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Differential games, continuous Lyapunov functions, and stabilisation of non-linear dynamical systems

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Abstract: In this study, the authors address the two-player zero-sum differential game problem for non-linear dynamical systems with non-linear-non-quadratic cost functions over the *infinite* time horizon. The pursuer's goal is to minimise the cost function and guarantee asymptotic stability of the closed-loop system, whereas the evader's goal is to maximise the cost function. Closed-loop asymptotic stability is certified by continuous Lyapunov functions that are viscosity solutions of the steady-state Hamilton–Jacobi–Isaacs equation for the controlled system. Since it is difficult to find viscosity solutions of partial differential equations for numerous problems of practical interest, they extend an inverse optimality framework to provide explicit closed-form solutions of differential game problems, which involve affine in the controls dynamical systems with quadratic cost functions and linear dynamical systems with Lagrangians in polynomial form. The authors' framework allows also to solve optimal robust control problems involving non-linear dynamical systems with non-linear-non-quadratic cost functionals and provides a generalisation of the mixed-norm $\mathscr{H}_2/\mathscr{H}_\infty$ optimal robust control framework. Two numerical examples illustrate the applicability of theoretical results provided.

1 Introduction

Numerous optimal and robust control problems in aerospace engineering [1, 2], electrical engineering [3], marine engineering [4], communication networks [5], and economics [6] have been successfully modelled and solved applying differential game theory [7, 8]. Two-player zero-sum differential games involve two control inputs, generally named *pursuer* and *evader*; the pursuer attempts to minimise a given cost function, whereas the evader strives to maximise this cost function. The game ends when the system's trajectory verifies some conditions, such as entering in *finite* time a given neighbourhood of the controlled system's equilibrium point. Several variations of the differential game problem, such as games involving more than two players [9], multiple cost functions [10], and cooperations among players [11, 12], have drawn considerable attention as well.

In this paper, we address the two-player zero-sum differential game problem for non-linear dynamical systems with non-linear-non-quadratic cost functions over the *infinite* time horizon. Specifically, we characterise the pursuer's and evader's non-linear state feedback control laws that enforce both the saddle point condition on the cost function and asymptotic stability of the closed-loop dynamical system; if the end-of-game condition requires steering the system's trajectory to an equilibrium point, then closed-loop asymptotic stability is critical to guarantee this condition. Our framework is not primarily designed to address pursuit games, i.e. differential games wherein the pursuer's goal is to enter a given neighbourhood of the evader within some time that is specified a priori [8, Ch. 1]. However, since the notion of asymptotic convergence is equivalent to the notion of finite-time convergence to any arbitrarily small neighbourhood of the equilibrium point [13, Def. 4.1], our framework can be used to address differential game problems that end when the system's trajectory enters a given neighbourhood of an equilibrium point within some time interval that is finite and not assigned a priori. Both the pursuer and the evader have complete knowledge of the system's state at any time instant, but do not cooperate to achieve Lyapunov stability or convergence of the closed-loop system to the equilibrium point. Indeed, the pursuer guarantees closed-loop asymptotic stability for a set of evader's admissible controls, some of which may cause instabilities of the closed-loop system, if applied with other pursuer's admissible controls.

Two-player differential games involving linear dynamical systems and quadratic cost functions over the *infinite* time horizon are discussed in [14]. In addition, differential games on the infinite horizon for non-linear systems have been explored in [15, 16], without accounting for any form of stability of the closed-loop system. Further advances on differential games on the infinite time interval occurred in the contexts of classic $\mathscr{H}_\infty$ robust control [17–20], non-linear optimal robust control [21–25], and risk-sensitive optimal control [26–29]. The connection between these topics is given by the fact that both robust and risk-sensitive control problems can be casted as deterministic differential games, where the noise plays the evader's role [30, 31].

The study of differential games is deeply connected with the problem of solving the Hamilton–Jacobi–Isaacs equation [8, Ch. 4]. Generally, continuously differentiable solutions of the Hamilton–Jacobi equation do not exist [32], and hence one needs to resort to non-smooth analysis [33, Ch. 4] and generalised solutions, such as minimax [34, Ch. 2, 35], proximal [36], and viscosity solutions [37, 38]. Alternative approaches to the differential game problem involve viability theory [39], which is based on set-valued analysis; for details, see [40, 41] and the numerous references therein.

The continuous-time, non-linear-non-quadratic optimal control problem for state-feedback asymptotic stabilisation was addressed in [42] by showing that a *continuously differentiable* solution of the steady-state Hamilton–Jacobi–Bellman equation is a Lyapunov function for the non-linear system and thus guarantees both stability and optimality. In this paper, we apply a similar framework and prove for the first time that if there exists a *continuous* Lyapunov function that is a viscosity solution of the steady-state Hamilton–Jacobi–Isaacs equation for the controlled system, then there exists a solution of the differential game on the infinite horizon, wherein the end-of-game condition is the asymptotic stabilisation of the closed-loop system. We also provide an analytic expression for the cost function evaluated at the saddle point and characterise the corresponding evader's and pursuer's control laws.

To find the viscosity solution of the steady-state Hamilton–Jacobi–Isaacs equation and solve a two-player zero-sum differential game problem, it is necessary to determine the value of the game as well as the pursuer's and evader's control strategies in state-feedback form that verify the saddle point condition. In general, the players' feedback control laws are not continuous and hence, the existence and uniqueness of the closed-loop system's trajectory cannot be guaranteed a priori. Sufficient conditions that ensure the existence and uniqueness of solutions in forward time of ordinary differential equations that are not continuous or Lipschitz continuous are given in [43–45].

By modelling the pursuer as the desired control action and the evader as an exogenous disturbance acting on the dynamical system, the differential game framework presented in this paper allows tackling the optimal robust control problem for closed-loop asymptotic stabilisation of non-linear dynamical systems. Specifically, this paper is the first to address the optimal robust control problem for closed-loop asymptotic stabilisation of non-linear dynamical systems using continuous, but not continuously differentiable, Lyapunov functions. The least upper bound on the system's cost function is estimated by providing an analytical expression for the best worst-case system's performance over the class of admissible input disturbances. Remarkably, it is shown that our framework specialises to the classic mixed-norm $\mathscr{H}_2/\mathscr{H}_\infty$ framework developed in [46, 47] to solve the optimal robust control problem for linear dynamical systems with quadratic cost functions. It is also worth to note the connections between this work and [48, 49], where the differential game problem and the optimal robust control problem over the infinite time horizon are addressed under the restrictive assumption that the Hamilton–Jacobi–Isaacs equation has a continuously differentiable solution.

Computing viscosity solutions of the Hamilton–Jacobi–Isaacs equation and the corresponding pursuer's and evader's control laws is a daunting task for most problems of practical interest. In the second part of this paper, we extend for the first time an inverse optimal control approach [50–55] to parameterise a family of asymptotically stabilising controllers that guarantee the existence of a saddle point for a derived cost functional. Specifically, we do not attempt to find a value function and feedback control laws that satisfy the Hamilton–Jacobi–Isaacs equation, but given a family of asymptotically stabilising controllers, we find those cost functions that satisfy the Hamilton–Jacobi–Isaacs equation. Applying our converse differential game framework to solve the inverse optimal robust control problem, we achieve the same results as in [46], which were deduced using a mixed-norm $\mathscr{H}_2/\mathscr{H}_\infty$ approach, and in [47] using a dissipativity-based approach.

Two numerical examples illustrate the features and the applicability of the theoretical results proven. Specifically, the first example concerns a differential game problem, wherein a missile is used to defend an aircraft from an attacking missile [56]. The second example illustrates our converse differential game framework by addressing a biological pest control problem modelled by using the Lotka–Volterra equations [13, p. 198].

2 Mathematical background

In this section, we establish notation, definitions, and review some basic results. Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, $\mathbb{R}^{n\times m}$ denotes the set of $n \times m$ real matrices, and $\mathbb{C}$ denotes the set of complex numbers. We write $\|\cdot\|$ for the Euclidean vector norm, I_n or I for the $n \times n$ identity matrix, $0_{n\times m}$ or 0 for the zero $n \times m$ matrix, and A^{T} for the transpose of the matrix A. Given $f: X \times Y \to \mathbb{R}$, where $X \subseteq \mathbb{R}^{m_1}$ and $Y \subseteq \mathbb{R}^{m_2}$, we define

$$\arg\min_{(x,y)\in(X,Y)} \max f(x,y) \triangleq \{(x^*,y^*)\in(X,Y): f(x^*,y^*) \le f(x,y^*), \forall x\in X, \text{ and } f(x^*,y^*) \ge f(x^*,y), \forall y \in Y\}$$

and

$$\min_{(x,y)\in(X,Y)} \max f(x,y) \triangleq f(x^*,y^*),$$

for all $(x^*,y^*) \in \arg\min_{(x,y)\in(X,Y)}\max f(x,y)$. If $(x^*,y^*) \in \arg\min_{(x,y)\in(X,Y)}\max f(x,y)$, then we say that (x^*,y^*) is a *saddle point* for $f(\cdot,\cdot)$ on $X \times Y$.

