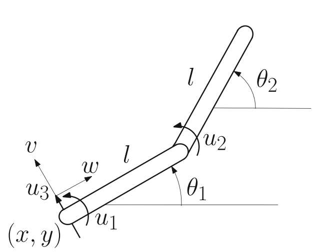

Fig. 1 A two-links robotic manipulator subject to nonholonomic constraints

The control moments $u_1 \in \mathbb{R}$ and $u_2 \in \mathbb{R}$ are applied to the first and second hinge, respectively. The control force $u_3 \in \mathbb{R}$ is applied to the first hinge along the same direction as the knife-edge constraint. Hence, $u_{1,tran} = u_3[-\sin\theta_1, \cos\theta_1, 0]^{\mathrm{T}}$, $u_{1,rot} = [0, 0, u_1]^{\mathrm{T}}$, $u_{2,tran} = 0$, and $u_{2,rot} = [0, 0, u_2]^{\mathrm{T}}$. By setting $|u_1| \le \rho_{1,4}$, $|u_2| \le \rho_{2,4}$, and $|u_3| \le \rho_{1,2}$, where $\rho_{1,2}, \rho_{1,4}, \rho_{2,4} > 0$, it follows that

$$\begin{aligned}\mathcal{G}_{1,\mathrm{tran}} &= \left\{z \in \mathbb{R}^3 : 0 \le \|z\|_2 \le \rho_{1,2}\right\},\\ \mathcal{G}_{1,\mathrm{rot}} &= \left\{z \in \mathbb{R}^3 : 0 \le \|z\|_2 \le \rho_{1,4}\right\},\\ \mathcal{G}_{2,\mathrm{tran}} &= \{0\}, \qquad \mathcal{G}_{2,\mathrm{rot}} = \left\{z \in \mathbb{R}^3 : 0 \le \|z\|_2 \le \rho_{2,4}\right\}.\end{aligned}$$

It follows from (5) that the equations of motion of this two-body system are given by [24, Ch. 5]

$$\begin{aligned}u_3(t) &= 2m\dot v(t) + \frac{3}{2}ml\ddot\theta_1(t) + \frac{1}{2}ml\ddot\theta_2(t)\cos(\theta_2(t)\\ &\quad -\theta_1(t)) - \frac{1}{2}ml^2\dot\theta_2^2(t)\sin(\theta_2(t)-\theta_1(t)),\\ &\qquad v(t_1) = v_1, \quad t \in [t_1, t_2],\end{aligned} \tag{41}$$

$$\begin{aligned}u_1(t) &= \frac{4}{3}ml^2\ddot\theta_1(t) + \frac{1}{2}ml^2\ddot\theta_2(t)\cos(\theta_2(t)-\theta_1(t))\\ &\quad +\frac{3}{2}ml\dot v(t) - \frac{1}{2}ml^2\dot\theta_2^2(t)\sin(\theta_2(t)-\theta_1(t)),\\ &\qquad \begin{bmatrix}\theta_1(t_1)\\ \theta_1(t_2)\end{bmatrix} = \begin{bmatrix}\theta_{1,1}\\ \theta_{1,2}\end{bmatrix},\end{aligned} \tag{42}$$

$$\begin{aligned}u_2(t) &= \frac{1}{3}ml^2\ddot\theta_2(t) + \frac{1}{2}ml^2\ddot\theta_1(t)\cos(\theta_2(t)-\theta_1(t))\\ &\quad +\frac{1}{2}ml\dot v(t)\cos(\theta_2(t)-\theta_1(t)) + \frac{1}{2}ml\dot\theta_1(t)\\ &\quad \cdot\left[v(t) + l\dot\theta_1(t)\right]\sin(\theta_2(t)-\theta_1(t)),\\ &\qquad \begin{bmatrix}\theta_2(t_1)\\ \theta_2(t_2)\end{bmatrix} = \begin{bmatrix}\theta_{2,1}\\ \theta_{2,2}\end{bmatrix}.\end{aligned} \tag{43}$$

The effort needed to control the robotic manipulator is captured by

$$J[u(\cdot)] = \int_{t_1}^{t_2} \|u(t)\| \mathrm{d}t, \quad (44)$$

where $u = [u_1, u_2, u_3]^{\mathrm{T}}$. An analytical solution for the problem of finding $u : [t_1, t_2] \to \mathbb{R}^3$ such that (44) is minimized and the equations of motion (41)–(43) are satisfied does not exist. For this reason, the optimization toolbox GPOPS [46] has been employed.

