Semistabilization, Feedback Dissipativation, System Thermodynamics, and Limits of Performance in Feedback Control

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Abstract— **In this paper, we develop a thermodynamic framework for semistabilization of linear and nonlinear dynamical systems. The proposed framework unifies system thermodynamic concepts with feedback dissipativity and control theory to provide a thermodynamic-based semistabilization framework for feedback control design. Specifically, we consider feedback passive and dissipative systems since these systems are not only widespread in systems and control, but also have clear connections to thermodynamics. In addition, we define the notion of entropy for a nonlinear feedback dissipative dynamical system. Then, we develop a state feedback control design framework that minimizes the time-averaged system entropy and show that, under certain conditions, this controller also minimizes the time-averaged system energy. The main result is cast as an optimal control problem characterized by an optimization problem involving two linear matrix inequalities.**

I. Introduction

System thermodynamics, in the sense of [1], involves open interconnected dynamical systems that exchange matter and energy with their environment in accordance with the first law (conservation of energy) and the second law (nonconservation of entropy) of thermodynamics. Self-organization can spontaneously occur in such systems by invoking the two fundamental axioms of the science of heat. Namely, *i*) if the energies in the connected subsystems of an interconnected system are equal, then energy exchange between these subsystems is not possible, and *ii*) energy flows from more energetic subsystems to less energetic subsystems. These axioms establish the existence of a system entropy function as well as *equipartition of energy* [1] in system thermodynamics and *information consensus* [2] in cooperative networks; an emergent behavior in thermodynamic systems as well as swarm systems.

Using system-theoretic thermodynamic concepts, an energy and entropy-based hybrid controller architecture was proposed in [3], [4] as a means for achieving enhanced energy dissipation in lossless and dissipative dynamical systems. These dynamic controllers combined a logical switching architecture with continuous dynamics to guarantee that the system plant energy is strictly decreasing across switchings. The general framework developed in [3] leads to closed-loop systems described by impulsive differential equations [4]. In particular, the authors in [3], [4] construct hybrid dynamic controllers that guarantee that the closed-loop system is consistent with basic thermodynamic principles. Specifically, the existence of an entropy function for the closed-loop system is established that satisfies a hybrid Clausius-type inequality. Special cases of energy-based and entropy-based hybrid controllers involving state-dependent switching were also developed.

Recent technological advances in communications and computation have spurred a broad interest in control of networks and control over networks [5]. Network systems involve distributed decision-making for coordination of networks of dynamic agents and address a broad area of applications including cooperative control of unmanned air vehicles, microsatellite clusters, mobile robotics, and congestion control in communication networks. In many applications involving multiagent systems, groups of agents are required to agree on certain quantities of interest. In particular, it is important to develop information consensus protocols for networks of dynamic agents, wherein a unique feature of the closed-loop dynamics under any control algorithm that achieves consensus is the existence of a continuum of equilibria representing a state of equipartitioning or consensus [6], [7], [8], [9], [10]. Under such dynamics, the limiting consensus state achieved is not determined completely by the dynamics, but depends on the initial system state as well. For such systems possessing a continuum of equilibria, *semistability* [11], [12], and not asymptotic stability, is the relevant notion of stability. In addition, system-theoretic thermodynamic concepts [1], [6], [7], [8], [9] have proved invaluable in addressing Lyapunov stability and convergence for nonlinear dynamical networks.

Semistability and state equipartitioning also arise in numerous complex large-scale dynamical networks that demonstrate a degree of synchronization. System synchronization typically involves coordination of events that allows a dynamical system to operate in unison resulting in system self-organization. The onset of synchronization in populations of coupled dynamical networks have been studied for various complex networks including network models for mathematical biology, statistical physics, kinetic theory, bifurcation theory, as well as plasma physics [13]. Synchronization of firing neural oscillator populations also appears in the neuroscience literature [14], [15].

In this paper, we develop a thermodynamic framework for semistabilization of linear and nonlinear dynamical systems. The proposed framework unifies system thermodynamic concepts with feedback dissipativity and control theory to provide a thermodynamic-based semistabilization framework for feedback control design. Specifically, we consider feedback passive and dissipative systems [12], [16], [17], [18] since these systems are not only widespread in system engineering, but also have clear connections to thermodynamics [1], [16]. In addition, using ideas from [1], we define the notion of entropy for a nonlinear feedback dissipative dynamical system. Then, we develop a state feedback control design framework that minimizes the time-averaged system entropy and show that, under certain conditions, this controller also minimizes the time-averaged system energy. The main result is cast as an optimal control problem characterized by an optimization

This work was supported in part by the Air Force Office of Scientific Research under Grant FA9550-12-1-0192, the Defense Threat Reduction Agency under Grant HDTRA1-10-1-0090, and the Domenica Rea D'Onofrio Fellowship.

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problem involving two linear matrix inequalities.

II. Notation and Mathematical Preliminaries

The notation used in this paper is fairly standard. Specifically, $\mathbb{R}$ (resp., $\mathbb{C}$) denotes the set of real (resp., complex) numbers, $\overline{\mathbb{R}}_+$ denotes the set of nonnegative numbers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, $\mathbb{R}^{n\times m}$ denotes the set of $n\times m$ real matrices, $\mathbb{S}^n$ denotes the set of $n\times n$ real symmetric matrices, $(\cdot)^{\mathrm{T}}$ denotes transpose, $(\cdot)^{\#}$ denotes the group generalized inverse, spec$(\cdot)$ denotes the spectrum of a square matrix including multiplicity, I_n or I denotes the $n\times n$ identity matrix, and $\mathbf{e}$ denotes the ones vector of order n, that is, $\mathbf{e} = [1, \ldots, 1]^{\mathrm{T}}$, $\mathbf{e} \in \mathbb{R}^n$. Furthermore, we write $\|\cdot\|$ for the Euclidean vector norm, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ for the range space and the null space of a matrix A, respectively, $\mathrm{tr}(\cdot)$ for the trace operator, and $A \geq 0$ (resp., $A > 0$) to denote the fact that the Hermitian matrix A is nonnegative (respectively, positive) definite.

