Barrier Lyapunov Functions and Constrained Model Reference Adaptive Control

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***Abstract*—In classical model reference adaptive control, the closed-loop system's ability to track a given reference signal can be tuned by choosing the adaptive rates and parameterizing the solution of an algebraic Lyapunov equation that appears in the adaptive law. The projection operator can be employed to impose user-defined constraints on the adaptive gains. However, using the projection operator and quadratic Lyapunov functions to certify uniform ultimate boundedness of the trajectory tracking error, the bounds on the trajectory tracking error can only be estimated, but not explicitly imposed *a priori*. In this letter, we provide an adaptive control law for the same class of nonlinear dynamical systems as classical model reference adaptive control. A barrier Lyapunov function guarantees that user-defined constraints on both the trajectory tracking error and the adaptive gains are verified.**

***Index Terms*—Adaptive control, aerospace, constrained control, Lyapunov methods.**

I. Introduction

GIVEN a nonlinear dynamical system, whose uncontrolled linearized dynamics is unknown and which is affected by parametric and matched uncertainties, a model reference adaptive control (MRAC) law guarantees both uniform boundedness of the trajectory tracking error and its asymptotic convergence to zero. An MRAC law also guarantees uniform boundedness of the adaptive gains [1]–[4].

In classical MRAC, both the adaptive gain rates and the solution of an algebraic Lyapunov equation, which appears in the adaptive laws, can be tuned to guarantee satisfactory trajectory tracking performance, such as high responsiveness to large trajectory tracking errors [4, Ch. 9]. However, large adaptive gain rates induce large and sudden variations of the adaptive gains. Consequently, the control inputs may experience large and rapid excursions that could overly stress the system's actuators or exceed the actuators' saturation limits. Additionally, large variations of the control input may induce large excursions of the trajectory tracking error during the transient period, and the closed-loop system's trajectory may exceed the system's operating range. Conversely, small adaptive gain rates imply slow convergence of the trajectory tracking error to zero and longer transient periods. Moreover, an explicit relation between the solution of the Lyapunov equation underlying the adaptive law and the tracking error can not be found for complex problems.

The most popular approach to constrain adaptive gains consists in modifying their dynamics by means of the projection operator [5], [6]. Specifically, in the presence of parametric and matched uncertainties, this operator guarantees that the adaptive gains lay within some constraint set, while retaining boundedness of the trajectory tracking error. However, the use of the projection operator suffers from several limitations. As proven in [7], using the projection operator and quadratic Lyapunov functions to certify uniform ultimate boundedness of the trajectory tracking error, the bounds on the trajectory tracking error can not be imposed *a priori* and can only be estimated in a conservative manner. Additionally, the set constraining the adaptive gains must be convex [4, pp. 329–341]. Only recently, barrier Lyapunov functions have been utilized to impose user-defined constraints on the closed-loop system's trajectory tracking error at all time, while the adaptive gains are constrained by using the projection operator [7]. An alternative to the use of projection operators has been proposed in [8] using an optimal control framework.

In this letter, we consider partly unknown nonlinear dynamical systems and provide an adaptive control law, which guarantees that both the trajectory tracking error and the adaptive gains are contained at all time within user-defined constraint sets, which are not necessarily convex. Additionally, the proposed control law allows to correlate explicitly the bounds on the trajectory tracking error with the bounds on the adaptive gains and hence, for instance, it allows to impose smaller control inputs for smaller trajectory tracking errors. A barrier Lyapunov function certifies the effectiveness of the proposed approach, and no projection operator is used to enforce constraints on the adaptive gains.

The nonlinear dynamical systems considered in this letter are affected by matched and parametric uncertainties and their uncontrolled linearized dynamics is unknown. It can be proven that if these systems' dynamical models were known perfectly, then asymptotic convergence of the trajectory tracking error could be achieved by a linear control law, whose constant gains verify the matching conditions [4, p. 282]. A unique feature of the proposed control law is that it can be used to bound the estimated adaptive gain error, that is, the difference between the adaptive gains and an estimate of the gains that verify the matching conditions. Additionally, the proposed framework is unique for its ability to correlate explicitly the

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bounds on the trajectory tracking error with the bounds on the estimated adaptive gain error.

II. Notation, Definitions, and Mathematical Preliminaries

In this section, we establish notation, definitions, and review some basic results. Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}^n$ the set of $n\times 1$ real column vectors, and $\mathbb{R}^{n\times m}$ the set of $n\times m$ real matrices. The *interior* of the set $\mathcal{C}\subset\mathbb{R}^n$ is denoted by $\mathring{\mathcal{C}}$ and the *boundary* of $\mathcal{C}$ is denoted by $\partial\mathcal{C}$.

The *identity matrix* in $\mathbb{R}^{n\times n}$ is denoted by I_n or I, the *zero* $n\times m$ *matrix* in $\mathbb{R}^{n\times m}$ is denoted by $0_{n\times m}$ or 0, the *zero vector* in $\mathbb{R}^n$ is denoted by 0_n or 0, and the *trace* of $A\in\mathbb{R}^{n\times n}$ is denoted by $\mathrm{tr}(A)$. We write $\|\cdot\|$ for the *Euclidean vector norm* and the corresponding *equi-induced matrix norm* and $\|B\|_{\mathrm{F}}\triangleq[\mathrm{tr}(BB^{\mathrm{T}})]^{\frac{1}{2}}$, $B\in\mathbb{R}^{n\times m}$, for the *Frobenius norm.*

The *Fréchet derivative* of the continuously differentiable function $V:\mathcal{D}\to\mathbb{R}$ at $x\in\mathcal{D}\subseteq\mathbb{R}^n$ is denoted by $V'(x)\triangleq\frac{\partial V(x)}{\partial x}$. The *Fréchet derivative* of the continuously differentiable function $h:\mathcal{X}\to\mathbb{R}$ at $X\in\mathcal{X}\subseteq\mathbb{R}^{n\times m}$ is given by [9, Ch. 5], [10]

$$\frac{\partial h(X)}{\partial X}\triangleq\begin{bmatrix}\frac{\partial h(X)}{\partial X_{1,1}} & \cdots & \frac{\partial h(X)}{\partial X_{n,1}}\\ \vdots & \ddots & \vdots\\ \frac{\partial h(X)}{\partial X_{1,m}} & \cdots & \frac{\partial h(X)}{\partial X_{n,m}}\end{bmatrix},\qquad(1)$$

where $X_{i,j}$ denotes the element of X on the ith row and jth column. Moreover, given $X:[t_0,\infty)\to\mathcal{X}$, it holds that [9, Ch. 5], [10]

$$\dot{h}(X(t))=\mathrm{tr}\left(\dot{X}(t)\frac{\partial h(X(t))}{\partial X}\right),\qquad t\geq t_0.\qquad(2)$$

III. Constrained Adaptive Control

A. Problem Statement

In this section, we address the problem of regulating a nonlinear dynamical system, whose uncontrolled linearized dynamics is unknown and which is affected by matched and parametric uncertainties. In particular, we present an adaptive control law so that the closed-loop system's trajectory tracks the trajectory of a given reference dynamical model within some user-defined bounds on the tracking error, and the adaptive gains remain sufficiently close to an estimate of the gains that verify the matching conditions. In the absence of constraints, the proposed adaptive control law reduces to the classical MRAC law [4, p. 286].