The next result states a key property of saddle points, known as *minimax equality*.

Lemma 1 [21]: Let $f: X \times Y \to \mathbb{R}$, where $X \subseteq \mathbb{R}^{m_1}$ and $Y \subseteq \mathbb{R}^{m_2}$. If the set $\arg\min_{(x,y)\in(X,Y)}\max f(x,y)$ is not empty, then

$$\min_{x\in X}\max_{y\in Y} f(x,y) = \max_{y\in Y}\min_{x\in X} f(x,y). \tag{1}$$

In this paper, we consider non-linear autonomous dynamical systems of the form

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \quad t \ge 0, \tag{2}$$

where for every $t \ge 0$, $x(t) \in \mathscr{D} \subseteq \mathbb{R}^n$, $\mathscr{D}$ is an open set with $0 \in \mathscr{D}$, and $f: \mathscr{D} \to \mathbb{R}^n$ is such that $f(0) = 0$ and $f(\cdot)$ is locally Lipschitz continuous in x. Next, we introduce the definition of total derivative of a continuous function along the trajectory of (2).

Definition 1 [57, p. 422]: Consider the *continuous* function $V: \mathscr{D} \to \mathbb{R}$ and let $x(t)$, $t \ge 0$, denote the solution of (2). Then, the *total derivative* of $V: \mathscr{D} \to \mathbb{R}$ along the trajectory of (2) is defined as

$$\dot{V}(x(t)) \triangleq \lim_{\tau\to t^-} \frac{V(x(t)) - V(x(\tau))}{t-\tau}, \quad t \ge 0, \tag{3}$$

whenever this limit exists.

If $V(\cdot)$ is *continuously differentiable*, then Definition 1 reduces to the classic definition of total derivative [13, p. 137]. Specifically, if $V(\cdot)$ is continuously differentiable, then $\dot{V}(x(t)) = \partial V(x(t))/\partial x f(x(t))$, where $\partial V(x)/\partial x$ denotes the Fréchet derivative of $V(\cdot)$ at x.

In the following, we introduce the notion of viscosity solution of a first-order partial differential equation. To this goal, we need to define first order sub- and super-jets, and proper continuous functions.

Definition 2 [58, p. 23]: Consider the continuous function $V: \mathscr{D} \to \mathbb{R}$ and let $\xi \in \mathbb{R}^n$ be such that $x + \xi \in \mathscr{D}$. The *first-order sub-jets* of $V(\cdot)$ is defined as

$$\mathscr{J}^{1,+}V(x) \triangleq \{p \in \mathbb{R}^{1\times n}: V(x+\xi) \le V(x) + p\xi + o(\|\xi\|), \text{ as } \xi \to 0\}, \tag{4}$$

and the *first-order super-jets* of $V(\cdot)$ is defined as

$$\mathscr{J}^{1,-}V(x) \triangleq \{p \in \mathbb{R}^{1\times n}: V(x+\xi) \ge V(x) + p\xi + o(\|\xi\|), \text{ as } \xi \to 0\}. \tag{5}$$

Definition 3 [58, p. 57]: Let $G: \mathscr{D} \times \mathbb{R} \times \mathbb{R}^{1\times n} \to \mathbb{R}$ be a continuous function. If

$$G(x,r,p) \ge G(x,s,p), \quad (x,p) \in \mathscr{D} \times \mathbb{R}^{1\times n}, \tag{6}$$

for all $r \ge s$, then $G(\cdot)$ is a *proper continuous function*.

Sub- and super-jets allowed Crandall and Lions [38] to introduce the concept of viscosity solution of a partial differential equation, which is presented by the next definition.

Definition 4 [58, pp. 22–23]: Let $V: \mathscr{D} \to \mathbb{R}$ be a continuous function on $\mathscr{D}$ and $G: \mathscr{D} \times \mathbb{R} \times \mathbb{R}^{1\times n} \to \mathbb{R}$ be a proper continuous function on $\mathscr{D} \times \mathbb{R} \times \mathbb{R}^{1\times n}$. If

$$G(x, V(x), p) \ge 0, \quad p \in \mathscr{J}^{1,+}V(x), \quad x \in \mathscr{D}, \tag{7}$$

then $V(\cdot)$ is a viscosity subsolution of

$$G(x, V(x), V'(x)) \geq 0 \tag{8}$$

on $\mathscr{D}$. If

$$G(x, V(x), p) \leq 0, \quad p \in \mathscr{J}^{1,-}V(x), \quad x \in \mathscr{D}, \tag{9}$$

then $V(\cdot)$ is a viscosity supersolution of

$$G(x, V(x), V'(x)) \leq 0 \tag{10}$$

on $\mathscr{D}$. Finally, if $V(\cdot)$ is both a viscosity subsolution of (8) and a viscosity supersolution of (10), then $V(\cdot)$ is a *viscosity solution* of

$$G(x, V(x), V'(x)) = 0, \quad x \in \mathscr{D}. \tag{11}$$

The next theorem provides sufficient conditions for asymptotic stability of the non-linear dynamical system (2), which involve continuous Lyapunov functions. For the statement of this result, recall that $z: [0, \infty) \to \mathbb{R}$ is *absolutely continuous* if for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$\sum_{k=1}^{N} |z(b_k) - z(a_k)| < \varepsilon, \quad \sum_{k=1}^{N} (b_k - a_k) < \delta(\varepsilon), \tag{12}$$

where $(a_k, b_k) \subset [0, \infty)$, $k = 1, \ldots, N$, are disjoint intervals [59, p. 127].

Theorem 1: Consider the non-linear dynamical system (2) and let $V: \mathscr{D} \to \mathbb{R}$ be a continuous function such that

$$V(0) = 0, \tag{13}$$

$$V(x) > 0, \quad x \in \mathscr{D} \setminus \{0\}. \tag{14}$$

If $V(x(t))$, $t \geq 0$, is absolutely continuous along the trajectory of (2), $\dot{V}(x(t))$ exists for all $t \geq 0$, and there exists a class $\mathscr{K}$ function $\gamma: [0, \infty) \to [0, \infty)$ such that

$$\dot{V}(x(t)) < -\gamma(\| x(t) \|), \quad x(t) \neq 0, \quad t \geq 0, \tag{15}$$

then the equilibrium point $x(t) \equiv 0$ of (2) is asymptotically stable. Finally, if $\mathscr{D} = \mathbb{R}^n$ and $V(\cdot)$ is radially unbounded, i.e.

$$V(x) \to \infty, \quad \| x \| \to \infty, \tag{16}$$

then the zero solution $x(t) \equiv 0$, $t \geq 0$, to (2) is globally asymptotically stable.

Proof: Since $V(x(t))$, $t \geq 0$, is absolutely continuous along the trajectory of (2), it follows from (15) and Theorem 3.11 of [59] that $V(x(t))$ is a decreasing function of time, i.e. $V(x(t)) < V(x(\tau))$, for all $t > \tau \geq 0$. The result now directly follows from Theorem 12.1 of [57] applied to the time-invariant Lipschitz continuous dynamical system (2). □

If the Lyapunov function $V(\cdot)$ in Theorem 1 is continuously differentiable, then Theorem 1 specialises to Lyapunov's sufficient conditions for asymptotic stability [13, Th. 3.1].

In this paper, we consider controlled non-linear dynamical systems of the form

$$\dot{x}(t) = F(x(t), u(t), w(t)), \quad x(0) = x_0, \quad t \geq 0, \tag{17}$$

where $F: \mathscr{D} \times U \times W \to \mathbb{R}^n$ is locally Lipschitz continuous in x, u, and w, $\mathscr{D}$ is an open set with $0 \in \mathscr{D} \subseteq \mathbb{R}^n$, $U \subseteq \mathbb{R}^{m_1}$ with $0 \in U$, $W \subseteq \mathbb{R}^{m_2}$ with $0 \in W$, and $F(0, 0, 0) = 0$. The controls $u(\cdot)$ and $w(\cdot)$ in (17) are restricted to the class of *admissible* controls consisting of integrable functions $u: [0, \infty) \to U$ and $w: [0, \infty) \to W$, and we assume that $x(t) \in \mathscr{D}$, $t \geq 0$, for all admissible controls $u(\cdot)$ and $w(\cdot)$. A solution $t \mapsto x(t)$ of (17) with admissible controls $u(\cdot)$ and $w(\cdot)$ is said to be *right maximally* defined if $x(\cdot)$ cannot be extended (either uniquely or non-uniquely) forward in time. We assume that a right maximal solution of (17) with admissible controls $u(\cdot)$ and $w(\cdot)$ exists on $[0, \infty)$ and is unique, and hence, we assume that (17) with admissible controls $u(\cdot)$ and $w(\cdot)$ is *forward complete*.