In this paper, we consider nonlinear dynamical systems $\mathcal{G}$ of the form

$$\dot{x}(t) = f(x(t)) + G(x(t))u(t), \quad x(0) = x_0, \quad t \geq 0, \quad (1)$$

$$y(t) = h(x(t)) + J(x(t))u(t), \quad (2)$$

where, for each $t \geq 0$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$ denotes the state vector, $u(t) \in U \subseteq \mathbb{R}^m$ denotes the control input, $y(t) \in Y \subseteq \mathbb{R}^l$ denotes the system output, and $f : \mathcal{D} \to \mathbb{R}^n$, $G : \mathcal{D} \to \mathbb{R}^{n\times m}$, $h : \mathcal{D} \to \mathbb{R}^l$, and $J : \mathcal{D} \to \mathbb{R}^{l\times m}$. For the dynamical system $\mathcal{G}$ given by (1) and (2) defined on the state space $\mathcal{D} \subseteq \mathbb{R}^n$, $\mathcal{U}$ and $\mathcal{Y}$ define input and output spaces, respectively, consisting of continuous bounded U-valued and Y-valued functions on the semi-infinite interval $[0, \infty)$. The spaces $\mathcal{U}$ and $\mathcal{Y}$ are assumed to be closed under the shift operator. The mappings $f(\cdot)$, $G(\cdot)$, $h(\cdot)$, and $J(\cdot)$ are assumed to be continuously differentiable and $f(\cdot)$ has at least one equilibrium point $x_{\mathrm{e}} \in \mathcal{D}$ so that $f(x_{\mathrm{e}}) + G(x_{\mathrm{e}})u_{\mathrm{e}} = 0$ and $y_{\mathrm{e}} = h(x_{\mathrm{e}}) + J(x_{\mathrm{e}})u_{\mathrm{e}}$ for some $u_{\mathrm{e}} \in U$. Finally, we assume that $\mathcal{G}$ is completely reachable [12].

III. Feedback Dissipativation and Thermodynamics

In this section, a thermodynamic state feedback control framework is proposed based on the notion of the thermodynamic entropy developed in [1]. The following definition of *feedback dissipativation* is needed for developing the main results in the paper. Feedback dissipative systems define a class of dynamical systems for which a continuously differentiable feedback transformation exists that renders the system $\mathcal{G}$ dissipative and is a generalization of the *feedback passivation* notion introduced in [18].

Definition 3.1: $\mathcal{G}$ is called *state feedback dissipative* if there exists a state feedback transformation $u = \phi(x) + \beta(x)v$, where $\phi : \mathcal{D} \to \mathbb{R}^m$ and $\beta : \mathcal{D} \to \mathbb{R}^{m\times m}$ are continuously differentiable, with $\det\beta(x) \neq 0$, $x \in \mathcal{D}$, such that the nonlinear dynamical system $\mathcal{G}_{\mathrm{s}}$ given by

$$\dot{x}(t) = f(x(t)) + G(x(t))\phi(x(t)) + G(x(t))\beta(x(t))v(t), \quad x(0) = x_0, \quad t \geq 0, \quad (3)$$

$$y(t) = h(x(t)) + J(x(t))\phi(x(t)) + J(x(t))\beta(x(t))v(t), \quad (4)$$

is dissipative with respect to the supply rate $r(v, y)$, where $r : U \times Y \to \mathbb{R}$ is locally integrable for all input-output pairs satisfying (3) and (4), and $r(0, 0) = 0$. If $r(v, y) = v^{\mathrm{T}}y$, then $\mathcal{G}$ is *state feedback passive*.

For simplicity of exposition, in the reminder of the paper we will assume that $\beta(x) = I_m$.

Remark 3.1: The nonlinear dynamical system $\mathcal{G}$ given by (1) and (2) is feedback equivalent to a passive system with a C^2 storage function if and only if $\mathcal{G}$ has (vector) relative degree $\{1, \ldots, 1\}$ at $x = 0$ and is weakly minimum phase. Alternatively, the Kalman-Yakubovich-Popov lemma [12] can be used to construct smooth state feedback controllers that guarantee feedback passivation as well as feedback dissipativation [18].

The following result is a direct consequence of dissipativity theory [12]. For this result as well as for the reminder of the paper we assume that all storage functions $V_{\mathrm{s}}(\cdot)$ of the nonlinear dynamical system $\mathcal{G}_{\mathrm{s}}$ are continuously differentiable.

Proposition 3.1 ([12]): Consider the nonlinear dynamical system $\mathcal{G}$ given by (1) and (2), and assume that $\mathcal{G}$ is state feedback dissipative. Then there exist functions $V_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}$, $\ell : \mathbb{R}^n \to \mathbb{R}^p$, and $\mathcal{W} : \mathbb{R}^n \to \mathbb{R}^{p\times m}$ such that $V_{\mathrm{s}}(\cdot)$ is continuously differentiable and nonnegative definite, $V_{\mathrm{s}}(x_{\mathrm{e}}) = V_{\mathrm{se}}$, and $\dot{V}_{\mathrm{s}}(x) = r(v, y) - [\ell(x) + \mathcal{W}(x)v]^{\mathrm{T}}[\ell(x) + \mathcal{W}(x)v]$.

Defining $d(x, v) \triangleq [\ell(x) + \mathcal{W}(x)v]^{\mathrm{T}}[\ell(x) + \mathcal{W}(x)v]$, where $d : \mathcal{D} \times U \to \overline{\mathbb{R}}_+$ is a continuous, nonnegative-definite dissipation rate function, and $\mathrm{d}Q(t) \triangleq [r(v(t), y(t)) - d(x(t), v(t))]\mathrm{d}t$, where $\mathrm{d}Q(t)$ is the amount of energy (heat) received or dissipated by the state feedback dissipative system over the infinitesimal time interval $\mathrm{d}t$, we arrive at a *Clausius-type equality* for $\mathcal{G}_{\mathrm{s}}$. For the next result $\oint$ denotes a cyclic integral evaluated along an arbitrary closed path of $\mathcal{G}_{\mathrm{s}}$, that is, $\oint \triangleq \int_{t_0}^{t_{\mathrm{f}}}$ with $t_{\mathrm{f}} \geq t_0$ and $v(\cdot) \in \mathcal{U}$ such that $x(t_{\mathrm{f}}) = x(t_0) = x_0 \in \mathcal{D}$.

Proposition 3.2: Consider the nonlinear dynamical system $\mathcal{G}$ given by (1) and (2), and assume that $\mathcal{G}$ is state feedback dissipative. Then, for all $t_{\mathrm{f}} \geq t_0 \geq 0$ and $v(\cdot) \in \mathcal{U}$ such that $V_{\mathrm{s}}(x(t_{\mathrm{f}})) = V_{\mathrm{s}}(x(t_0))$,

$$\int_{t_0}^{t_{\mathrm{f}}} \frac{r(v(t), y(t)) - d(x(t), v(t))}{c + V_{\mathrm{s}}(x(t))}\mathrm{d}t = \oint \frac{\mathrm{d}Q(t)}{c + V_{\mathrm{s}}(x(t))} = 0, \quad (5)$$

where $c > 0$.