Consider the nonlinear dynamical system

$$\dot{x}(t)=Ax(t)+B\big[u(t)+\Theta^{\mathrm{T}}\Phi(x(t))\big],$$
$$x(t_0)=x_0,\qquad t\geq t_0,\qquad(3)$$

where $x(t)\in\mathcal{D}\subseteq\mathbb{R}^n$, $t\geq t_0$, denotes the *system's trajectory*, $u(t)\in U\subseteq\mathbb{R}^m$ denotes the *control input*, $A\in\mathbb{R}^{n\times n}$ is unknown, $B\in\mathbb{R}^{n\times m}$, $\Theta\in\mathbb{R}^{N\times m}$, and the *regressor vector* $\Phi:\mathcal{D}\to\mathbb{R}^N$ is Lipschitz continuous. We assume that the pair (A,B) is controllable; in problems of practical interest, this assumption is verified, since the entries of A can be estimated and its structure is usually known [4, p. 281]. The term $\Theta^{\mathrm{T}}\Phi(x)$, $x\in\mathcal{D}$, captures both the *parametric* and the *matched uncertainties* by mean of the unknown matrix Θ; the choice of $\Phi(\cdot)$ is based on some prior knowledge of the system's dynamics [4, Ch. 9].

Consider the *reference dynamical model*

$$\dot{x}_{\mathrm{ref}}(t)=A_{\mathrm{ref}}x_{\mathrm{ref}}(t)+B_{\mathrm{ref}}r(t),$$
$$x_{\mathrm{ref}}(t_0)=x_{\mathrm{ref},0},\qquad t\geq t_0,\qquad(4)$$

where $x_{\mathrm{ref}}(t)\in\mathbb{R}^n$, $t\geq t_0$, $r(t)\in\mathbb{R}^m$ is bounded and denotes the *command input*, $A_{\mathrm{ref}}\in\mathbb{R}^{n\times n}$ is Hurwitz, $B_{\mathrm{ref}}\in\mathbb{R}^{n\times m}$, and the pair $(A_{\mathrm{ref}},B_{\mathrm{ref}})$ is controllable. Let $e(t)\triangleq x(t)-x_{\mathrm{ref}}(t)$, $t\geq t_0$, denote the *trajectory tracking error*, let $(K_x,K_r)\in\mathbb{R}^{n\times m}\times\mathbb{R}^{m\times m}$ be such that

$$A_{\mathrm{ref}}=A+BK_x^{\mathrm{T}},\qquad(5)$$
$$B_{\mathrm{ref}}=BK_r^{\mathrm{T}},\qquad(6)$$

and let $K\triangleq\big[K_x^{\mathrm{T}},K_r^{\mathrm{T}},-\Theta^{\mathrm{T}}\big]$; the *matching conditions* (5) and (6) are standard assumptions in MRAC [4, p. 282] and guarantee the existence of a control law for tracking the reference model (4). Since both A and Θ are unknown, $K\in\mathbb{R}^{m\times(n+m+N)}$ is unknown. However, it is always possible to find $K_e\in\mathbb{R}^{m\times(n+m+N)}$ that provides an estimate of K, that is, such that $\|K_e-K\|_{\mathrm{F}}\leq\varepsilon$, for some $\varepsilon\geq 0$.

Consider the feedback control law

$$\phi(\pi,\hat{K})=\hat{K}\pi,\qquad(\pi,\hat{K})\in\mathbb{R}^{n+m+N}\times\mathbb{R}^{m\times(n+m+N)},\qquad(7)$$

let $\Delta K(t)\triangleq\hat{K}(t)-K_e$ denote the *estimated adaptive gain's error*, and let $\widetilde{\Delta K(t)}\triangleq\hat{K}(t)-K$ denote the *adaptive gain's error*. Note that $\|\widetilde{\Delta K}(t)-\Delta K(t)\|_{\mathrm{F}}\leq\varepsilon$, $t\geq t_0$.

Consider the compact, connected *constraint set*

$$\mathcal{C}\triangleq\Big\{(e,\Delta K)\in\mathbb{R}^n\times\mathbb{R}^{m\times(n+m+N)}:$$
$$h(e^{\mathrm{T}}Me,\Delta K\Gamma^{-1}\Delta K^{\mathrm{T}})\geq 0\Big\},\qquad(8)$$

where $M\in\mathbb{R}^{n\times n}$ is symmetric and positive-definite, $\Gamma\in\mathbb{R}^{(n+m+N)\times(n+m+N)}$ is symmetric and positive-definite, and $h:\mathbb{R}\times\mathbb{R}^{m\times m}\to\mathbb{R}$ is continuously differentiable and such that $h(0,0)>0$. The compactness of $\mathcal{C}$ allows to capture bounded constraint sets, and the connectedness of $\mathcal{C}$ guarantees that there exists a subset of $\mathring{\mathcal{C}}$ containing both $(e(t_0),\Delta K(t_0))$ and $(0_n,0_{m\times(n+m+N)})$ that cannot be expressed as two disjoint nonempty sets. The matrices M and Γ can be chosen to weigh the constraints on the components of e and the columns of ΔK, respectively.

Define $h_e:\mathbb{R}\times\mathbb{R}^{m\times m}\to\mathbb{R}$ and $h_X:\mathbb{R}\times\mathbb{R}^{m\times m}\to\mathbb{R}^{m\times m}$ so that

$$h_e(e^{\mathrm{T}}Me,\Delta K\Gamma^{-1}\Delta K^{\mathrm{T}})\triangleq\left.\frac{\partial h(\beta,X)}{\partial\beta}\right|_{\substack{\beta=e^{\mathrm{T}}Me\\ X=\Delta K\Gamma^{-1}\Delta K^{\mathrm{T}}}},\qquad(9)$$

$$h_X(e^{\mathrm{T}}Me,\Delta K\Gamma^{-1}\Delta K^{\mathrm{T}})\triangleq\left.\frac{\partial h(\beta,X)}{\partial X}\right|_{\substack{\beta=e^{\mathrm{T}}Me\\ X=\Delta K\Gamma^{-1}\Delta K^{\mathrm{T}}}},\qquad(10)$$

for all $(e,\Delta K)\in\mathring{\mathcal{C}}$, and assume that

$$h_e(e^{\mathrm{T}}Me,\Delta K\Gamma^{-1}\Delta K^{\mathrm{T}})\leq 0,\qquad(e,\Delta K)\in\mathring{\mathcal{C}},\qquad(11)$$
$$h_X(e^{\mathrm{T}}Me,\Delta K\Gamma^{-1}\Delta K^{\mathrm{T}})\leq 0,\qquad(12)$$

that is, $h_X(\cdot,\cdot)$ is symmetric and nonpositive-definite, and

$$(0_n,0_{m\times(n+m+N)})=\underset{(e,\Delta K)\in\mathbb{R}^n\times\mathbb{R}^{m\times(n+m+N)}}{\arg\max}h(e^{\mathrm{T}}Me,\Delta K\Gamma^{-1}\Delta K^{\mathrm{T}}).\qquad(13)$$