Integrable functions $\phi: \mathscr{D} \to U$ and $\psi: \mathscr{D} \to W$ satisfying $\phi(0) = 0$ and $\psi(0) = 0$ are called *control laws*. If $u(t) = \phi(x(t))$, $t \geq 0$, and $w(t) = \psi(x(t))$, where $\phi(\cdot)$ and $\psi(\cdot)$ are control laws and $x(t)$ satisfies (17), then we call $u(\cdot)$ and $w(\cdot)$ *feedback control laws*. Given control laws $\phi(\cdot)$ and $\psi(\cdot)$, and feedback control laws $u(t) = \phi(x(t))$, $t \geq 0$, and $w(t) = \psi(x(t))$, the *closed-loop system* (17) is given by

$$\dot{x}(t) = F(x(t), \phi(x(t)), \psi(x(t))), \quad x(0) = x_0, \quad t \geq 0. \tag{18}$$

Next, we introduce the notion of asymptotically stabilising feedback control laws. To this goal, consider the controlled non-linear dynamical system (17) and define the set of regulation controllers

$$\begin{aligned} \mathscr{S}(x_0) \triangleq \{(u(\cdot), w(\cdot)) : u(\cdot) \text{ and } w(\cdot) \text{ are admissible and (17)} \\ \text{has a unique solution } x(\cdot) \text{ that satisfies} \\ x(t) \to 0 \text{ as } t \to \infty\}. \end{aligned}$$

In addition, given the control law $\psi(\cdot)$, let $\mathscr{S}_\psi(x_0) \triangleq \{u(\cdot) : (u(\cdot), \psi(x(\cdot))) \in \mathscr{S}(x_0)\}$ and given the control law $\phi(\cdot)$, let $\mathscr{S}_\phi(x_0) \triangleq \{w(\cdot) : (\phi(x(\cdot)), w(\cdot)) \in \mathscr{S}(x_0)\}$.

Definition 5 [48]: Consider the controlled dynamical system (17). The feedback control law $u(\cdot) = \phi(x(\cdot))$ is *asymptotically stabilising* if the closed-loop system (18) is asymptotically stable for all admissible controls $w(\cdot) \in \mathscr{S}_\phi(x_0)$. Furthermore, the feedback control law $u(\cdot) = \phi(x(\cdot))$ is *globally asymptotically stabilising* if the closed-loop system (18) is globally asymptotically stable for all admissible controls $w(\cdot) \in \mathscr{S}_\phi(x_0)$.

Assuming that $(\phi(x(\cdot)), \psi(\cdot)) \in \mathscr{S}(x_0)$ does not necessarily imply that both players are actively pursuing the stability of the closed-loop system. In this paper, the case wherein the pursuer guarantees closed-loop stability in spite of the evader's input, i.e. $u = \phi(x)$, $x \in \mathscr{D}$, and $w(\cdot) \in \mathscr{S}_\phi(x_0)$, is addressed within the context of optimal robust control. Note that if $(\phi(x(\cdot)), \psi(\cdot)) \in \mathscr{S}(x_0)$, then there exists a unique solution $x(t)$, $t \geq 0$, of (17). This assumption is commonly employed in the study of control Lyapunov functions and asymptotic controllability of non-linear dynamical systems [60].

3 Differential games and closed-loop asymptotic stabilisation

In this section, we solve differential game problems involving non-linear-non-quadratic cost functions and non-linear dynamical systems, whose end-of-game condition is the asymptotic stabilisation of the closed-loop system in spite of the evader's disturbing action. The next theorem, which is the main result of this section, characterises feedback controllers that guarantee asymptotic closed-loop stability of (17), and minimise with respect to $u(\cdot)$ and maximise with respect to $w(\cdot)$ a non-linear-non-quadratic performance functional; sufficient conditions for the existence of a saddle point are given by considering viscosity solutions of the steady-state Hamilton–Jacobi–Isaacs equation. For the statement of this result, let $L: \mathscr{D} \times U \times W \to \mathbb{R}$ be continuous in x, u, and w.

Theorem 2: Consider the controlled non-linear dynamical system (17) with

$$J(x_0, u(\cdot), w(\cdot)) \triangleq \int_0^\infty L(x(t), u(t), w(t))\,\mathrm{d}t, \tag{19}$$

where $u(\cdot)$ and $w(\cdot)$ are admissible controls. Let $V: \mathscr{D} \to \mathbb{R}$ be a continuous function, and let $\phi: \mathscr{D} \to U$ and $\psi: \mathscr{D} \to W$ be control laws such that

$$\phi(0) = 0, \tag{20}$$

$$\psi(0) = 0, \tag{21}$$

$$V(0) = 0, \tag{22}$$

$$V(x) > 0, \quad x \in \mathscr{D} \setminus \{0\}. \tag{23}$$

Assume that there exists a class $\mathscr{K}$ function $\gamma: [0, \infty) \to [0, \infty)$ such that

$$\dot{V}(x(t)) < -\gamma(\| x(t) \|), \quad x(t) \neq 0, \quad t \geq 0, \tag{24}$$

$$L(x(t), \phi(x(t)), \psi(x(t))) + \dot{V}(x(t)) = 0, \tag{25}$$

along the trajectory of the closed-loop system (18). Suppose that $V(x(t))$, $t \geq 0$, is absolutely continuous along the trajectory of

$$\dot{x}(t) = F(x(t), u(t), \psi(x(t))), \quad x(0) = x_0, \quad t \geq 0, \tag{26}$$

for all $u(\cdot) \in \mathscr{S}_\psi(x_0)$, $\dot{V}(x(t))$ exists for all $t \geq 0$, and

$$L(x(t), u(t), \psi(x(t))) + \dot{V}(x(t)) \geq 0. \tag{27}$$

Moreover, assume that $V(x(t))$, $t \geq 0$, is absolutely continuous along the trajectory of

$$\dot{x}(t) = F(x(t), \phi(x(t)), w(t)), \quad x(0) = x_0, \quad t \geq 0, \tag{28}$$

for all $w(\cdot) \in \mathscr{S}_\phi(x_0)$, $\dot{V}(x(t))$ exists for all $t \geq 0$, and

$$L(x(t), \phi(x(t)), w(t)) + \dot{V}(x(t)) \leq 0. \tag{29}$$

Then with the feedback controls $u = \phi(x)$ and $w = \psi(x)$, the closed-loop system (18) is asymptotically stable and there exists a neighbourhood $\mathscr{D}_0 \subseteq \mathscr{D}$ of $x = 0$ such that

$$J(x_0, \phi(x(\cdot)), \psi(x(\cdot))) = V(x_0), \quad x_0 \in \mathscr{D}_0. \tag{30}$$

In addition, if $x_0 \in \mathscr{D}_0$, then

$$\begin{aligned} &J(x_0, \phi(x(\cdot)), \psi(x(\cdot))) \\ &\quad = \min \max{}_{(u(\cdot), w(\cdot)) \in \mathscr{S}_\psi(x_0) \times \mathscr{S}_\phi(x_0)} J(x_0, u(\cdot), w(\cdot)). \end{aligned} \tag{31}$$

Finally, if $\mathscr{D} = \mathbb{R}^n$, $U = \mathbb{R}^{m_1}$, $W = \mathbb{R}^{m_2}$, and

$$V(x) \to \infty, \quad \| x \| \to \infty, \tag{32}$$

then the closed-loop system (18) is globally asymptotically stable.