Proof: It follows from Proposition 3.1 that

$$\oint \frac{\mathrm{d}Q(t)}{c + V_{\mathrm{s}}(x(t))} = \int_{t_0}^{t_{\mathrm{f}}} \frac{r(v(t), y(t)) - d(x(t), v(t))}{c + V_{\mathrm{s}}(x(t))}\mathrm{d}t = \log_e \frac{c + V_{\mathrm{s}}(x(t_{\mathrm{f}}))}{c + V_{\mathrm{s}}(x(t_0))}, \quad (6)$$

which proves the assertion. ∎

In light of Proposition 3.2, we give a definition of entropy for a feedback dissipative system.

Definition 3.2: For the nonlinear dynamical system $\mathcal{G}_{\mathrm{s}}$ given by (3) and (4) a function $S : \mathcal{D} \to \mathbb{R}$ satisfying

$$S(x(t_2)) \geq S(x(t_1)) + \int_{t_1}^{t_2} \frac{\mathrm{d}Q(t)}{c + V_{\mathrm{s}}(x(t))} \quad (7)$$

for every $t_2 \geq t_1 \geq 0$ and $v(\cdot) \in \mathcal{U}$ is called the *entropy* function of $\mathcal{G}_{\mathrm{s}}$.

Recalling that $\mathrm{d}Q(t) = [r(v(t), y(t)) - d(x(t), v(t))]\mathrm{d}t$ is the infinitesimal amount of the net energy received or dissipated by $\mathcal{G}_{\rm s}$ over the infinitesimal time interval $\mathrm{d}t$, it follows from (7) that

$$\mathrm{d}S(x(t)) \geq \frac{\mathrm{d}Q(t)}{c+V_{\rm s}(x(t))}, \quad t \geq t_0. \tag{8}$$

Inequality (8) is analogous to the classical thermodynamic inequality for the variation of entropy during an infinitesimal irreversible transformation with the shifted system energy $c+V_{\rm s}(x)$ playing the role of the thermodynamic temperature. Specifically, note that since $\frac{\mathrm{d}S}{\mathrm{d}Q} = \frac{1}{c+V_{\rm s}}$, it follows that $\frac{\mathrm{d}S}{\mathrm{d}Q}$ defines the reciprocal of the system thermodynamic temperature $T_{\rm e}$. That is, $\frac{1}{T_{\rm e}} \triangleq \frac{\mathrm{d}S}{\mathrm{d}Q}$ and $T_{\rm e} > 0$.

The next result shows that all entropy functions for a nonlinear dynamical system $\mathcal{G}_{\rm s}$ given by (3) and (4) are continuous on $\mathcal{D}$. For stating this result, recall that the nonlinear dynamical system $\mathcal{G}_{\rm s}$ given by (3) and (4) with $\hat{x} \in \mathbb{R}^n$ and $\hat{v} \in \mathbb{R}^m$ such that $x(t) \equiv \hat{x}$ and $v(t) \equiv \hat{v}$, $t \geq 0$, satisfying (3), is *locally controllable* at $\hat{x}$ if, for every $T > 0$ and $\varepsilon > 0$, the set of points that can be reached from and to $\hat{x}$ in finite time T using admissible inputs $v : [0, T] \to U$, satisfying $\|v(t) - \hat{v}\| < \varepsilon$, contains a neighborhood of $\hat{x}$ [12, p. 333].

Theorem 3.1: Consider the dissipative nonlinear dynamical system $\mathcal{G}_{\rm s}$ given by (3) and (4). Assume that $\mathcal{G}_{\rm s}$ is completely reachable and assume that for every $x_{\rm e} \in \mathcal{D}$, there exists $v_{\rm e} \in \mathbb{R}^m$ such that $x(t) \equiv x_{\rm e}$ and $v(t) \equiv v_{\rm e}$, $t \geq 0$, satisfy (3), and $\mathcal{G}_{\rm s}$ is locally controllable at every $x_{\rm e} \in \mathcal{D}$. Then every entropy function $S(x)$, $x \in \mathcal{D}$, of $\mathcal{G}_{\rm s}$ is continuous on $\mathcal{D}$.

Proof: Let $x_{\rm e} \in \mathcal{D}$ be an equilibrium point of $\mathcal{G}_{\rm s}$ with $v(t) \equiv v_{\rm e}$, that is, $f(x_{\rm e}) + G(x_{\rm e})\phi(x_{\rm e}) + G(x_{\rm e})v_{\rm e} = 0$. Now, let $\delta > 0$ and note that it follows from the continuity of $f(\cdot)$, $G(\cdot)$, and $\phi(\cdot)$ that there exist $T > 0$ and $\varepsilon > 0$ such that for every $v : [0, T) \to \mathbb{R}^n$ and $\|v(t) - v_{\rm e}\| < \varepsilon$, $\|x(t) - x_{\rm e}\| < \delta$, $t \in [0, T)$, where $v(\cdot) \in \mathcal{U}$ and $x(t)$, $t \in [0, T)$, denotes the solution to (1) with the initial condition $x_{\rm e}$. Furthermore, it follows from the local controllability of $\mathcal{G}_{\rm s}$ that for every $\hat{T} \in (0, T]$, there exists a strictly increasing, continuous function $\gamma : \mathbb{R} \to \mathbb{R}$ such that $\gamma(0) = 0$, and for every $x_0 \in \mathcal{D}$ such that $||x_0 - x_{\rm e}|| \leq \gamma(\hat{T})$, there exist $\hat{t} \in [0, \hat{T}]$ and an input $v : [0, \hat{T}] \to \mathbb{R}^m$ such that $\|v(t) - v_{\rm e}\| < \varepsilon$, $t \in [0, \hat{t})$, and $x(\hat{t}) = x_0$. Hence, there exists $\rho > 0$ such that for every $x_0 \in \mathcal{D}$ such that $\|x_0 - x_{\rm e}\| \leq \rho$, there exists $\hat{t} \in [0, \gamma^{-1}(\|x_0 - x_{\rm e}\|)]$ and an input $v : [0, \hat{t}] \to \mathbb{R}^n$ such that $\|v(t) - v_{\rm e}\| < \varepsilon$, $t \in [0, \hat{t}]$, and $x(\hat{t}) = x_0$.