Note that the interior of $\mathcal{C}$, that is,

$$\mathring{\mathcal{C}}=\Big\{(e,\Delta K)\in\mathbb{R}^n\times\mathbb{R}^{m\times(n+m+N)}:$$
$$h(e^{\mathrm{T}}Me,\Delta K\Gamma^{-1}\Delta K^{\mathrm{T}})>0\Big\},$$

is not empty, since $h(0,0) > 0$, and $\mathring{\mathcal{C}} \setminus \{0\}$ is not empty, since $h(\cdot,\cdot)$ is continuous. Conditions (11)–(13) guide the design of the function $h(\cdot,\cdot)$ that captures the constraint set $\mathcal{C}$. Specifically, it follows from (8) and (11)–(13) that $h(e^{\mathrm{T}}Me,\cdot)$ must be chosen as a non-increasing function of $e^{\mathrm{T}}Me = \|M^{\frac{1}{2}}e\|^2$, $(e,\Delta K) \in \mathring{\mathcal{C}}$, where $M^{\frac{1}{2}}$ denotes the square root of M [11, p. 474], and $h(\cdot, \Delta K\Gamma^{-1}\Delta K^{\mathrm{T}})$ must be chosen so that it attains its maximum for $\Delta K = 0$. As an example, consider

$$h(e^{\mathrm{T}}Me, \Delta K\Gamma^{-1}\Delta K^{\mathrm{T}}) = h_{\max} - \|M^{\frac{1}{2}}e\|^2 - \|\Delta K\|^2_{\mathrm{F},\Gamma^{-1}}, \quad (e,\Delta K) \in \mathbb{R}^m \times \mathbb{R}^{m\times(n+m+N)} \tag{14}$$

where $h_{\max} > 0$; note that (14) verifies (11)–(13), since $h(0,0) = h_{\max} > 0$, $h_e(e^{\mathrm{T}}Me, \Delta K\Gamma^{-1}\Delta K^{\mathrm{T}}) = -1$, and $h_X(e^{\mathrm{T}}Me, \Delta K\Gamma^{-1}\Delta K^{\mathrm{T}}) = -I_m$.

B. Main Result

The next theorem is the main result of this section and provides an adaptive law for the control law (7) so that both the trajectory tracking error $e(\cdot)$ and the estimated adaptive gain's error $\Delta K(\cdot)$ verify the user-defined constraints captured by (8). To state this theorem, note that it follows from (3) with $u(t) = \phi(\pi(t), \hat{K}(t))$, $t \ge t_0$, and (4) that

$$\dot{e}(t) = A_{\mathrm{ref}}e(t) + B\widetilde{\Delta K}(t)\pi(t), \quad e(t_0) = x_0 - x_{\mathrm{ref},0}, \qquad t \ge t_0. \tag{15}$$

In addition, define the positive-definite function

$$V(e,\Delta K) \triangleq \frac{e^{\mathrm{T}}Pe + \mathrm{tr}\big(\Delta K\Gamma^{-1}\Delta K^{\mathrm{T}}\big)}{h(e^{\mathrm{T}}Me, \Delta K\Gamma^{-1}\Delta K^{\mathrm{T}})}, \quad (e,\Delta K) \in \mathring{\mathcal{C}}. \tag{16}$$

Given $\alpha > 0$ and $\pi \in \mathbb{R}^{n+m+N}$, define

$$\mathcal{S}_{\alpha,\pi} \triangleq \Big\{(e,\Delta K) \in \mathbb{R}^n \times \mathbb{R}^{m\times(n+m+N)} : -\alpha e^{\mathrm{T}}e + 2\varepsilon\, \mathrm{tr}\big(R_\pi(e,\Delta K)R_\pi^{\mathrm{T}}(e,\Delta K)\big)^{1/2} \ge 0\Big\}, \tag{17}$$

where $R_\pi(e,\Delta K) \triangleq \pi e^{\mathrm{T}}[P - V(e,\Delta K)h_e(e^{\mathrm{T}}Me, \Delta K\Gamma^{-1}\Delta K^{\mathrm{T}})M]B$. Lastly, note that $x_{\mathrm{ref}}(t)$, $t \ge t_0$, is bounded, since $r(t)$ is bounded and A_{ref} is Hurwitz [12, p. 245], and if $e(t)$ is bounded, then $\pi(t) = [x^{\mathrm{T}}(t), r^{\mathrm{T}}(t), -\Phi^{\mathrm{T}}(x(t))]^{\mathrm{T}} \in \Pi$, where $\Pi \subset \mathbb{R}^{n+m+N}$ is compact. Thus, if both $e(\cdot)$ and $\Delta K(\cdot)$ are bounded, then it follows from Weierstrass theorem [12, Th. 2.13] that

$$\pi^* \triangleq \mathrm{argmax}_{\pi\in\Pi}\Big[-\alpha e^{\mathrm{T}}e + 2\varepsilon\, \mathrm{tr}\big(R_\pi(e,\Delta K) \times R_\pi^{\mathrm{T}}(e,\Delta K)\big)^{1/2}\Big] \tag{18}$$

exists and is finite.

Theorem 1: Consider the uncertain nonlinear dynamical system (3), the reference model (4), the control law $\phi(\cdot,\cdot)$ given by (7), the constraint set (8), and the set $\mathcal{S}_{\alpha,\pi}$ given by (17). Let $x_0 \in \mathbb{R}^n$, $x_{\mathrm{ref},0} \in \mathbb{R}^n$, and $\Delta K_0 \in \mathbb{R}^{m\times(n+m+N)}$ be such that $(x_0 - x_{\mathrm{ref},0}, \Delta K_0) \in \mathring{\mathcal{C}} \setminus \{0\}$. Assume there exist $\alpha > 0$, $P \in \mathbb{R}^{n\times n}$, which is symmetric and positive-definite, and $Q : \mathbb{R} \times \mathbb{R}^{m\times m} \to \mathbb{R}^{n\times n}$ such that

$$\begin{aligned}-Q(e^{\mathrm{T}}Me, \Delta K\Gamma^{-1}\Delta K^{\mathrm{T}}) &= A_{\mathrm{ref}}^{\mathrm{T}}\Big[P - V(e,\Delta K)h_e(e^{\mathrm{T}}Me, \Delta K\Gamma^{-1}\Delta K^{\mathrm{T}})M\Big] \\ &\quad + \Big[P - V(e,\Delta K)h_e(e^{\mathrm{T}}Me, \Delta K\Gamma^{-1}\Delta K^{\mathrm{T}})M\Big]A_{\mathrm{ref}}, \quad (e,\Delta K) \in \mathring{\mathcal{C}},\end{aligned} \tag{19}$$