Proof: Note that $V(x(t))$ is absolutely continuous and $\dot{V}(x(t))$ exists along the trajectory of (18), since $V(x(t))$, $t \geq 0$, is absolutely continuous and $\dot{V}(x(t))$ exists both along the trajectories of (26) and (28) for all $u(\cdot) \in \mathscr{S}_\psi(x_0)$ and for all $w(\cdot) \in \mathscr{S}_\phi(x_0)$. Hence, local and global asymptotic stabilities are direct consequences of (22)–(24), and (32) by applying Theorem 1 to the closed-loop system given by (18).
Since $V(x(t))$, $t \geq 0$, is absolutely continuous, it follows from (25) and Theorem 3.11 of [59] that

$$\begin{aligned} &\int_0^\infty L(x(t), \phi(x(t)), \psi(x(t)))\, \mathrm{d}t \\ &\quad = -\int_0^\infty \dot{V}(x(t))\, \mathrm{d}t = V(x(0)) - \lim_{t \to \infty} V(x(t)), \end{aligned} \tag{33}$$

where $x(t)$, $t \geq 0$, satisfies (18). Equation (30) now follows from (33), since $V(\cdot)$ is a continuous function and hence

$$\lim_{t \to \infty} V(x(t)) = V\Big(\lim_{t \to \infty} x(t)\Big) = 0, \tag{34}$$

along the trajectory of (18).
Next, let $x_0 \in \mathscr{D}_0$, let $u(\cdot)$ and $w(\cdot)$ be admissible controls, and let $x(t)$, $t \geq 0$, be the solution of (17). Then, it holds that

$$\begin{aligned} L(x(t), u(t), w(t)) = {} & \dot{V}(x(t)) + L(x(t), u(t), w(t)) \\ & - \dot{V}(x(t)), \quad t \geq 0, \end{aligned} \tag{35}$$

along the trajectory of (17). Now, since $V(\cdot)$ is continuous, (34) is satisfied along the trajectory of (26) for every $u(\cdot) \in \mathscr{S}_\psi(x_0)$, and since $V(x(t))$, $t \geq 0$, is absolutely continuous along the trajectory of (26), it follows from (35), (34), (27), and Theorem 3.11 of [59] that

$$\begin{aligned} J(x_0, u(\cdot), \psi(x(\cdot))) &= \int_0^\infty L(x(t), u(t), \psi(x(t)))\, \mathrm{d}t \\ &= \int_0^\infty \dot{V}(x(t))\, \mathrm{d}t \\ &\quad + \int_0^\infty L(x(t), u(t), \psi(x(t)))\, \mathrm{d}t \\ &\quad - \int_0^\infty \dot{V}(x(t))\, \mathrm{d}t \\ &\geq -\int_0^\infty \dot{V}(x(t))\, \mathrm{d}t \\ &= -\lim_{t \to \infty} V(x(t)) + V(x_0) \\ &= J(x_0, \phi(x(\cdot)), \psi(x(\cdot))). \end{aligned} \tag{36}$$

Similarly, since $V(x(t))$, $t \geq 0$, is absolutely continuous along the trajectory of (28), (34) is satisfied along the trajectory of (28) for every $w(\cdot) \in \mathscr{S}_\phi(x_0)$, and it follows from (35), (34), (29), and Theorem 3.11 of [59] that

$$\begin{aligned} J(x_0, \phi(x(\cdot)), w(\cdot)) &= \int_0^\infty L(x(t), \phi(x(t)), w(t))\, \mathrm{d}t \\ &\leq -\int_0^\infty \dot{V}(x(t))\, \mathrm{d}t \\ &= -\lim_{t \to \infty} V(x(t)) + V(x_0) \\ &= J(x_0, \phi(x(\cdot)), \psi(x(\cdot))). \end{aligned} \tag{37}$$

Hence (36) and (37) yield (31). Finally, it follows from (31) that $(\phi(x(\cdot)), \psi(x(\cdot)))$ is a saddle point for $J(x_0, u(\cdot), w(\cdot))$ on $\mathscr{S}_\psi(x_0) \times \mathscr{S}_\phi(x_0)$, $x_0 \in \mathscr{D}_0$. □

Conditions (22)–(24) imply that the continuous function $V(\cdot)$, which verifies the partial differential equation (25), is a Lyapunov function that guarantees asymptotic stability of the closed-loop system (18). Conditions (25), (27), and (29) guarantee that the saddle point condition (31) is satisfied. Therefore, if the conditions of Theorem 2 are satisfied, then the pursuer's control law $u = \phi(x)$, $x \in \mathscr{D}$, and the evader's control law $w = \psi(x)$ guarantee successful completion of a differential game, whose terminal condition requires the asymptotic stabilisation of the underlying dynamical system. The next result provides a connection between (25) and the *Hamilton–Jacobi–Isaacs equation*

$$L(x, \phi(x), \psi(x)) + V'(x) F(x, \phi(x), \psi(x)) = 0, \quad x \in \mathscr{D}, \tag{38}$$

where $V'(\cdot)$ verifies Definition 4 with

$$G(x, q, p) = L(x, \phi(x), \psi(x)) + p^{\mathrm{T}} F(x, \phi(x), \psi(x)),$$

$(x, q, p) \in \mathscr{D} \times \mathbb{R} \times \mathbb{R}^n$.

Proposition 1: Let $V: \mathscr{D} \to \mathbb{R}$ be a viscosity solution of (38). Then

$$\dot{V}(x(t)) = V'(x(t)) \dot{x}(t), \quad t \geq 0, \tag{39}$$

along the trajectory of the closed-loop system (18), and (25) is verified along the trajectory of (18).

Proof: If $V(\cdot)$ is a viscosity solution of (38), then it follows from Definitions 2 and 4 that

$$\lim_{\|\xi\|\to 0}\frac{V(x+\xi)-V(x)-V'(x)\xi}{\|\xi\|}=0, \tag{40}$$

where $V'(x)\in\left[\mathscr{J}^{1,+}V(x)\right]\cap\left[\mathscr{J}^{1,-}V(x)\right]$, $\xi\in\mathbb{R}^n$, and $x+\xi\in\mathscr{D}$. Now, let $x(t)$, $t\ge 0$, denote the solution of (18) and $\xi(t)=x(t+\tau)-x(t)$, where $\tau\ge 0$. Then it follows from (40) that

$$\begin{aligned}\lim_{\tau\to 0^-}\Big[&\frac{V(x(t+\tau))-V(x(t))}{\tau}\frac{\tau}{\|x(t+\tau)-x(t)\|}\\ &-V'(x(t))\frac{x(t+\tau)-x(t)}{\tau}\frac{\tau}{\|x(t+\tau)-x(t)\|}\Big]=0,\end{aligned} \tag{41}$$

for all $t\ge 0$, which implies that (39) is satisfied along the trajectory of (18), and the result ensues. □

Proposition 1 allows casting the framework outlined in Theorem 2 within the context of generalised solutions of the Hamilton–Jacobi–Isaacs equation. Specifically, this result proves that viscosity solutions of (38) satisfy (25). However, the converse is not necessarily true. Remarkably, if a viscosity solution of the Dirichlet problem associated to (38) exists, then it is unique [58, Ch. 6, Th. 2].

It holds that $\mathscr{S}_\psi(x_0)\times\mathscr{S}_\phi(x_0)\subseteq\mathscr{S}(x_0)$, for given $\phi(\cdot)$ and $\psi(\cdot)$, and considering control inputs corresponding to null convergent solutions, i.e. such that $(u(\cdot),w(\cdot))\in\mathscr{S}(x_0)$, is equivalent to incorporating a system detectability condition through the cost. However, an explicit characterisation of the sets $\mathscr{S}(x_0)$, $\mathscr{S}_\psi(x_0)$, and $\mathscr{S}_\phi(x_0)$ is not necessary.

The feedback control laws $u=\phi(x)$ and $w=\psi(x)$ are independent of the initial condition x_0 and, using (38), (27), and (29) are given by

$$\begin{aligned}&\left[\phi^{\mathrm{T}}(x),\ \psi^{\mathrm{T}}(x)\right]^{\mathrm{T}}\\ &\in\underset{(u(\cdot),w(\cdot))\in\mathscr{S}_\psi(x_0)\times\mathscr{S}_\phi(x_0)}{\arg\min\max}\ [L(x,u,w)+V'(x)F(x,u,w)],\end{aligned} \tag{42}$$

where $V'(x)$ belongs to the intersection of the sub- and super-jets of $V(\cdot)$, i.e. $V'(x)\in\left[\mathscr{J}^{1,+}V(x)\right]\cap\left[\mathscr{J}^{1,-}V(x)\right]$, $x\in\mathscr{D}$. It follows from Theorem 2 that the pair $(\phi(\cdot),\psi(\cdot))$ guarantees asymptotic stability of the closed-loop system. However, $\psi(\cdot)$ may generate instabilities in the sense that, given an admissible control $u(\cdot)\notin\mathscr{S}_\psi(x_0)$, the solution $x(t)=0$, $t\ge 0$, of the non-linear differential equation

$$\dot{x}(t)=F(x(t),u(t),\psi(x(t))),\quad x(0)=x_0,\quad t\ge 0, \tag{43}$$

is not asymptotically stable and may even be unstable.