Since $r(\cdot, \cdot)$ is locally integrable for all input-output pairs satisfying (3) and (4), there exists $M \in (0, \infty)$ such that

$$\left| \int_0^{\hat{t}} \frac{r(v(\sigma), y(\sigma)) - d(x(\sigma), v(\sigma))}{c + V_{\rm s}(x(\sigma))} \mathrm{d}\sigma \right| \leq \int_0^{\hat{t}} \left| \frac{\mathrm{d}Q(\sigma)}{c + V_{\rm s}(x(\sigma))} \right| \leq M\gamma^{-1}(\|x_0 - x_{\rm e}\|). \tag{9}$$

Now, if $S(\cdot)$ is an entropy function of $\mathcal{G}_{\rm s}$, then

$$-\int_0^{\hat{t}} \frac{\mathrm{d}Q(\sigma)}{c + V_{\rm s}(x(\sigma))} \geq S(x_{\rm e}) - S(x(\hat{t})). \tag{10}$$

If $S(x_{\rm e}) \geq S(x(\hat{t}))$, then combining (9) and (10) yields

$$|S(x_{\rm e}) - S(x(\hat{t}))| \leq M\gamma^{-1}(\|x_0 - x_{\rm e}\|). \tag{11}$$

Alternatively, if $S(x(\hat{t})) \geq S(x_{\rm e})$, then (11) can be derived by reversing the roles of $x_{\rm e}$ and $x(\hat{t})$ and using the assumption that $\mathcal{G}_{\rm s}$ is locally controllable from and to $x_{\rm e}$. Hence, since $\gamma(\cdot)$ is continuous and $x(\hat{t})$ is arbitrary, it follows that $S(\cdot)$ is continuous on $\mathcal{D}$. ■

Next, we characterize a continuously differentiable entropy function for state feedback dissipative systems.

Proposition 3.3: Consider the dissipative nonlinear dynamical system $\mathcal{G}_{\rm s}$ given by (3) and (4). Then the continuously differentiable function $S : \mathcal{D} \to \mathbb{R}$ given by

$$S(x) \triangleq \log_e [c + V_{\rm s}(x)] - \log_e c, \tag{12}$$

where $c > 0$, is an entropy function of $\mathcal{G}_{\rm s}$.

Proof: Using Proposition 3.1 it follows that

$$\dot{S}(x(t)) = \frac{\dot{V}_{\rm s}(x(t))}{c + V_{\rm s}(x(t))} = \frac{\dot{Q}(t)}{c + V_{\rm s}(x(t))}, \quad t \geq 0. \tag{13}$$

Now, integrating (13) over $[t_1, t_2]$ yields (7). ■

Remark 3.2: In [1], the authors show that the entropy function for an energy balance equation involving a large-scale, compartmental thermodynamic model is unique. However, whether or not there exists a unique continuously differentiable entropy function for $\mathcal{G}_{\rm s}$ given by (3) and (4) is an open problem.

Finally, the following result presenting an upper and lower bound of the entropy function for a state feedback dissipative system is needed for later developments.

Proposition 3.4: Consider the nonlinear dynamical system $\mathcal{G}_{\rm s}$ given by (3) and (4), and let $S : \mathcal{D} \to \mathbb{R}$ given by (12) be an entropy function of $\mathcal{G}_{\rm s}$. Then,

$$\frac{V_{\rm s}(x)}{c + V_{\rm s}(x)} \leq S(x) \leq \frac{1}{c} V_{\rm s}(x), \quad x \in \mathcal{D}. \tag{14}$$

Proof: Note that (12) can be rewritten as $S(x) = \log_e[1 + V_{\rm s}(x)/c]$. The assertion is a direct consequence of the inequality $z/(1+z) \leq \log_e(1+z) \leq z$, $z > -1$. ■

IV. THERMODYNAMIC SEMISTABILIZATION

In this section, we use the results of Section III to present a framework for *semistabilization* of nonlinear systems. Semistabilization is the property of controlled dynamical systems possessing a continuum of equilibria whereby every closed-loop system trajectory that starts in a neighborhood of a Lyapunov stable equilibrium converges to a (possibly different) Lyapunov stable equilibrium [12].

To address the state feedback, thermodynamic-based semistabilization problem, consider the nonlinear dynamical system $\mathcal{G}_{\rm s}$ given by (3) and (4) with performance criterion

$$J(x_0, \phi(\cdot)) = \lim_{t \to \infty} \left[\frac{1}{t} \int_0^t S(x(\sigma)) \mathrm{d}\sigma \right]. \tag{15}$$

The performance criterion $J(x_0, \phi(\cdot))$ can be interpreted as the time-average of the entropy function for the dissipative nonlinear dynamical system $\mathcal{G}_{\rm s}$. The key feature of this optimal control problem is that it addresses semistability instead of asymptotic stability. In the absence of energy exchange with the environment, a thermodynamically consistent nonlinear dynamical system model possesses a continuum of equilibria, and hence, is semistable; that is, the system states converge to Lyapunov energy equilibria determined by the system initial conditions [1]. A key question that arises is whether or not this optimal control problem is well defined; that is, whether $J(x_0, \phi(\cdot))$ is finite and if there exists a state feedback controller such that $J(x_0, \phi(\cdot))$ is minimized. The first question is addressed by the following proposition.

Proposition 4.1: Consider the nonlinear dissipative dynamical system $\mathcal{G}_{\rm s}$ given by (3) and (4). If there exists

$\phi : \mathcal{D} \to \mathbb{R}^m$ such that (3), with $v(t) \equiv 0$, is semistable, then $|J(x_0, \phi(\cdot))| < \infty$.

Proof: Since (3) with $v(t) \equiv 0$ is semistable, $x(t)$ is bounded for all $t \geq 0$. It follows from Theorem 3.1 that $S(\cdot)$ is a continuous entropy function on $\mathcal{D}$ for $\mathcal{G}_{\rm s}$. Hence, $S(x(t))$ is bounded for all $t \geq 0$. Now, let $|S(x(t))| \leq c$ for all $t \geq 0$. Then, $-c \leq (1/t)\int_0^t S(x(\sigma)){\rm d}\sigma \leq c$ for all $t \geq 0$, which proves the result. ∎

To address the question of existence of a semistabilizing controller such that $J(x_0, \phi(\cdot))$ given by (15) is minimized, we consider an *auxiliary minimization problem* involving the performance criterion

$$\mathcal{J}(x_0, \phi(\cdot)) = \lim_{t\to\infty}\left[\frac{1}{t}\int_0^t V_{\rm s}(x(\sigma)){\rm d}\sigma\right]. \qquad (16)$$

Hence, it follows from the auxiliary minimization problem that we seek feedback controllers that minimize the stored energy in the system in order to attain a stable energy level determined by the system initial conditions and the control system effort.