$Q(e^{\mathrm{T}}Me, \Delta K\Gamma^{-1}\Delta K^{\mathrm{T}}) = Q^{\mathrm{T}}(e^{\mathrm{T}}Me, \Delta K\Gamma^{-1}\Delta K^{\mathrm{T}})$, and $Q(e^{\mathrm{T}}Me, \Delta K\Gamma^{-1}\Delta K^{\mathrm{T}}) \ge \alpha I_n$, where $V(\cdot,\cdot)$ is given by (16). Let

$$\begin{aligned}\dot{\hat{K}}^{\mathrm{T}}(t) &= -\Gamma\pi(t)e^{\mathrm{T}}(t)\Big[P - V(e(t),\Delta K(t)) \times h_e(e^{\mathrm{T}}(t)Me(t), \Delta K(t)\Gamma^{-1}\Delta K^{\mathrm{T}}(t))M\Big]B \\ &\quad \times \Big[I_m - V(e(t),\Delta K(t)) \times h_X(e^{\mathrm{T}}(t)Me(t), \Delta K(t)\Gamma^{-1}\Delta K^{\mathrm{T}}(t))\Big]^{-1}, \\ \hat{K}(t_0) &= \Delta K_0, \qquad t \ge t_0.\end{aligned} \tag{20}$$

If $\mathcal{S}_{\alpha,\pi^*} \subset \mathring{\mathcal{C}}$, where π^* is given by (18), then (3) with $u = \phi(x,\hat{K})$ and (20) are such that $(e(t),\Delta K(t)) \in \mathring{\mathcal{C}}$, $t \ge t_0$.

The proof of Theorem 1 is presented in the Appendix.

C. Significance of Theorem 1

Theorem 1 provides sufficient conditions to constrain both the trajectory tracking error $e(\cdot)$ and the estimated adaptive gain error $\Delta K(\cdot)$. Constraints on the trajectory tracking error are captured by the continuously differentiable function $h(e^{\mathrm{T}}Me,\cdot) = h(\|M^{\frac{1}{2}}e\|^2,\cdot)$, $e \in \mathbb{R}^m$. Since $\|M^{\frac{1}{2}}e\|$ is a *weighted Euclidean norm* of e [13, p. 18], it captures a measure of the trajectory tracking error and hence, Theorem 1 provides sufficient conditions for the control law (7) and the adaptive law (20) to guarantee satisfactory trajectory tracking for the unknown dynamical system (3).

The relevance of imposing constraints on the estimated adaptive gain error $\Delta K(\cdot)$ can be explained as follows.

Proposition 1 [4, pp. 281–282]: Consider the nonlinear dynamical system (3) and the reference dynamical model (4). Assume that both A and Θ are known, and there exist $K_x \in \mathbb{R}^{n\times m}$ and $K_r \in \mathbb{R}^{m\times m}$ such that the matching conditions (5) and (6) are verified. If $u = \phi_{\mathrm{ideal}}(\pi, K)$, where

$$\phi_{\mathrm{ideal}}(\pi, K) = K\pi, \quad (\pi, K) \in \mathbb{R}^{n+m+N} \times \mathbb{R}^{m\times(n+m+N)}, \tag{21}$$

and $K = [K_x^{\mathrm{T}}, K_r^{\mathrm{T}}, -\Theta^{\mathrm{T}}]$, then $\lim_{t\to\infty} e(t) = 0$.

It follows from Proposition 1 that if the dynamics of (4) were known, then the control law (21) would guarantee asymptotic convergence of the trajectory tracking error. Thus, a desirable design specification on the control law (7) is that the adaptive gain's error $\widetilde{\Delta K}(t) = \hat{K}(t) - K$, $t \ge t_0$, verifies some constraints assigned *a priori*. However, since both A and Θ are unknown, $\widetilde{\Delta K}(\cdot)$ cannot be computed. Theorem 1 provides sufficient conditions to constrain $\Delta K(t) = \hat{K}(t) - K_e$, $t \ge t_0$, where $K_e \in \mathbb{R}^{m\times(n+m+N)}$ denotes an estimate of K, that is, K_e is such that $\|K_e - K\|_{\mathrm{F}} \le \varepsilon$, for some $\varepsilon \ge 0$. Indeed, Theorem 1 allows to impose constraints on the closed-loop system's dynamics captured by the continuously differentiable function $h(\cdot, \Delta K\Gamma^{-1}\Delta K^{\mathrm{T}})$, $\Delta K \in \mathbb{R}^{m\times(n+m+N)}$.

Remark 1: If $\varepsilon = 0$, then $K_e = K$ and it follows from the proof of Theorem 1 that $\mathring{\mathcal{S}}_{\alpha,\pi^*} = \{\emptyset\}$ and the adaptive control law (20) guarantees that $(e(t),\Delta K(t)) \in \mathring{\mathcal{C}}$, $t \ge t_0$, and $e(t) \to 0$ as $t \to \infty$.

Remark 2: It follows from (7), the definition of equi-induced norm [11, p. 608], and the triangle inequality that

$$\|\phi(\pi(t),\hat{K}(t))\| \le \|\hat{K}(t)\|\|\pi(t)\| \le (\|\Delta K(t)\| + \|K_e\|)\|\pi(t)\|, \quad t \ge t_0. \tag{22}$$

Therefore, Theorem 1 provides sufficient conditions to constrain the adaptive control law $\phi(\cdot,\cdot)$ at all time, while the trajectory tracking error lays in a user-define constraint.

Remark 3: One can prove that in the presence of unmatched uncertainties [4, p. 317], (7) and (20) allow to impose user-defined constraints both on the trajectory tracking error and the adaptive gains. To show this result, the set $\mathcal{S}_{\alpha,\pi}$ must be redefined to account for an upper bound on the magnitude of the external disturbance, since $\mathcal{S}_{\alpha,\pi^*}$ captures the set where the total derivative of the Lyapunov function (16) is not guaranteed to be negative-definite. It can also be shown that in the presence of unmatched uncertainties, (7) and (20) guarantee satisfactory trajectory tracking for smaller values of ε, $\|M\|$, and $\|\Gamma^{-1}\|$.