If we consider the input $w(\cdot)$ in (17) as a disturbance, then the framework developed in Theorem 2 allows addressing optimal robust control problems, since it provides an analytical expression for the least upper bound with respect to $u(\cdot)$ of the cost function (19) over a class of non-disruptive exogenous disturbances $w(\cdot)\in\mathscr{S}_\phi(x_0)$. Specifically, it follows from (30) and (31) that

$$\begin{aligned}J(x_0,\phi(x(\cdot)),w(\cdot))&\le J(x_0,\phi(x(\cdot)),\psi(x(\cdot)))\\ &=V(x_0),\quad x_0\in\mathscr{D}_0,\end{aligned} \tag{44}$$

for all admissible inputs $w(\cdot)$ such that $\lim_{t\to\infty}x(t)=0$, where $x(\cdot)$ is the solution of (17) with $u=\phi(x)$. Hence, we provide an expression for the best worst-case system performance over the class of admissible input disturbances.

The next remark explains how the proposed framework, which is designed to address differential games on the *infinite* time horizon, can be employed also to address differential games, such as pursuit games, evolving over a *finite-time* interval that is not assigned a priori.

Remark 1: If the assumptions of Theorem 2 are verified, then the closed-loop dynamical system is asymptotically stable, which implies, per definition, that for every $l>0$, there exists $\hat{t}_{\mathrm{f}}\ge 0$ such that if $t>\hat{t}_{\mathrm{f}}$, then $\|x(t)\|<l$, where $x(\cdot)$ denotes the solution of (18). Now, consider a game involving the non-linear dynamical system (17) and the cost function (19), wherein the closed-loop system trajectory has to be steered to a neighbourhood of the equilibrium point $x=0$ in finite-time, i.e. such that $\|x(t_{\mathrm{f}})\|=l$, where $t_{\mathrm{f}}\ge 0$ is finite and not specified a priori. In this case, Theorem 2 provides sufficient conditions to solve this game. Therefore, Theorem 2 can be applied to address games of degree [8, p. 12], such as the *homicidal chauffeur game* [8, pp. 232–237], whose end-of-game condition is that the system's trajectory enters a given neighbourhood of the origin within a *finite* interval not assigned a priori.

Next, we apply Theorem 2 to address the linear-quadratic differential game problem and show clear connections between our differential game framework and the mixed-norm $\mathscr{H}_2/\mathscr{H}_\infty$ frameworks developed in [46, 47]. For the statement of the next result, consider the linear time-invariant dynamical system

$$\dot{x}(t)=Ax(t)+B_u u(t)+B_w w(t),\quad x(0)=x_0,\quad t\ge 0, \tag{45}$$

with cost function

$$\begin{aligned}J(x_0,u(\cdot),w(\cdot))=\int_0^\infty&[x^{\mathrm{T}}(t)R_1x(t)+u^{\mathrm{T}}(t)R_{2u}u(t)\\ &-\gamma^2w^{\mathrm{T}}(t)w(t)]\mathrm{d}t,\end{aligned} \tag{46}$$

where $x(t)\in\mathbb{R}^n$, $t\ge 0$, $u(t)\in\mathbb{R}^{m_1}$, $w(t)\in\mathbb{R}^{m_2}$, $A\in\mathbb{R}^{n\times n}$, $B_u\in\mathbb{R}^{n\times m_1}$, $B_w\in\mathbb{R}^{n\times m_2}$, $R_1\in\mathbb{R}^{n\times n}$, $R_{2u}\in\mathbb{R}^{m_1\times m_1}$, and $\gamma>0$. In addition, we assume that $R_1>0$, $R_{2u}>0$, and $B_uR_{2u}^{-1}B_u^{\mathrm{T}}-\gamma^{-2}B_wB_w^{\mathrm{T}}\ge 0$.

Corollary 1: Consider the linear time-invariant dynamical system (45) with quadratic cost function (46), where $u(\cdot)$ and $w(\cdot)$ are admissible controls. If there exists $P\in\mathbb{R}^{n\times n}$ such that $P>0$ and

$$0=A^{\mathrm{T}}P+PA+R_1-PB_uR_{2u}^{-1}B_u^{\mathrm{T}}P+\gamma^{-2}PB_wB_w^{\mathrm{T}}P, \tag{47}$$

then, with the feedback controls

$$u=\phi(x)=-R_{2u}^{-1}B_u^{\mathrm{T}}Px, \tag{48}$$

$$w=\psi(x)=\gamma^{-2}B_w^{\mathrm{T}}Px, \tag{49}$$

the dynamical system (45) is globally asymptotically stable

$$J(x_0,\phi(x(\cdot)),\psi(x(\cdot)))=x_0^{\mathrm{T}}Px_0,\quad x_0\in\mathbb{R}^n, \tag{50}$$

Equation (31) is satisfied with $x_0\in\mathbb{R}^n$, and

$$J(x_0,\phi(x(\cdot)),w(\cdot))\le x_0^{\mathrm{T}}Px_0,\quad w(\cdot)\in\mathscr{S}_\phi(x_0),\quad x_0\in\mathbb{R}^n. \tag{51}$$

In addition, the zero solution $x(t)\equiv 0$, $t\ge 0$, of (45) with $u=\phi(x)$ and $w=0$ is globally asymptotically stable and

$$J(x_0,\phi(x(\cdot)),0)\le J(x_0,\phi(x(\cdot)),\psi(x(\cdot))),\quad x_0\in\mathbb{R}^n. \tag{52}$$

Proof: The result is a consequence of Theorem 2 with $F(x,u,w)=Ax+B_uu+B_ww$, $L(x,u,w)=x^{\mathrm{T}}R_1x+u^{\mathrm{T}}R_{2u}u-\gamma^2w^{\mathrm{T}}w$, $V(x)=x^{\mathrm{T}}Px$, $\mathscr{D}=\mathbb{R}^n$, $U=\mathbb{R}^{m_1}$, and $W=\mathbb{R}^{m_2}$. Specifically, (22), (23), and (32) are trivially satisfied, and since $V(\cdot)$ is continuously differentiable, (48) and (49) directly follow from (42) by setting

$$0 = \frac{\partial}{\partial u}\frac{\partial}{\partial w}\big[x^{\mathrm{T}}R_1x + u^{\mathrm{T}}R_{2u}u - \gamma^2 w^{\mathrm{T}}w + V'(x)(Ax + B_u u + B_w w)\big].$$

Moreover, it follows from (47) that

$$\begin{aligned} &x^{\mathrm{T}}R_1x + \phi^{\mathrm{T}}(x)R_{2u}\phi(x) - \gamma^2\psi^{\mathrm{T}}(x)\psi(x) \\ &\quad + V'(x)[Ax + B_u\phi(x) + B_w\psi(x)] \\ &= x^{\mathrm{T}}\big[A^{\mathrm{T}}P + PA + R_1 - PB_uR_{2u}^{-1}B_u^{\mathrm{T}}P \\ &\quad + \gamma^{-2}PB_wB_w^{\mathrm{T}}P\big]x \\ &= 0, \quad x \in \mathbb{R}^n, \end{aligned} \tag{53}$$

which verifies (25), and since $R_1 > 0$ and $B_uR_{2u}^{-1}B_u^{\mathrm{T}} - \gamma^{-2}B_wB_w^{\mathrm{T}} \geq 0$, (24) is satisfied with

$$\gamma(\parallel x \parallel) = x^{\mathrm{T}}\big(R_1 + B_uR_{2u}^{-1}B_u^{\mathrm{T}} - \gamma^{-2}B_wB_w^{\mathrm{T}}\big)x.$$

Finally, since

$$\begin{aligned} &L(x, u, \psi(x)) + V'(x)F(x, u, \psi(x)) \\ &= L(x, u, \psi(x)) + V'(x)[Ax + B_u u + B_w\psi(x)] \\ &\quad - L(x, \phi(x), \psi(x)) - V'(x)[Ax + B_u\phi(x) + B_w\psi(x)] \\ &= [u - \phi(x)]^{\mathrm{T}}R_{2u}(x)[u - \phi(x)] \\ &\geq 0, \quad (x, u) \in \mathbb{R}^n \times \mathbb{R}^{m_1}, \end{aligned} \tag{54}$$

and

$$\begin{aligned} L(x, \phi(x), w) + V'(x)F(x, u, w) = -\gamma^2 \parallel w - \psi(x) \parallel^2 \leq 0, \\ (x, w) \in \mathbb{R}^n \times \mathbb{R}^{m_2}, \end{aligned} \tag{55}$$

conditions (27) and (29) hold.