The following lemma is necessary for proving the main result of this section.

Lemma 4.1: Consider the dissipative nonlinear dynamical system $\mathcal{G}_{\rm s}$ given by (3) and (4) with continuously differentiable storage function $V_{\rm s} : \mathcal{D} \to \mathbb{R}_+$. Suppose there exists $\phi^* : \mathcal{D} \to \mathbb{R}^m$ such that (3), with $v(t) \equiv 0$, is semistable, $\dot{V}_{\rm s}(x(t)) \leq 0$, $t \geq 0$, and $\mathcal{J}(x_0, \phi(\cdot))$ is minimized. If $\mathcal{J}(x_0, \phi^*(\cdot)) = 0$, then $\arg\min_{\phi(\cdot)\in\mathbb{R}^m} \mathcal{J}(x_0, \phi(\cdot)) = \arg\min_{\phi(\cdot)\in\mathbb{R}^m} J(x_0, \phi(\cdot))$ and $J(x_0, \phi^*(\cdot)) = 0$. Alternatively, if $\mathcal{J}(x_0, \phi^*(\cdot)) \neq 0$, then $J(x_0, \phi^*(\cdot)) = S(x_{\rm e})$, where $x_{\rm e} = \lim_{t\to\infty} x(t)$.

Proof: It follows from Proposition 3.4 and $\dot{V}_{\rm s}(x(t)) \leq 0$, $t \geq 0$, that

$$\frac{V_{\rm s}(x(t))}{c + V_{\rm s}(x(0))} \leq S(x(t)) \leq \frac{V_{\rm s}(x(t))}{c}, \quad t \geq 0. \qquad (17)$$

Hence, $\frac{\mathcal{J}(x_0,\phi(\cdot))}{(c+V_{\rm s}(x(0)))} \leq J(x_0, \phi(\cdot)) \leq \frac{\mathcal{J}(x_0,\phi(\cdot))}{c}$. Now, if $\mathcal{J}(x_0, \phi^*(\cdot)) = 0$, then $J(x_0, \phi(\cdot))$ is minimized and $J(x_0, \phi^*(t)) = 0$, $t \geq 0$.

Alternatively, if $\mathcal{J}(x_0, \phi^*(\cdot)) \neq 0$, then it follows from the definition of $\mathcal{J}(x_0, \phi^*(\cdot))$ that there exists $c^* > 0$ such that $\mathcal{J}(x_0, \phi^*(t)) \geq c^*$ for all $t \geq 0$, and hence, $\lim_{t\to\infty} \mathcal{J}(x_0, \phi^*(t))t/(c + V_{\rm s}(x(0))) = \infty$. Thus, $\lim_{t\to\infty}\int_0^t S(x(\sigma)){\rm d}\sigma = \infty$. It follows from l'Hôpital's rule that $J(x_0, \phi^*(t)) = S(x_{\rm e})$, where $x_{\rm e} = \lim_{t\to\infty} x(t)$. ∎

Theorem 4.1: Consider the dissipative nonlinear dynamical system $\mathcal{G}_{\rm s}$ given by (3) and (4) with continuously differentiable storage function $V_{\rm s} : \mathcal{D} \to \mathbb{R}_+$. Assume that there exists $\phi^* : \mathcal{D} \to \mathbb{R}^m$ such that (16) is minimized, (3), with $v(t) \equiv 0$, is semistable, and $\dot{V}_{\rm s}(x(t)) \leq 0$, $t \geq 0$. Then, for $S : \mathcal{D} \to \mathbb{R}$ given by (12), $\arg\min_{\phi(\cdot)\in\mathbb{R}^m} \mathcal{J}(x_0, \phi(\cdot)) = \arg\min_{\phi(\cdot)\in\mathbb{R}^m} J(x_0, \phi(\cdot))$.

Proof: If $\mathcal{J}(x_0, \phi^*(\cdot)) = 0$, then it follows from Lemma 4.1 that $\phi^*(\cdot) = \arg\min_{\phi(\cdot)\in\mathbb{R}^m} J(x_0, \phi(\cdot))$. Alternatively, if $\mathcal{J}(x_0, \phi^*(\cdot)) \neq 0$, then, using similar arguments as in the proof of Lemma 4.1, $\lim_{t\to\infty} t\mathcal{J}(x_0, \phi^*(t)) = \infty$, and hence, $\int_0^t V_{\rm s}(x(\sigma)){\rm d}\sigma = \infty$ as $t \to \infty$. Hence, using l'Hôpital's rule, it follows that $\mathcal{J}(x_0, \phi^*(\cdot)) = V_{\rm s}(x_{\rm e})$.

Next, since for all $\phi : \mathcal{D} \to \mathbb{R}^m$ such that $\mathcal{J}(x_0, \phi(\cdot))$ is finite, $\lim_{t\to\infty} \mathcal{J}(x_0, \phi(t))t/(c+V_{\rm s}(x(0))) = \infty$, and hence, using (17), it follows that $\lim_{t\to\infty}\int_0^t S(x(\sigma)){\rm d}\sigma = \infty$. Consequently, for all $\phi : \mathcal{D} \to \mathbb{R}^m$ such that $\mathcal{J}(x_0, \phi(\cdot))$ is finite and $\mathcal{G}_{\rm s}$ is semistable, it follows from l'Hôpital's rule that $J(x_0, \phi(\cdot)) = S(x_{\rm e}) = \log_e(1 + V_{\rm s}(x_{\rm e}))$, where $x_{\rm e} = \lim_{t\to\infty} x(t)$. Next, assume that $\phi^*_{\rm J} : \mathcal{D} \to \mathbb{R}^m$ is such that $\mathcal{G}_{\rm s}$ is semistable, $\dot{V}_{\rm s}(x(t)) \leq 0$, $t \geq 0$, and $J(x_0, \phi(\cdot))$ is minimized. Then, it follows that $J(x_0, \phi^*_{\rm J}(\cdot)) = S(x_{\rm e}) = \log_e(1 + V_{\rm s}(x_{\rm e}))$. By uniqueness of solutions of $x(\cdot)$ it follows that $\phi^*(\cdot)$ uniquely determines $x_{\rm e}$ and $\dot{V}_{\rm s}(x(t))$, $t \geq 0$. Choosing $V_{\rm s}(x_{\rm e}) = V_{\rm se}$, where $V_{\rm se} \in \mathbb{R}$, it follows that $\phi^*(\cdot)$ uniquely determines $V_{\rm s}(x_{\rm e})$, and hence, $J(x_0, \phi^*(\cdot)) = \log_e(1 + V_{\rm s}(x_{\rm e}))$, which proves the result. ∎

It follows from Theorem 4.1 that an optimal semistable controller minimizing $\mathcal{J}(x_0, v(\cdot))$ given by (16) also minimizes the entropy functional $J(x_0, v(\cdot))$ given by (15). Since quadratic cost functions arise naturally in dissipativity theory [17], [12], [19], addressing the auxiliary cost (16) can be simpler than addressing the entropy (logarithmic) cost functional (15).