D. Applicability of Theorem 1

Theorem 1 relies on two assumptions. Specifically $\mathcal{S}_{\alpha,\pi^*}$ must be a proper subset of $\mathring{\mathcal{C}}$, that is, $\mathcal{S}_{\alpha,\pi^*} \subset \mathring{\mathcal{C}}$. Moreover, there must exist $P \in \mathbb{R}^{n\times n}$, which is symmetric and positive-definite, and a symmetric matrix function $Q(\cdot,\cdot)$ such that (19) is verified and $Q(e^{\mathrm{T}}Me, \Delta K\Gamma^{-1}\Delta K^{\mathrm{T}}) \geq \alpha I_n$, for some given $\alpha > 0$. The set $\mathcal{S}_{\alpha,\pi^*}$ captures the effect of not being able to determine exactly K on the ability of enforcing satisfactory trajectory tracking. Indeed, it follows from the proof of Theorem 1 that if $K_e \neq K$, then $\varepsilon > 0$, $\mathring{\mathcal{S}}_{\alpha,\pi^*} \neq \{\emptyset\}$, and $\dot{V}(e,\Delta K) \geq 0$, $(e,\Delta K) \in \mathcal{S}_{\alpha,\pi^*}$. Moreover, if $\mathcal{S}_{\alpha,\pi^*}$ is not a proper subset of $\mathring{\mathcal{C}}$, then $(e(\cdot),\Delta K(\cdot))$ escapes the interior of the constraint set $\mathcal{C}$ and Theorem 1 can not be applied to impose the desired constraints. One can enforce that $\mathcal{S}_{\alpha,\pi^*} \subset \mathring{\mathcal{C}}$ by choosing $\alpha > 0$ sufficiently large, M and Γ so that $\|M\|$ and $\|\Gamma^{-1}\|$ are small, and K_e sufficiently close to K so that ε is small.

The next result, whose proof is reported in the Appendix, provides sufficient conditions to find $P \in \mathbb{R}^{n\times n}$, and a symmetric, positive-definite matrix function $Q(\cdot,\cdot)$ such that, given $\alpha > 0$, (19) is verified.

Theorem 2: Consider the constraint set $\mathcal{C}$ given by (8). Let $\alpha > 0$, let $A_{\mathrm{ref}} \in \mathbb{R}^{n\times n}$ be Hurwitz, let $Q_1 \in \mathbb{R}^{n\times n}$ be symmetric, positive-definite, and such that $Q_1 \geq \alpha I_n$, let $P \in \mathbb{R}^{n\times n}$ be the symmetric and positive-definite solution of the algebraic Lyapunov equation

$$0 = A_{\mathrm{ref}}^{\mathrm{T}}P + PA_{\mathrm{ref}} + Q_1, \tag{23}$$

let $M \in \mathbb{R}^{n\times n}$ be symmetric and positive-definite, and define

$$\begin{aligned} &Q_2(e^{\mathrm{T}}Me, \Delta K\Gamma^{-1}\Delta K^{\mathrm{T}}) \\ &\quad \triangleq V(e,\Delta K)h_e(e^{\mathrm{T}}Me, \Delta K\Gamma^{-1}\Delta K^{\mathrm{T}}) \\ &\quad \times \left(A_{\mathrm{ref}}^{\mathrm{T}}M + MA_{\mathrm{ref}}\right), \end{aligned} \tag{24}$$

for all $(e,\Delta K) \in \mathring{\mathcal{C}}$, where $V(\cdot,\cdot)$ is given by (16). Then, (19) is verified by the symmetric matrix function

$$\begin{aligned} &Q(e^{\mathrm{T}}Me, \Delta K\Gamma^{-1}\Delta K^{\mathrm{T}}) \\ &\quad = Q_1 + Q_2(e^{\mathrm{T}}Me, \Delta K\Gamma^{-1}\Delta K^{\mathrm{T}}), \quad (e,\Delta K) \in \mathring{\mathcal{C}}, \end{aligned} \tag{25}$$

and $Q(e^{\mathrm{T}}Me, \Delta K\Gamma^{-1}\Delta K^{\mathrm{T}}) \geq \alpha I_n$.

Note that the adaptive law (20) involves the inverse of the matrix function $[I_m - V(e,\Delta K)h_X(e^{\mathrm{T}}Me, \Delta K\Gamma^{-1}\Delta K^{\mathrm{T}})]$, $(e,\Delta K) \in \mathring{\mathcal{C}}$. The identity matrix I_m is symmetric and positive-definite, $V(e,\Delta K)$, $(e,\Delta K) \in \mathring{\mathcal{C}}$, is positive definite, and (12) holds by assumption. Therefore, $[I_m - V(e,\Delta K)h_X(e^{\mathrm{T}}Me, \Delta K\Gamma^{-1}\Delta K^{\mathrm{T}})]$, $(e,\Delta K) \in \mathring{\mathcal{C}}$, is symmetric and positive-definite, hence invertible, and the right-hand side of (20) has no singularity.

E. Connections to Classical MRAC

The next result specializes Theorem 1 to the case wherein no constraint is imposed on the trajectory tracking error $e(\cdot)$ and the gain matrix $\Delta K(\cdot)$. In [4], this corollary is stated as Theorem 9.2 and provides the classical direct MRAC law for the nonlinear dynamical system (3).

Corollary 1: Consider the nonlinear uncertain dynamical system (3), the control law $\phi(\cdot,\cdot)$ given by (7), and the trajectory tracking error dynamics (15). Let $P \in \mathbb{R}^{n\times n}$ be the symmetric, positive-definite solution of (23) Then, the solution $x(t)$, $t \geq t_0$, of (3) with $u = \phi(x,\hat{K})$ and

$$\dot{\hat{K}}^{\mathrm{T}}(t) = -\Gamma\pi(t)e^{\mathrm{T}}(t)PB, \quad \hat{K}(t_0) = \Delta K_0, \quad t \geq t_0, \tag{26}$$

where $\Delta K_0 \in \mathbb{R}^{m\times(n+m+N)}$, is such that $\lim_{t\to\infty} e(t) = 0$.