Now, since the conditions of Theorem 2 are satisfied, (50) and (51) directly follow from (30) and (44), respectively, (31) with $x_0 \in \mathbb{R}^n$ is satisfied, and (45) with $u = \phi(x)$ and $w = \psi(x)$ is globally asymptotically stable. Lastly, it follows from Theorem 8.3 of [13] that (45) with $u = \phi(x)$ and $w = 0$ is globally asymptotically stable, which implies that $x(t) \to 0$ as $t \to \infty$ and the constant function $w(t) = 0$ is such that $w(\cdot) \in \mathcal{S}_\phi(x_0)$. Thus, (52) directly follows from (51). □

Corollary 1 provides sufficient conditions for global asymptotic stability of (45) with state feedback control laws (48) and (49). Since the closed-loop linear dynamical system

$$\begin{aligned} \dot{x}(t) = \big(A - B_uR_{2u}^{-1}B_u^{\mathrm{T}}P + \gamma^{-2}B_wB_w^{\mathrm{T}}P\big)x(t), \\ x(0) = x_0, \quad t \geq 0, \end{aligned} \tag{56}$$

is globally asymptotically stable, (56) is globally exponentially stable [61]. If the conditions of Corollary 1 are verified, then Theorem 6.3.1 of [62] implies that

$$|||G(s)|||_\infty \leq \gamma, \quad s \in \mathbb{C}, \tag{57}$$

where $|||G(s)|||_\infty \triangleq \sup_{\omega \in \mathbb{R}} \sigma_{\max}[G(\jmath\omega)]$ denotes the $\mathscr{H}_\infty$ norm, $\sigma_{\max}[\cdot]$ denotes the maximum singular value,

$$G(s) \triangleq (C - D_uR_{2u}^{-1}B_u^{\mathrm{T}}P)^{\mathrm{T}}(sI_n - A + B_uR_{2u}^{-1}B_u^{\mathrm{T}}P)B_w \tag{58}$$

denotes the closed-loop transfer function of (45) with output

$$z(t) = Cx(t) + D_u u(t),$$

$R_1 = C^{\mathrm{T}}C$, and $R_{2u} = D_u^{\mathrm{T}}D_u$. Moreover, the bounded real lemma [13, Th. 5.15] implies that $G(s)$, $s \in \mathbb{C}$, is bounded real [13, Def. 5.19] and non-expansive [13, Def. 5.12].

Lastly, Corollary 1 provides clear connections with the mixed-norm $\mathscr{H}_2/\mathscr{H}_\infty$ frameworks developed in Theorem 3.1 of [46] and Corollary 4.1 of [47]. Specifically, under the same assumptions as in Corollary 1, the authors in [47] prove global exponential stability and non-expansivity of the linear time-invariant dynamical system (45) with $u = \phi(x)$ and $w = 0$. Moreover, the authors in [47] prove that both (50) and (51) are satisfied.

4 Converse differential games

Theorem 2 provides sufficient conditions to solve differential games over the infinite time horizon. In most cases of practical interest, finding the state-feedback controls $u = \phi(x)$ and $w = \psi(x)$, and the continuous function $V(\cdot)$, which satisfy the steady-state Hamilton–Jacobi–Isaacs equation (38) may be impossible.

In this section, we address the differential game problem over the infinite time horizon, in a way that does not require solving (38) directly. Specifically, we parameterise a family of asymptotically stabilising controllers $u = \phi(x)$ and $w = \psi(x)$, and provide an explicit characterisation of the cost functions, such that both the saddle point condition (31) and the Hamilton–Jacobi–Isaacs equation (38) are satisfied by $u = \phi(x)$ and $w = \psi(x)$. Once the connection between asymptotically stabilising controllers and cost functions has been established, one can vary the parameters in the performance integrands and seamlessly compute the feedback controllers that guarantee a successful completion of the game.

The approach presented in this section extends the notion of *inverse optimal control problem*, which was introduced in [51–54, 63] to address optimal control problems involving non-linear affine in the control dynamical systems with quadratic cost functions. A similar approach has been developed also in [14, Ch. 1] to parameterise the feedback control laws that solve differential games involving linear dynamical systems with quadratic cost functions.

Next, we prove a main theorem characterising converse differential games. For the statement of the next result, consider the affine in the control dynamical system

$$\begin{aligned} \dot{x}(t) = f(x(t)) + G_u(x(t))u(t) + G_w(x(t))w(t), \\ x(0) = x_0, \quad t \geq 0, \end{aligned} \tag{59}$$

with cost function

$$\begin{aligned} J(x_0, u(\cdot), w(\cdot)) = \int_0^\infty &[L_1(x(t)) + L_{2u}(x(t))u(t) \\ &+ L_{2w}(x(t))w(t) + u^{\mathrm{T}}(t)R_{2u}(x(t))u(t) \\ &+ w^{\mathrm{T}}(t)R_{2w}(x(t))w(t)]\mathrm{d}t, \end{aligned} \tag{60}$$

where $x(t) \in \mathbb{R}^n$, $t \geq 0$, $u(t) \in \mathbb{R}^{m_1}$, $w(t) \in \mathbb{R}^{m_2}$, $L_1: \mathbb{R}^n \to \mathbb{R}$, $L_{2u}: \mathbb{R}^n \to \mathbb{R}^{1 \times m_1}$, $L_{2w}: \mathbb{R}^n \to \mathbb{R}^{1 \times m_2}$, $R_{2u}: \mathbb{R}^n \to \mathbb{R}^{m_1 \times m_1}$, and $R_{2w}: \mathbb{R}^n \to \mathbb{R}^{m_2 \times m_2}$ are continuous on $\mathbb{R}^n$. Moreover, let $\partial V(x)/\partial x$ denote the Fréchet derivative of the continuously differentiable function $V(\cdot)$ at x.

Theorem 3: Consider the controlled non-linear affine dynamical system (59) with cost function (60), where $u(\cdot)$ and $w(\cdot)$ are admissible controls. Assume there exist a continuously differentiable function $V: \mathbb{R}^n \to \mathbb{R}$ such that

$$V(0) = 0, \tag{61}$$

$$V(x) > 0, \quad x \in \mathbb{R}^n \setminus \{0\}, \tag{62}$$

$$
\begin{aligned}
0 > {} & \frac{\partial V(x)}{\partial x} f(x) - \frac{1}{2}\frac{\partial V(x)}{\partial x}\big[G_u(x)R_{2u}^{-1}(x)L_{2u}^{\mathrm{T}}(x) \\
& + G_w(x)R_{2w}^{-1}(x)L_{2w}^{\mathrm{T}}(x)\big] \\
& - \frac{1}{2}\frac{\partial V(x)}{\partial x}\big[G_u(x)R_{2u}^{-1}(x)G_u^{\mathrm{T}}(x) \\
& + G_w(x)R_{2w}^{-1}(x)G_w^{\mathrm{T}}(x)\big]\left[\frac{\partial V(x)}{\partial x}\right]^{\mathrm{T}}, \quad x \in \mathbb{R}^n \setminus \{0\},
\end{aligned} \tag{63}
$$

$$L_{2u}(0) = 0, \tag{64}$$

$$L_{2w}(0) = 0, \tag{65}$$

$$V(x) \to \infty, \quad \| x \| \to \infty. \tag{66}$$

Then, with the feedback controls

$$u = \phi(x) = -\frac{1}{2}R_{2u}^{-1}(x)\left[\frac{\partial V(x)}{\partial x}G_u(x) + L_{2u}(x)\right]^{\mathrm{T}}, \tag{67}$$

$$w = \psi(x) = -\frac{1}{2}R_{2w}^{-1}(x)\left[\frac{\partial V(x)}{\partial x}G_w(x) + L_{2w}(x)\right]^{\mathrm{T}}, \tag{68}$$

the dynamical system (59) is globally asymptotically stable. In addition, the performance functional (60) with

$$
\begin{aligned}
L_1(x) = {} & \phi^{\mathrm{T}}(x)R_{2u}(x)\phi(x) + \psi^{\mathrm{T}}(x)R_{2w}(x)\psi(x) \\
& - \frac{\partial V(x)}{\partial x} f(x)
\end{aligned} \tag{69}
$$

is such that

$$J(x_0, \phi(x(\cdot)), \psi(x(\cdot))) = V(x_0), \quad x_0 \in \mathbb{R}^n, \tag{70}$$

and (31) and (44) are verified with $x_0 \in \mathbb{R}^n$. Lastly, if $L_1(x) > 0$, $x \in \mathbb{R}^n \setminus \{0\}$, and $L_{2u}(x) = 0$, $x \in \mathbb{R}^n$, then the zero solution $x(t) \equiv 0$, $t \geq 0$, of (59) with $u = \phi(x)$ and $w = 0$ is asymptotically stable and

$$J(x_0, \phi(x(\cdot)), 0) \leq V(x_0), \quad x_0 \in \mathbb{R}^n. \tag{71}$$

Proof: The proof follows as in the proofs of Theorems 2, and is omitted for brevity. □

Remarkably, Theorem 3 involves a *continuously differentiable* scalar function $V(\cdot)$. The same result was proven in Corollary 10.6 of [13], whose proof relies on a dissipativity-based approach.