V. Thermodynamic Semistabilization of Linear Systems

In this section, we address the problem of semistabilizing optimal controllers for linear systems so that $\mathcal{G}$ is given by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad t \geq 0, \qquad (18)$$

$$y(t) = Cx(t) + Du(t), \qquad (19)$$

where, for each $t \geq 0$, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^l$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{l\times n}$, and $D \in \mathbb{R}^{l\times m}$. Given $u = Kx+v$, $K \in \mathbb{R}^{m\times n}$, we assume that $\mathcal{G}$ is state feedback dissipative, that is, the nonlinear dynamical systems $\mathcal{G}_{\rm s}$ given by (3) and (4) takes the form

$$\dot{x}(t) = \tilde{A}x(t) + Bv(t), \quad x(0) = x_0, \quad t \geq 0, \qquad (20)$$

$$y(t) = \tilde{C}x(t) + Dv(t), \qquad (21)$$

where $\tilde{A} \triangleq A+BK$, $\tilde{C} \triangleq C+DK$, and $\mathcal{G}_{\rm s}$ is dissipative with respect to the supply rate $r(v, y)$, where $r : \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}$ is locally integrable for all input-output pairs satisfying (20) and (21), and $r(0, 0) = 0$. For the reminder of the paper define $\mathcal{K} \triangleq \{K \in \mathbb{R}^{m\times n} : A + BK \text{ is semistable}\}$. In this case, Theorem 4.1 specializes to the following result.

Theorem 5.1: Consider the dissipative dynamical system $\mathcal{G}_{\rm s}$ given by (20) and (21) with continuously differentiable storage function $V_{\rm s} : \mathbb{R}^n \to \mathbb{R}_+$. Assume there exists $K^* \in \mathcal{K}$ that minimizes (16) and $\dot{V}_{\rm s}(x(t)) \leq 0$, $t \geq 0$, where $x(t)$, $t \geq 0$, satisfies (20) with $v(t) \equiv 0$. Then, for $S : \mathbb{R}^n \to \mathbb{R}$ given by (12), $\arg\min_{K\in\mathcal{K}} \mathcal{J}(x_0, K) = \arg\min_{K\in\mathcal{K}} J(x_0, K)$.

For the reminder of the paper, we consider the special case of dissipative systems $\mathcal{G}_{\rm s}$ with quadratic supply rates. Specifically, we set $\mathcal{D} = \mathbb{R}^n$, $U = \mathbb{R}^m$, and $Y = \mathbb{R}^l$, and let

$$r(v, y) = y^{\rm T}Qy + 2y^{\rm T}Zv + v^{\rm T}Rv, \qquad (22)$$

where $Q \in \mathbb{S}^n$, $Z \in \mathbb{R}^{l\times m}$, and $R \in \mathbb{S}^m$ [17]. It follows from Theorem 5.9 of [12] that in this case the linear system $\mathcal{G}_{\rm s}$ given by (20) and (21) possesses a quadratic storage function $V_{\rm s}(x) = x^{\rm T}Px$, where $P = P^{\rm T} \geq 0$ satisfies

$$0 = \tilde{A}^{\rm T}P + P\tilde{A} - \tilde{C}^{\rm T}Q\tilde{C} + L^{\rm T}L, \qquad (23)$$

$$0 = PB - \tilde{C}^{\rm T}(QD + Z) + L^{\rm T}W, \qquad (24)$$

$$0 = \tilde{R} - W^{\rm T}W, \qquad (25)$$

where $L \in \mathbb{R}^{p\times n}$, $W \in \mathbb{R}^{p\times m}$, and $\tilde{R} \triangleq R + Z^{\mathrm{T}}D + D^{\mathrm{T}}Z + D^{\mathrm{T}}ZD$. In this case, $\mathcal{J}(x_0, K)$ has the form

$$\mathcal{J}(x_0, K) = \lim_{t\to\infty} \frac{1}{t}\int_0^t x^{\mathrm{T}}(s)Px(s)\mathrm{d}s.$$

To eliminate the dependence of the initial condition x_0 on $\mathcal{J}(x_0, K)$ and $J(x_0, K)$, we assume that the initial state x_0 is a random variable such that $\mathbb{E}[x_0] = 0$ and $\mathbb{E}[x_0 x_0^{\mathrm{T}}] = V$, where $\mathbb{E}$ denotes the expectation operator.

Proposition 5.1: Assume that $\mathcal{G}_{\mathrm{s}}$ given by (20) and (21) is dissipative with respect to the quadratic supply rate (22) and suppose there exists $K \in \mathcal{K}$ and $\dot{V}_{\mathrm{s}}(x(t)) \leq 0$, $t \geq 0$, where $x(t)$, $t \geq 0$, satisfies (20) with $v(t) \equiv 0$. Then there exists an $n \times n$ nonnegative-definite matrix P such that (23)–(25) hold and, with $v(t) \equiv 0$,

$$\begin{aligned}\mathcal{J}(K) &= x_0^{\mathrm{T}}[I_n - \tilde{A}^{\mathrm{T}}(\tilde{A}^{\mathrm{T}})^{\#}]P[I_n - \tilde{A}\tilde{A}^{\#}]x_0 \\ &= \mathrm{tr}[I_n - \tilde{A}^{\mathrm{T}}(\tilde{A}^{\mathrm{T}})^{\#}]P[I_n - \tilde{A}\tilde{A}^{\#}]V. \qquad (26)\end{aligned}$$