Proof: The result follows from Theorems 1 and 2 and Remark 1 with $h(e^{\mathrm{T}}Me, \Delta K\Gamma^{-1}\Delta K^{\mathrm{T}}) = 1$, $(e,\Delta K) \in \mathbb{R}^n \times \mathbb{R}^{m\times(n+m+N)}$. ■

IV. Illustrative Numerical Example

In this section, we provide a numerical example to illustrate the applicability of the theoretical results presented in this letter. The roll dynamics of an unmanned aerial vehicle is captured by [7, Example 5.3], [14, pp. 59–64]

$$\begin{aligned} \begin{bmatrix} \dot{\varphi}(t) \\ \dot{p}(t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ \theta_1 & \theta_2 \end{bmatrix} \begin{bmatrix} \varphi(t) \\ p(t) \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \left(u(t) + \Theta^{\mathrm{T}}\Phi(\varphi(t), p(t))\right), \\ &\quad [\varphi(0), p(0)]^{\mathrm{T}} = [\varphi_0, p_0]^{\mathrm{T}}, \qquad t \geq 0, \end{aligned} \tag{27}$$

where $\varphi(t) \in \mathbb{R}$, $t \geq 0$, denotes the roll angle, $p(t) \in \mathbb{R}$ denotes the roll rate, $u(t) \in \mathbb{R}$ denotes the aileron deflection angle, $\Theta \in \mathbb{R}^6$ is unknown, and $\Phi(\varphi,p) = [\sin t, \varphi, p, |\varphi|p, |p|p, \varphi^3]^{\mathrm{T}}$, $(\varphi,p) \in \mathbb{R}\times\mathbb{R}$. In particular, both θ_1 and θ_2 capture the effect of aerodynamic moments acting on the vehicle, and $\Theta^{\mathrm{T}}\Phi(\varphi,p)$, $(\varphi,p) \in \mathbb{R}\times\mathbb{R}$, captures undesired aerodynamic moments induced by the aileron deflection; since aerodynamic coefficients can only be estimated through wind-tunnel tests, we consider θ_1, θ_2, and Θ as unknown. Note that the nonlinear plant (27) is in the same form as (3) with $n = 2$, $m = 1$, $N = 3$, $\mathcal{D} = \mathbb{R}^n$, $x = [\varphi, p]^{\mathrm{T}}$, $A = \begin{bmatrix} 0 & 1 \\ \theta_1 & \theta_2 \end{bmatrix}$, $B = [0, 1]^{\mathrm{T}}$, $\Phi(x) = \Phi(\varphi,p)$, $x_0 = [\varphi_0, p_0]^{\mathrm{T}}$, and $t_0 = 0$.

Our goal is to design an adaptive gain $\hat{K} : [0,\infty) \to \mathbb{R}^{m\times(n+m+N)}$ for the control law $\phi(\cdot,\cdot)$ given by (7) so that the solution $x(\cdot)$ of (27) with $u = \phi(\pi,\hat{K})$ follows the trajectory $x_{\mathrm{ref}}(\cdot)$ of the reference dynamical model (4) with

$$A_{\mathrm{ref}} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\zeta\omega \end{bmatrix}, \qquad B_{\mathrm{ref}} = \begin{bmatrix} 0 \\ \omega^2 \end{bmatrix}, \tag{28}$$

$\zeta \in (0,1)$, $\omega > 0$, and $r(t) = \mathrm{sign}(\sin t)$, $t \geq 0$, where $\mathrm{sign}(\cdot)$ denotes the signum function [15, p. 19]. In particular, we wish to constrain both the trajectory tracking error $e(\cdot)$ and the adaptive gain $\hat{K}(\cdot)$ so that the pair $(e(\cdot),\Delta K(\cdot))$ lays in the constraint set (8) with $h(\cdot,\cdot)$ given by (14). We will meet these design goals by applying Theorem 1.

The pair (A,B) is controllable and the matching conditions (5) and (6) are verified by

$$K_x = -\begin{bmatrix} \omega^2 + \theta_1 \\ 2\zeta\omega + \theta_2 \end{bmatrix}, \qquad K_r = \omega^2. \tag{29}$$

Let $Q_1 \in \mathbb{R}^{n\times n}$ be symmetric and positive-definite, and let $P \in \mathbb{R}^{n\times n}$ be the symmetric and positive-definite solution of (23).

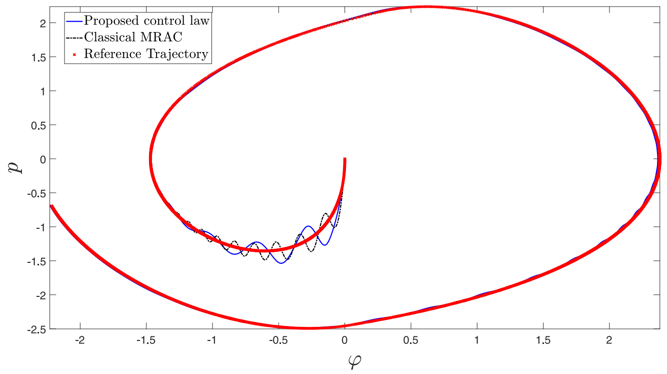

Fig. 1. Closed-loop system's trajectories obtained applying the control law (7) with proposed adaptive law (20) and the control law (7) with classical MRAC law (26). The closed-loop system's trajectory obtained applying (20) experiences smaller oscillations than the closed-loop system's trajectory obtained applying (26), but converges to the reference system's trajectory later than the closed-loop system's trajectory obtained applying (26).

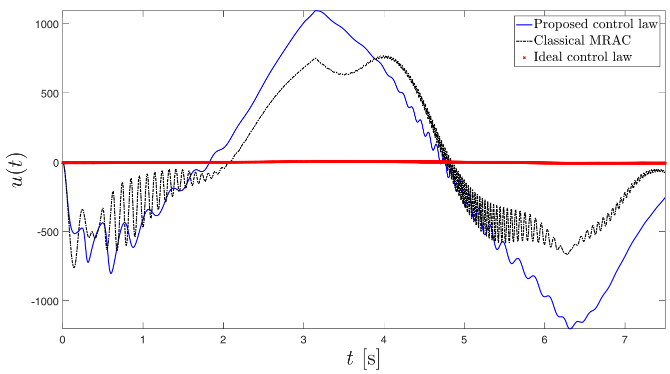

Fig. 2. Control law (7) with proposed adaptive law (20), control law (7) with classical model reference adaptive control (MRAC) law (26), and ideal control law (21). The proposed control law experiences smaller and lower-frequency oscillations than the classical MRAC law.

It follows from Theorem 2 that given $\alpha > 0$, there exists $Q(\cdot,\cdot)$ such that the conditions of Theorem 1 are verified. Thus, the control law (7) and the adaptive law (20) guarantee that $\|M^{\frac{1}{2}}e\|^2 + \|\Delta K(t)\|^2_{\mathrm{F},\Gamma^{-1}} \le h_{\max}$, $t \ge 0$.

Let $\theta_1 = 0$, $\theta_2 = 0$, $\Theta^{\mathrm{T}} = [0.25, 0.5, 1, -1, 1, 1]$, $\zeta = 0.7$, $\omega = 1$, $h_{\max} = 30$, $x_0 = 0$, $x_{\mathrm{ref},0} = x_0$, $M = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}$, $\Gamma = 24 \cdot 10^3 \cdot I_6$, $Q = I_2$, $K_e = K + 0.1[1, 1, 0, 1, 1, 1, 1, 1, 1]^{\mathrm{T}}$, and $\varepsilon = 0.2828$. Figure 1 shows the solution $x(\cdot)$ of (27) with control law (7) and proposed adaptive law (20), the solution $x(\cdot)$ of (27) with control law (7) and classical adaptive law (26), and the solution $x_{\mathrm{ref}}(\cdot)$ of (4). Figure 2 shows the control input $u(\cdot)$ obtained applying both (7) and (20), the control input $u(\cdot)$ obtained applying both (7) and (26), and the control input $u(\cdot)$ obtained applying the ideal control law (21). Finally, Figure 3 shows $\|M^{\frac{1}{2}}e(t)\|^2 + \|\Delta K(t)\|^2_{\mathrm{F},\Gamma^{-1}}$, $t \in [0, 7.5]$ s, obtained applying both the proposed adaptive law (20) and the classical adaptive law (26).