Next, we specialise Theorem 3 to linear time-invariant dynamical systems, whose control inputs are non-linear and whose Lagrangian is in *polynomial form*. For the statement of the next result, consider the linear dynamical system (45), and let $R_1 \in \mathbb{R}^{n \times n}$, $R_{2u} \in \mathbb{R}^{m_1 \times m_1}$, $R_{2w} \in \mathbb{R}^{m_2 \times m_2}$, and $\hat{R}_q \in \mathbb{R}^{n \times n}$, $q = 2, \ldots, r$, where r is a positive integer, be such that $R_1 > 0$, $R_{2u} > 0$, $R_{2w} < 0$, $\hat{R}_q \geq 0$, $q = 2, \ldots, r$, and $S \triangleq B_u R_{2u}^{-1} B_u^{\mathrm{T}} + B_w R_{2w}^{-1} B_w^{\mathrm{T}} \geq 0$.

Corollary 2: Consider the controlled linear dynamical system (45), where $u(\cdot)$ and $w(\cdot)$ are admissible controls. Assume that there exist $P \in \mathbb{R}^{n \times n}$ and $M_q \in \mathbb{R}^{n \times n}$, $q = 2, \ldots, r$, such that $P > 0$, $M_q \geq 0$, $q = 2, \ldots, r$,

$$0 = A^{\mathrm{T}}P + PA + R_1 - PSP, \tag{72}$$

and

$$0 = (A - SP)^{\mathrm{T}}M_q + M_q(A - SP) + \hat{R}_q, \quad q = 2, \ldots, r. \tag{73}$$

Then the zero solution $x(t) \equiv 0$, $t \geq 0$, of the closed-loop system

$$\dot{x}(t) = Ax(t) + B_u\phi(x) + B_w\psi(x), \quad x(0) = x_0, \quad t \geq 0 \tag{74}$$

is globally asymptotically stable with feedback controls

$$\phi(x) = -R_{2u}^{-1}B_u^{\mathrm{T}}\left(P + \sum_{q=2}^{r}(x^{\mathrm{T}}M_q x)^{q-1}M_q\right)x \tag{75}$$

and

$$\psi(x) = -R_{2w}^{-1}B_w^{\mathrm{T}}\left(P + \sum_{q=2}^{r}(x^{\mathrm{T}}M_q x)^{q-1}M_q\right)x. \tag{76}$$

In addition, the performance functional (60) with $L_{2u}(x) = 0$, $L_{2w}(x) = 0$, $R_{2u}(x) = R_{2u}$, $R_{2w}(x) = R_{2w}$, and

$$
\begin{aligned}
L_1(x) = x^{\mathrm{T}}\Bigg(& R_1 + \sum_{q=2}^{r}(x^{\mathrm{T}}M_q x)^{q-1}\hat{R}_q \\
& + \left[\sum_{q=2}^{r}(x^{\mathrm{T}}M_q x)^{q-1}M_q\right]^{\mathrm{T}} \\
& \times S\left[\sum_{q=2}^{r}(x^{\mathrm{T}}M_q x)^{q-1}M_q(t)\right]\Bigg)x
\end{aligned} \tag{77}
$$

is such that

$$J(x_0, \phi(\cdot), \psi(\cdot)) = x_0^{\mathrm{T}}Px_0 + \sum_{q=2}^{r}\frac{1}{q}\left(x_0^{\mathrm{T}}M_q x_0\right)^q, \quad x_0 \in \mathbb{R}^n. \tag{78}$$

Equation (31) is verified with $x_0 \in \mathbb{R}^n$, and

$$
\begin{aligned}
J(x_0, \phi(x(\cdot)), w(\cdot)) \leq J(x_0, \phi(\cdot), \psi(\cdot)), \\
w(\cdot) \in \mathcal{S}_\phi(x_0), \quad x_0 \in \mathbb{R}^n.
\end{aligned} \tag{79}
$$

Lastly, the zero solution $x(t) \equiv 0$, $t \geq 0$, of (45) with $u = \phi(x)$ and $w = 0$, where $\phi(\cdot)$ is given by (75), is globally asymptotically stable and

$$J(x_0, \phi(x(\cdot)), 0) \leq J(x_0, \phi(x(\cdot)), \psi(x(\cdot))), \quad x_0 \in \mathbb{R}^n. \tag{80}$$

Proof: The result is a direct consequence of Theorem 3 with $f(x) = Ax$, $G_u(x) = B_u$, $G_w(x) = B_w$, $L_{2u}(x) = 0$, $L_{2w}(x) = 0$, $R_{2u}(x) = R_{2u}$, $R_{2w}(x) = R_{2w}$, and

$$V(x) = x^{\mathrm{T}}Px + \sum_{q=2}^{r}\frac{1}{q}\left(x^{\mathrm{T}}M_q x\right)^q. \tag{81}$$

Specifically, (61), (62), and (64)–(66) are trivially satisfied, and (75) and (76) directly follow from (67) and (68). Next, it follows from (72), (73), (75), and (76) that

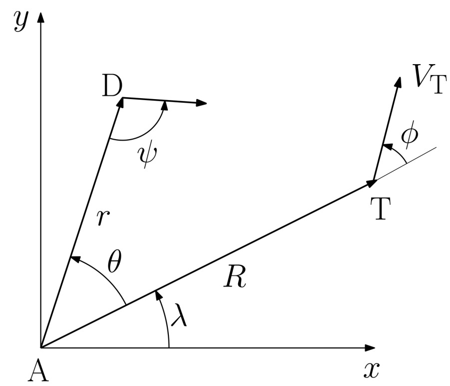

Fig. 1 *Geometry of the optimal aircraft defence problem*

$$
\begin{aligned}
&\frac{\partial V(x)}{\partial x} f(x) - \frac{1}{2}V'(x)\Big[G_u(x)R_{2u}^{-1}G_u^{\mathrm{T}}(x) \\
&\quad + G_w(x)R_{2w}^{-1}G_w^{\mathrm{T}}(x)\Big]\left[\frac{\partial V(x)}{\partial x}\right]^{\mathrm{T}} \\
&= -x^{\mathrm{T}}R_1x - \sum_{q=2}^{r}\left(x^{\mathrm{T}}M_qx\right)^{q-1}x^{\mathrm{T}}\hat{R}_qx \\
&\quad -\phi^{\mathrm{T}}(x)R_{2u}\phi(x) - \psi^{\mathrm{T}}(x)R_{2w}\psi(x) \\
&\quad -x^{\mathrm{T}}\left[\sum_{q=2}^{r}\left(x^{\mathrm{T}}M_qx\right)^{q-1}M_q\right] \\
&\quad \cdot S\left[\sum_{q=2}^{r}\left(x^{\mathrm{T}}M_qx\right)^{q-1}M_q\right]x \\
&< -x^{\mathrm{T}}\left(P + \sum_{q=2}^{r}(x^{\mathrm{T}}M_qx)^{q-1}M_q\right) \\
&\quad \cdot S\left(P + \sum_{q=2}^{r}(x^{\mathrm{T}}M_qx)^{q-1}M_q\right)x \\
&\le 0, \quad x \in \mathbb{R}^n \setminus \{0\},
\end{aligned}
\tag{82}
$$

which implies (63). Lastly, since $R_1 > 0$, it holds that $L_1(x) > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$ and $L_1(x) = 0$ if and only if $x = 0$. Since all conditions of Theorem 3 are satisfied, the result follows immediately. □

Corollary 2 provides a closed-form expression for a *non-linear* state-feedback control that guarantees disturbance rejection for a linear dynamical system and asymptotic stability of the undisturbed system. Moreover, this result provides an analytic expression for the least upper bound on a quadratic cost function over the class of admissible input disturbances $S_\phi(x_0)$.

Remarkably, Corollary 2 generalises the result of Bass and Webber [50] to the differential game problem. Moreover, if we consider $w(\cdot)$ as a disturbance input, then Corollary 2 extends the results of [50] to optimal control of linear systems with bounded energy disturbances. The same result is achieved in [46] applying a mixed-norm $\mathscr{H}_2/\mathscr{H}_\infty$ approach and [47] using a dissipativity-based approach.

5 Illustrative numerical examples

In this section, we provide two examples to highlight the direct and converse approaches to the differential game problem developed in the paper.

5.1 Optimal aircraft defence from an attacking missile

To show the applicability of the proposed differential game framework, we consider a problem involving an aircraft, an attacking missile, and a defending missile [56]. Specifically, in this example a missile pursues an aircraft, while a defending missile strives to intercept the attacker. Thus, the attacker's goal is to minimise the distance from the target, whereas the defender-aircraft team tries to maximise this distance.