Proof: Since $\mathcal{G}_{\mathrm{s}}$ is dissipative with respect to the quadratic supply rate (22), it follows from Theorem 5.9 of [12] that there exists $P = P^{\mathrm{T}} \geq 0$ such that (23)–(25) hold and $V_{\mathrm{s}}(x) = x^{\mathrm{T}}Px$ is a storage function for $\mathcal{G}_{\mathrm{s}}$. Since for $v(t) \equiv 0$, $\dot{V}_{\mathrm{s}}(x(t)) \leq 0$, $t \geq 0$, $\tilde{A}$ is semistable, and $x(t) = e^{\tilde{A}t}x_0$, $t \geq 0$, it follows that

$$\mathcal{J}(K) = \lim_{t\to\infty}\frac{1}{t}\int_0^t [x_0^{\mathrm{T}}e^{\tilde{A}^{\mathrm{T}}\tau}Pe^{\tilde{A}\tau}x_0]\mathrm{d}\tau = \mathrm{tr}\,\tilde{\mathcal{A}}_{\tilde{A}}(P)V,$$

where

$$\tilde{\mathcal{A}}_{\tilde{A}}(P) \triangleq \lim_{t\to\infty}\frac{1}{t}\int_0^t e^{\tilde{A}^{\mathrm{T}}\tau}Pe^{\tilde{A}\tau}\mathrm{d}\tau.$$

Now, since $\lim_{t\to\infty} e^{\tilde{A}^{\mathrm{T}}\tau}Pe^{\tilde{A}\tau}$ is finite, $\tilde{\mathcal{A}}_{\tilde{A}}(P) = \lim_{t\to\infty} e^{\tilde{A}^{\mathrm{T}}t}Pe^{\tilde{A}t}$. In addition, since $\tilde{A}$ is semistable, $\lim_{t\to\infty} e^{\tilde{A}t} = I_n - \tilde{A}\tilde{A}^{\#}$ [20]. Hence, $\lim_{t\to\infty} e^{\tilde{A}^{\mathrm{T}}t}Pe^{\tilde{A}t} = [I_n - \tilde{A}^{\mathrm{T}}(\tilde{A}^{\mathrm{T}})^{\#}]P[I_n - \tilde{A}\tilde{A}^{\#}]$, which proves the result. ■

Remark 5.1: Define the operator $\mathcal{L}_{\tilde{A}} : \mathbb{S}^n \to \mathbb{S}^n$ by

$$\mathcal{L}_{\tilde{A}}(P) \triangleq \tilde{A}^{\mathrm{T}}P + P\tilde{A}. \qquad (27)$$

It follows from Proposition 4.1 of [21] that $\mathcal{N}(\mathcal{L}_{\tilde{A}}) = \mathcal{R}(\tilde{\mathcal{A}}_{\tilde{A}})$ and $\mathcal{N}(\tilde{\mathcal{A}}_{\tilde{A}}) = \mathcal{R}(\mathcal{L}_{\tilde{A}})$. This implies that $V_{\mathrm{s}}(x) = x^{\mathrm{T}}Px$ is an integral of motion of

$$\dot{x}(t) = \tilde{A}x(t), \qquad x(0) = x_0, \qquad t \geq 0, \qquad (28)$$

if and only if $x^{\mathrm{T}}\tilde{\mathcal{A}}_{\tilde{A}}(P)x$ is the average over $[0, \infty)$ of $V_{\mathrm{s}}(x) = x^{\mathrm{T}}Px$ along the solutions of $\dot{x}(t) = \tilde{A}x(t)$. Furthermore, the elements of $\mathcal{N}(\tilde{\mathcal{A}}_{\tilde{A}})$ are quadratic functions that have zero average along the trajectories of $\dot{x}(t) = \tilde{A}x(t)$ if and only if $x \mapsto x^{\mathrm{T}}\mathcal{L}_{\tilde{A}}(P)x$ is the Lie derivative of $x \mapsto x^{\mathrm{T}}Px$ along the trajectories of $\dot{x}(t) = \tilde{A}x(t)$ for every $P \in \mathbb{S}^n$ [21].

The following definition is needed.

Definition 5.1 ([10]): Let $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{m\times n}$. The pair (A, C) is *semiobservable* if

$$\bigcap_{k=1}^{n} \mathcal{N}(CA^{k-1}) = \mathcal{N}(A). \qquad (29)$$

The following lemma provides necessary and sufficient conditions for a feedback gain matrix K to belong to the set $\mathcal{K}$.

Lemma 5.1 ([10]): The linear dynamical system $\mathcal{G}_{\mathrm{s}}$ given by (20), with $v(t) \equiv 0$, is semistable if and only if for every semiobservable pair $(\tilde{A}, \hat{R})$, where $\hat{R} = \hat{R}^{\mathrm{T}} \geq 0$ and $\tilde{A} = A + BK$, there exists $\hat{P} \in \mathbb{R}^{n\times n}$ such that $\hat{P} = \hat{P}^{\mathrm{T}} > 0$ and

$$0 = \tilde{A}^{\mathrm{T}}\hat{P} + \hat{P}\tilde{A} + \hat{R}. \qquad (30)$$

It is worth recalling that $\hat{P}$ is not unique [10]. The next result characterizes state feedback thermodynamic semistabilizing controllers using linear matrix inequalities.

Theorem 5.2: Consider the linear dynamical system $\mathcal{G}_{\mathrm{s}}$ given by (20) and (21), let Q characterizing the supply rate $r(v, y)$ given by (22) be such that $Q \leq 0$, and let $\hat{R} \geq 0$. Then K^* minimizes

$$\mathcal{J}(K) = \mathrm{tr}[I_n - \tilde{A}^{\mathrm{T}}(\tilde{A}^{\mathrm{T}})^{\#}]P[I_n - \tilde{A}\tilde{A}^{\#}]V, \qquad (31)$$

subject to

$$(\tilde{A}, \hat{R}) \text{ is semiobservable}, \qquad (32)$$

$$0 \geq \begin{bmatrix} \tilde{A}^{\mathrm{T}}P + P\tilde{A} - \tilde{C}^{\mathrm{T}}Q\tilde{C} & PB - \tilde{C}^{\mathrm{T}}(QD + Z) \\ B^{\mathrm{T}}P - (QD + Z)^{\mathrm{T}}\tilde{C} & \tilde{R} \end{bmatrix}, \qquad (33)$$

$$0 \geq \tilde{A}^{\mathrm{T}}\hat{P} + \hat{P}\tilde{A}, \qquad (34)$$

where $P = P^{\mathrm{T}} \geq 0$, $P \in \mathbb{R}^{n\times n}$, and $\hat{P} = \hat{P}^{\mathrm{T}} > 0$, $\hat{P} \in \mathbb{R}^{n\times n}$, if and only if K^* minimizes $J(K)$ given by (15) subject to (32)–(34).