It appears from Figure 3 that the proposed adaptive law (20) verifies the constraint both on the trajectory tracking error and the estimated adaptive gain's error at all time, whereas the classical MRAC law (26) does not verify these constraints. Indeed, applying (20), $\|M^{\frac{1}{2}}e(t)\|^2 + \|\Delta K(t)\|^2_{F,\Gamma^{-1}} < h_{\max}$, $t \ge 0$ s,

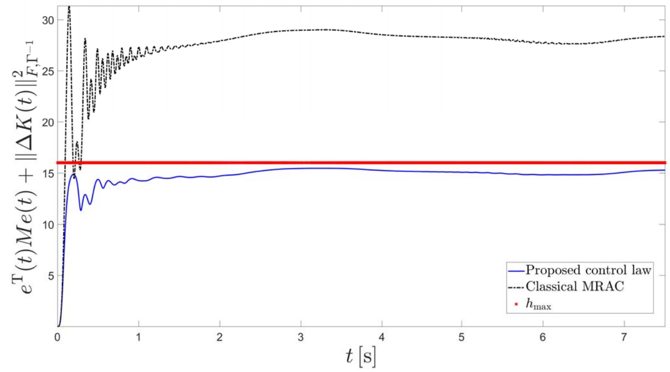

Fig. 3. Measure of the trajectory tracking error and the estimated adaptive gain error applying the proposed adaptive law (20) and the classical model reference adaptive control (MRAC) law (26).

whereas applying (26), $\|M^{\frac{1}{2}}e(t)\|^2 + \|\Delta K(t)\|^2_{F,\Gamma^{-1}} > h_{\max}$, $t > 0.0951$ s. Figure 2 shows that the control law (7) with the proposed adaptive law (20) experiences larger excursions, but considerably less oscillations than with the classical MRAC adaptive law (26), and this is an advantage, since rapid oscillations of the control input fatigue actuators. Figure 1 shows that in the transient phase, the closed-loop system's trajectory obtained applying the proposed adaptive law (20) experiences larger oscillations than the trajectory obtained applying the classical adaptive law (26). Indeed, for $t \in [0, 1.2364]$ s the average trajectory tracking error obtained applying (20) is 0.3416 and the average trajectory tracking error obtained applying (26) is 0.2843. For $t \in [1.2365, 7.5]$ s, the average trajectory tracking error obtained applying the proposed adaptive law (20) is 0.0014 and the average trajectory tracking error obtained applying the classical adaptive law (26) is 0.0011. Thus, after the transient period, the proposed adaptive law guarantees the same high-quality trajectory tracking as the classical adaptive law.

For a given M, both $h_{\max}$ and Γ were chosen through an iterative process. Firstly, a large value of $h_{\max}$ was chosen so that the effect of the constraints was negligible. Next, $h_{\max}$ was reduced, while Γ remained constant. As expected, the closed-loop system's converged later to the reference trajectory and hence, the trajectory tracking performance was improved by increasing $\|\Gamma\|$, while $h_{\max}$ remained constant. In classical MRAC, large values of $\|\Gamma\|$ are chosen to guarantee rapid convergence of the trajectory tracking error.

V. Conclusion

In this letter, we presented an MRAC law that guarantees satisfactory trajectory tracking for nonlinear dynamical systems affected by parametric and matched uncertainties, and whose uncontrolled linearized dynamics is unknown. In particular, the proposed control law bounds at all time, within user-defined constraint sets, both the trajectory tracking error and the difference between the adaptive gains and an estimate of the gains that verify the matching conditions. For its ability to constrain the adaptive gains within bounds assigned *a priori*, the proposed framework provides an alternative to those adaptive frameworks that involve the projection operator to limit the adaptive gain dynamics. Future work directions include both an analytical study of the trajectory tracking error obtained applying the proposed adaptive control framework and a comparative analysis of the proposed framework and other frameworks using the projection operator. Additional future work directions include an investigation of the fact that, as highlighted by the proposed numerical example, constraining both the trajectory tracking error and the estimated adaptive gain error, the proposed control law experiences smaller oscillations than the classical MRAC law.

Appendix

Proof of Theorem 1: This proof is divided in two parts. First, we assume that $(e(t), \Delta K(t)) \in \mathring{\mathcal{C}}$, $t \geq t_0$ and show that $\dot{V}(e, \Delta K) < 0$ for all $(e, \Delta K) \in \mathring{\mathcal{C}} \setminus \mathcal{S}_{\alpha,\pi^*}$. Then, we use a contradiction argument to prove that if $(x_0 - x_{\mathrm{ref},0}, \Delta K_0) \in \mathring{\mathcal{C}} \setminus \{0\}$, then $(e(t), \Delta K(t)) \in \mathring{\mathcal{C}}$, $t \geq t_0$.

Assume that $(e(t), \Delta K(t)) \in \mathring{\mathcal{C}}$, $t \geq t_0$. Since $\mathcal{C}$ is compact by assumption, both $e(\cdot)$ and $\Delta K(\cdot)$ are bounded and (18) exists and is finite. Additionally, it follows from (16), (15), and (2) that for all $(e, \Delta K) \in \mathring{\mathcal{C}}$,

$$\begin{aligned}\dot{V}(e, \Delta K) \leq& -\alpha h^{-1}(e^{\mathrm{T}}Me, \Delta K\Gamma^{-1}\Delta K^{\mathrm{T}})e^{\mathrm{T}}e \\ &+ 2h^{-1}(e^{\mathrm{T}}Me, \Delta K\Gamma^{-1}\Delta K^{\mathrm{T}}) \\ &\times \mathrm{tr}\Big(\widetilde{\Delta K}\pi e^{\mathrm{T}}\Big[P - V(e, \Delta K) \\ &\qquad \times h_e(e^{\mathrm{T}}Me, \Delta K\Gamma^{-1}\Delta K^{\mathrm{T}})M\Big]B \\ &+ \Delta K\Gamma^{-1}\dot{\hat{K}}^{\mathrm{T}}\Big[I_m - V(e, \Delta K) \\ &\qquad \times h_X(e^{\mathrm{T}}Me, \Delta K\Gamma^{-1}\Delta K^{\mathrm{T}})\Big]\Big), \quad (30)\end{aligned}$$