Let $t_1 = 0\,\mathrm{s}$, $t_2 = 1.50\,\mathrm{s}$, $\theta_1(t_1) = \theta_2(t_1) = 0$, $\theta_1(t_2) = \frac{\pi}{2}$, $\theta_2(t_2) = 0$, $v(t_1) = 0$, and $\rho_{1,2} = \rho_{1,4} = \rho_{2,4} = 4$. By setting a numerical tolerance of 10^{-3}, GPOPS evaluated the candidate optimal control $u^*(t)$, $t \in [t_1, t_2]$, the solution of the equations of motion (41)–(43), and the solution of the costate Eq. 17. Since numerical results show that $|p_0^*| + \|p^*_{\mathrm{dyn}}(t)\|_2 + \|p^*_{\mathrm{dot}}(t)\|_2 > 0$, $t \in [t_1, t_2]$, with $p_0^* = 0$, condition $i)$ of Theorem 2 is satisfied, and hence, the optimal control problem is abnormal.

Equations (28), (29), (38), and (39) have been evaluated numerically and it follows from Lemma 1 and Theorem 7 that the candidate optimal control moment $u_2^*(\cdot)$ is singular in the time interval $[\hat{t}_{1,2,\mathrm{rot}}, \hat{t}_{2,2,\mathrm{rot}}] = [0.53, 0.81]$. Therefore, as recommended in [45], in order to increase the accuracy of the numerical results, the second arm's optimal control are computed in a dedicated numerical simulation over the interval $[\hat{t}_{1,2,\mathrm{rot}}, \hat{t}_{2,2,\mathrm{rot}}]$. Figures 2 and 3 show the angular position and the candidate optimal control moments.

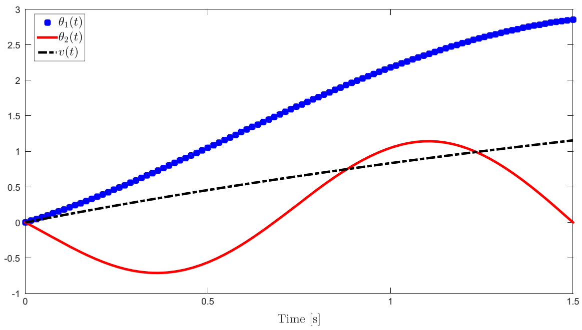

Fig. 2 Angular position of the robotic manipulator and translational velocity of the first link

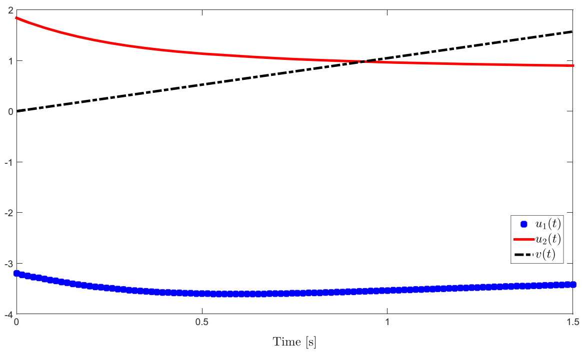

Fig. 3 Optimal controls

8 Conclusion

In this paper, we addressed the abnormal optimization problem for systems of N rigid bodies. Specifically, in the first part of the paper, we gave the equations of motion for a mechanical system subject to external forces and moments, and holonomic and nonholonomic constraints. One of the advantages of our formulation is that the equations of motion are *minimal*, that is, the system's dynamics cannot be uniquely expressed by a smaller set of differential equations. Moreover, we derived the equations of motion using the kinetic energy of the *constrained* dynamical system. In the second part of the paper, we provided a solution of the abnormal optimal control problems for systems of N rigid bodies. A numerical example illustrated the theoretical results achieved.

Since the second variation of the Hamiltonian function with respect to the control is not invertible along singular arcs [3], for the optimal control problem discussed in this paper it is not possible to prove the equivalence between the Lagrangian and the Hamiltonian formulations of the optimal control problems as shown in [9]. Future research will examine the equivalence between the Lagrangian and the Hamiltonian formulation for the problem discussed herein.

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