Proof: The existence of $P = P^{\mathrm{T}} \geq 0$ such that (33) holds guarantees that $\mathcal{G}_{\mathrm{s}}$ is dissipative with respect to the supply rate $r(v, y)$, whereas (32) and (34) guarantee that $\tilde{A}$ is semistable. The assertion follows as a direct consequence of Theorem 5.1, Proposition 5.1, and Lemma 5.1. ■

To guarantee that $(\tilde{A}, \hat{R})$ is semiobservable, let $\hat{R} = \tilde{A}^{\mathrm{T}}M\tilde{A}$, where $M = M^{\mathrm{T}} > 0$. In this case, $\mathcal{N}(\hat{R}\tilde{A}^{k-1}) = \mathcal{N}(\tilde{A}^{\mathrm{T}}M\tilde{A}^k) = \mathcal{N}(\tilde{A}^k)$, $k = 1, \ldots, n$. Since $\mathcal{N}(\tilde{A}) \subseteq \mathcal{N}(\tilde{A}^k)$ for every $k \in \{1, \ldots, n\}$ it follows that $\bigcap_{k=1}^{n} \mathcal{N}(\hat{R}\tilde{A}^{k-1}) = \bigcap_{k=1}^{n} \mathcal{N}(\tilde{A}^k) = \mathcal{N}(\tilde{A})$, which, by Definition 5.1, implies semiobservability of $(\tilde{A}, \hat{R})$.

The minimization problem given in Theorem 5.2 is complicated by the fact that $\mathcal{J}(K)$ involves $\tilde{A}^{\#}$ and $\tilde{A}$ which are functions of the feedback gain K. Next, we present a corollary to Theorem 5.2 that avoids this complexity. First, however, the following lemma is required.

Lemma 5.2: If $\tilde{A} = A + BK$ is semistable, then $Y \triangleq I_n - \tilde{A}\tilde{A}^{\#}$ is a unique matrix satisfying $\mathcal{N}(Y) = \mathcal{R}(\tilde{A})$, $\mathcal{R}(Y) = \mathcal{N}(\tilde{A})$, and $\mathcal{N}(\tilde{A}) \subseteq \mathcal{N}(Y - I_n)$.

Corollary 5.1: Consider the linear dynamical system $\mathcal{G}_{\mathrm{s}}$ given by (20) and (21), let Q characterizing the supply rate $r(v, y)$ given by (22) be such that $Q \leq 0$, and let $\hat{R} \geq 0$. Then K^* minimizes

$$\mathcal{J}(K) = \mathrm{tr}\, Y^{\mathrm{T}}PYV, \qquad (35)$$

subject to

$$(\tilde{A}, \hat{R}) \text{ is semiobservable}, \qquad (36)$$

$$\mathcal{N}(Y) = \mathcal{R}(\tilde{A}),\ \mathcal{R}(Y) = \mathcal{N}(\tilde{A}),\ \mathcal{N}(\tilde{A}) \subseteq \mathcal{N}(Y - I_n), \qquad (37)$$

$$0 \geq \begin{bmatrix} \tilde{A}^{\mathrm{T}}P + P\tilde{A} - \tilde{C}^{\mathrm{T}}Q\tilde{C} & PB - \tilde{C}^{\mathrm{T}}(QD + Z) \\ B^{\mathrm{T}}P - (QD + Z)^{\mathrm{T}}\tilde{C} & \tilde{R} \end{bmatrix}, \qquad (38)$$

$$0 \geq \tilde{A}^{\mathrm{T}}\hat{P} + \hat{P}\tilde{A}, \qquad (39)$$

where $P = P^{\mathrm{T}} \geq 0$, $P \in \mathbb{R}^{n\times n}$, and $\hat{P} = \hat{P}^{\mathrm{T}} > 0$, $\hat{P} \in \mathbb{R}^{n\times n}$, if and only if K^* minimizes $J(K)$ given by (15) subject to (36)–(39).

Proof: The result is a direct consequence of Theorem 5.2 and Lemma 5.2. ■

VI. Conclusion

Thermodynamics grew out of steam tables and the desire to design and build efficient heat engines, with its central problem involving hard limits on the efficiency of heat engines. Using the laws of thermodynamics, Carnot's principle states that it is impossible to perform a repeatable cycle in which the only result is the performance of positive work [1]. In particular, Carnot showed that the efficiency of a reversible cycle—that is, the ratio of the total work produced during the cycle and the amount of heat transferred from a boiler to a cooler—is bounded by a universal maximum, and this maximum is only a function of the temperatures of the boiler and the cooler. In other words, Carnot's principle shows that it is impossible to extract work from heat without at the same time discarding some heat, giving rise to an increasing quantity which has come to be known as (thermodynamic) entropy. From a system-theoretic point of view, entropy production places hard limits on system (heat engine) performance.

Fundamental limits of achievable performance in linear feedback control systems were first investigated by Bode [22]. Specifically, Bode's theorem states that for a single-input, single-output stable system transfer function with a stable loop-gain and relative degree greater than or equal to two, the integral over all frequencies of the natural logarithm of the magnitude of the sensitivity transfer function $S(s)$ vanishes, that is,

$$\int_0^\infty \log_e |S(\jmath\omega)| \mathrm{d}\omega = 0. \tag{40}$$

This result shows that it is not possible to decrease $|S(\jmath\omega)|$ below the value of 1 over all frequencies imposing fundamental limitations on achievable tracking and disturbance rejection performance for the closed-loop system.

Bode's integral limitation theorem has been extended to multi-input, multi-output unstable systems [23]. In particular, the authors in [23] show that the integral over all frequencies of the natural logarithm of the magnitude of the determinant of the sensitivity transfer function is proportional to the sum of the unstable loop-gain poles, that is,

$$\int_0^\infty \log_e |\det S(\jmath\omega)| \mathrm{d}\omega = \pi \sum_{i=1}^{n_{\mathrm{u}}} \mathrm{Re}\, p_i > 0, \tag{41}$$

where p_i, $i = 1, \ldots, n_{\mathrm{u}}$, denotes the ith unstable loop-gain pole. The unstable poles in the right-hand side of (41) worsen the achievable tracking and disturbance rejection performance for the closed-loop system. Nonlinear extensions of Bode's integral based on an information-theoretic interpretation, singular control, and Markov chains appear in [24], [25], [26]. In future research, we will merge the system thermodynamic semistabilization framework involving the singular control performance criterion (16) and the feedback limitation framework for nonlinear dynamical systems using Bode integrals and cheap control [24] to develop a unified nonlinear stabilization framework with a priori achievable system performance guarantees.

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