Therefore, since $\mathrm{tr}(X^{\mathrm{T}}Y)$ is an inner product of X and $Y \in \mathbb{R}^{n\times m}$ [11, p. 95], and, by assumption, $\|\widetilde{\Delta K}(t) - \Delta K(t)\|_{\mathrm{F}} \leq \varepsilon$, for some $\varepsilon \geq 0$, it follows from (20) and the Cauchy–Schwarz inequality [11, Fact 1.18.9] that

$$\begin{aligned}\dot{V}(e, \Delta K) \leq& h^{-1}(e^{\mathrm{T}}Me, \Delta K\Gamma^{-1}\Delta K^{\mathrm{T}}) \\ &\times \Big[-\alpha e^{\mathrm{T}}e + 2\varepsilon\,\mathrm{tr}\Big(\pi e^{\mathrm{T}}\Big[P - V(e, \Delta K) \\ &\qquad \times h_e(e^{\mathrm{T}}Me, \Delta K\Gamma^{-1}\Delta K^{\mathrm{T}})M\Big]B\Big)\Big], \\ &\qquad (e, \Delta K) \in \mathring{\mathcal{C}}, \quad (31)\end{aligned}$$

and hence $\dot{V}(e, \Delta K) < 0$, $(e, \Delta K) \in \mathring{\mathcal{C}} \setminus \mathcal{S}_{\alpha,\pi^*}$.

Next, assume that $(e(t_0), \Delta K(t_0)) \in \mathring{\mathcal{C}} \setminus \{0\}$, and suppose *ad absurdum* that there exists $T^* > 0$ such that $\lim_{t\to T^*} \mathrm{dist}((e(t), \Delta K(t)), \partial\mathcal{C}) = 0$, where $\mathrm{dist}(\cdot,\cdot)$ denotes the distance of a point from a set [16, p. 16]. In this case, $\lim_{t\to T^*} h(e^{\mathrm{T}}(t)Me(t), \Delta K(t)\Gamma^{-1}\Delta K^{\mathrm{T}}(t)) = 0$ along the trajectory of (15) and (20), and it follows from the continuity of $h(\cdot,\cdot)$, $e(\cdot)$, and $\Delta K(\cdot)$ that

$$\begin{aligned}&\lim_{t\to T^*} h(e^{\mathrm{T}}(t)Me(t), \Delta K(t)\Gamma^{-1}\Delta K^{\mathrm{T}}(t)) \\ &\quad = h(e^{\mathrm{T}}(T^*)Me(T^*), \Delta K(T^*)\Gamma^{-1}\Delta K^{\mathrm{T}}(T^*)), \quad (32)\end{aligned}$$

which implies that $(e(T^*), \Delta K(T^*)) \neq 0$, since $h(e^{\mathrm{T}}Me, \Delta K\Gamma^{-1}\Delta K^{\mathrm{T}}) > 0$ for all $(e, \Delta K) \in \mathring{\mathcal{C}}$ and $(0, 0) \in \mathring{\mathcal{C}}$ by assumption. Moreover, since $\mathrm{tr}(\cdot)$ is continuous and $e^{\mathrm{T}}Pe + \mathrm{tr}(\Delta K\Gamma^{-1}\Delta K^{\mathrm{T}})$ is positive-definite for all $(e, \Delta K) \in \mathbb{R}^n \times \mathbb{R}^{m\times(n+m+N)}$, it holds that $\lim_{t\to T^*}[e^{\mathrm{T}}(t)Pe(t) + \mathrm{tr}(\Delta K(t)\Gamma^{-1}\Delta K^{\mathrm{T}}(t))] \neq 0$. Therefore,

$$\begin{aligned}\infty &= \lim_{t\to T^*} V(e(t), \Delta K(t)) \\ &= \frac{e^{\mathrm{T}}(T^*)Pe(T^*) + \mathrm{tr}\big(\Delta K(T^*)\Gamma^{-1}\Delta K^{\mathrm{T}}(T^*)\big)}{h(e^{\mathrm{T}}(T^*)Me(T^*), \Delta K(T^*)\Gamma^{-1}\Delta K^{\mathrm{T}}(T^*))}. \quad (33)\end{aligned}$$

Now, if $(e(t_0), \Delta K(t_0)) \in \mathcal{S}_{\alpha,\pi^*}$, then there exists $T^{**} \geq t_0$ such that $(e(T^{**}), \Delta K(T^{**})) \in \partial\mathcal{S}_{\alpha,\pi^*}$. Since $\mathcal{S}_{\alpha,\pi^*} \subset \mathring{\mathcal{C}}$ by assumption, it holds that $T^{**} < T^*$ and it follows from (31) that for all t_1 and $t_2 \in [T^{**}, T^*)$ such that $t_2 \geq t_1$,

$$V(e(t_2), \Delta K(t_2)) \leq V(e(t_1), \Delta K(t_1)) < \infty, \quad (34)$$

along the trajectory of (15) and (20), which contradicts (33). Alternatively, if $(e(t_0), \Delta K(t_0)) \in \mathring{\mathcal{C}} \setminus \mathcal{S}_{\alpha,\pi^*}$, then there exists $T^{**} > t_0$ such that $(e(T^{**}), \Delta K(T^{**})) \in \partial\mathcal{S}_{\alpha,\pi^*}$, and (33) is contradicted by applying a similar argument as for the previous case. Therefore, if $(e(t_0), \Delta K(t_0)) \in \mathring{\mathcal{C}} \setminus \{0\}$, then $(e(t), \Delta K(t)) \in \mathring{\mathcal{C}}$, $t \geq t_0$, which concludes the proof. ■

Proof of Theorem 2: Since Q_1 is symmetric and positive-definite and A_{ref} is Hurwitz, there exists a unique symmetric, positive-definite $P \in \mathbb{R}^{n\times n}$ such that (23) is verified [11, Corollary 11.9.4]. The matrix function $Q_2(\cdot,\cdot)$ is symmetric and nonnegative-definite, since $V(e, \Delta K)$, $(e, \Delta K) \in \mathring{\mathcal{C}}$, is positive-definite, $A_{\mathrm{ref}}^{\mathrm{T}}M + MA_{\mathrm{ref}}$ is symmetric and nonpositive-definite [17], and (11) holds by assumption. Therefore, (19) follows from (23) and (24), $Q(\cdot,\cdot)$ is symmetric, since both Q_1 and $Q_2(\cdot,\cdot)$ are symmetric, and $Q(e^{\mathrm{T}}Me, \Delta K\Gamma^{-1}\Delta K^{\mathrm{T}}) = Q_1 + Q_2(e^{\mathrm{T}}Me, \Delta K\Gamma^{-1}\Delta K^{\mathrm{T}}) \geq \alpha I_n$, $(e, \Delta K) \in \mathring{\mathcal{C}}$. ■

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