To model this problem, we assume that the target aircraft, the attacker, and the defender are coplanar and their motion is captured in the moving reference frame $\{A; z_1, z_2\}$ centred in the attacker. We denote the target's, attacker's, and defender's heading velocities by v_{T}, v_{A}, and $v_{\mathrm{D}} > 0$, respectively, which are constant and such that $v_{\mathrm{T}} < v_{\mathrm{A}}$. The assumption that the three agents have constant velocity is realistic since in evasion-pursuit games, all agents must proceed at maximum forward velocity to meet their objectives; the assumption that the target is slower than the attacker is needed to guarantee the existence of a solution of this differential game for some non-trivial initial condition.

As illustrated in Fig. 1, the target's position can be uniquely captured in polar coordinates by the distance $R: [0, \infty) \to [0, \infty)$ from the attacker and the angle $\lambda: [0, \infty) \to \mathbb{R}$ between the z_1-axis and the target's position vector. The defender's position can be uniquely captured by the distance $r: [0, \infty) \to [0, \infty)$ from the attacker, the angle $\theta: [0, \infty) \to \mathbb{R}$ between the target's and the defender's position vectors, and the angle $\chi: [0, \infty) \to \mathbb{R}$ between the attacker's velocity vector and the defender's position vector. The target's heading velocity vector forms an angle $\delta: [0, \infty) \to \mathbb{R}$ with the target's position vector and the defender's heading velocity vector forms an angle $\zeta: [0, \infty) \to \mathbb{R}$ with the defender's position vector.

In this differential game, the agents' equations of motion are given by [56]

$$
\begin{aligned}
\dot{R}(t) &= \frac{v_{\mathrm{T}}}{v_{\mathrm{A}}}\cos\delta(t) - \cos(\theta(t) - \chi(t)), \\
R(0) &= R_0, \quad t \ge 0,
\end{aligned}
\tag{83}
$$

$$
\dot{r}(t) = -\cos\chi(t) - \cos\zeta(t), \quad r(0) = r_0, \tag{84}
$$

$$
\begin{aligned}
\dot{\theta}(t) &= \frac{1}{R}\left[\sin(\theta(t) - \chi(t)) - \frac{v_{\mathrm{T}}}{v_{\mathrm{A}}}\sin\delta(t)\right] \\
&\quad + \frac{1}{r(t)}[\sin\chi(t) - \sin\zeta(t)], \quad \theta(0) = \theta_0,
\end{aligned}
\tag{85}
$$

and the cost function is given by

$$
J_{t_{\mathrm{f}}}(x_0, u(\cdot), w(\cdot)) = \int_0^{t_{\mathrm{f}}}\dot{R}(t)\,\mathrm{d}t,
$$

where $x_0 = [R_0, r_0, \theta_0]^{\mathrm{T}}$, $u(t) = \chi(t)$, $t \ge 0$, $w(t) = [\delta(t), \zeta(t)]^{\mathrm{T}}$, and $t_{\mathrm{f}} \in [0, \infty)$ denotes the time needed by the aircraft or the defender to enter an open ball of radius $r_{\mathrm{c}} > 0$ centred in the attacker; t_{f} is not given. As discussed in Remark 1, this pursuit game can be cast as a differential game on the infinite time horizon, whose cost function is given by

$$
J(x_0, u(\cdot), \quad w(\cdot)) = \int_0^{\infty}\dot{R}(t)\,\mathrm{d}t, \tag{86}
$$

and Theorem 2 can be applied with $n = 3$, $m_1 = 1$, $m_2 = 2$, $x = [R, r, \theta]^{\mathrm{T}}$, $\mathscr{D} = [0, \infty) \times [0, \infty) \times \mathbb{R}$, $U = \mathbb{R}^{m_1}$, $W = \mathbb{R}^{m_2}$,

$$
F(x, u, w) = \begin{bmatrix} \dfrac{v_{\mathrm{T}}}{v_{\mathrm{A}}}\cos\delta - \cos(\theta - \chi) \\ -\cos\chi - \cos\zeta \\ \dfrac{1}{R}\left[\sin(\theta - \chi) - \dfrac{v_{\mathrm{T}}}{v_{\mathrm{A}}}\sin\delta\right] + \dfrac{1}{r}[\sin\chi - \sin\zeta] \end{bmatrix},
$$

and $L(x, u, w) = (v_{\mathrm{T}}/v_{\mathrm{A}})\cos\delta - \cos(\theta - \chi)$ to characterise the control laws $u = \phi(x)$ and $w = \psi(x)$, such that the saddle point condition (31) is verified for the cost function (86) and $x(t) \to 0$ as $t \to \infty$. In practise, by setting $u = \phi(x)$, $x \in \mathscr{D}$, and $w = \psi(x)$, the attacker is steered to a neighbourhood of the target within some finite-time interval $[0, t_f]$, where $t_f \ge 0$ is not specified a priori.

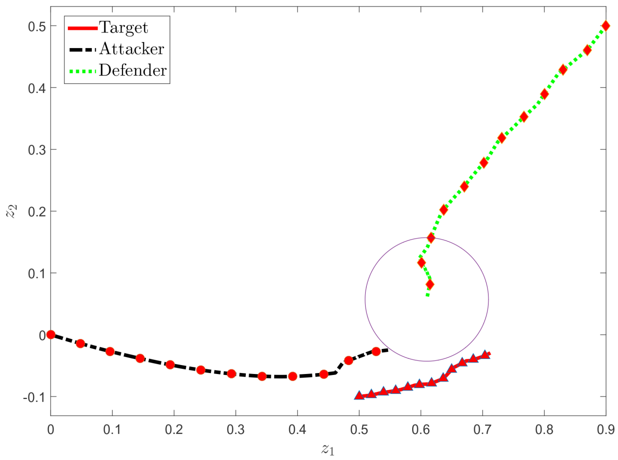

Fig. 2 *Positions of the target aircraft, attacking missile, and defending missile*

To apply Theorem 2 and solve this differential game problem numerically, we utilised a modified version of the 'Toolbox of Level Set Methods' software package [64]. In its original version, this toolbox, which applies Sethian's level set method [65] to compute viscosity solutions of the Hamilton–Jacobi equation, assumes prior knowledge of the pursuer's and evader's feedback control laws. Thus, we modified the 'Toolbox of Level Set Methods' to compute, for each pair of admissible controls $(\hat{u}, \hat{w}) \in \hat{U} \times \hat{W}$, the viscosity solution $V(x)$, $x \in \hat{\mathscr{D}}$, of the Hamilton–Jacobi–Isaacs equation (38) with $(\phi(x), \psi(x)) = (\hat{u}, \hat{w})$, where $\hat{\mathscr{D}} \subset \mathscr{D}$ denotes the discretised state space, $\hat{U} \subset U$ denotes the pursuer's discretised control space, and $\hat{W} \subset W$ denotes the evader's discretised control space. The pursuer's and evader's feedback control laws are eventually computed as the pairs $(\phi(x), \psi(x)) = (\hat{u}, \hat{w}) \in \hat{U} \times \hat{W}$ such that the Hamilton–Jacobi–Isaacs equation (38) and the saddle point condition (31) are verified for every $x \in \hat{\mathscr{D}}$. Equations (20)–(22) provide the boundary conditions at $x = 0$ for the level set method, whereas Equations (23), (24), (27), and (29) are verified *a posteriori* with $\gamma(\| x \|) = k \| x \|$, $x \in \hat{\mathscr{D}}$, and $k > 0$ arbitrarily small. Since our approach requires solving the Hamilton–Jacobi–Isaacs equation for every pair of admissible control inputs, parallel computing algorithms have been implemented to reduce the computational time.

Numerical solutions of (83)–(85) with $\chi(t) = \phi(x(t))$, $t \geq 0$, and $[\delta(t), \zeta(t)]^{\mathrm{T}} = \psi(x(t))$ are computed using the explicit Runge-Kutta (4, 5) integration method. Since the feedback control laws $\phi(x)$ and $\psi(x)$ are evaluated only at the points x of the discretised space $\hat{\mathscr{D}}$, if $x(t) \notin \hat{\mathscr{D}}$ for some $t > 0$, then $\phi(x(t))$ and $\psi(x(t))$ are computed by interpolating the values of $\phi(\cdot)$ and $\psi(\cdot)$ at the nearest adjacent points in $\hat{\mathscr{D}}$.

Let $(0.5, -0.1)$ denote the target's initial position and $(0.9, 0.5)$ denotes the defender's initial position expressed in the $\{A; z_1, z_2\}$ reference frame, $v_{\mathrm{A}} = v_{\mathrm{D}} = 5$, $v_{\mathrm{T}} = 0.2$, and $r_{\mathrm{c}} = 0.1$. Fig. 2 shows the three agents' trajectories obtained integrating (83)–(85) using a time step of 5×10^{-5} and partitioning the state space $\hat{D} = [0, 1] \times [0, 1] \times [-0.8, 0.8]$ in hyper-intervals, whose sides are 0.1 long. The analytical solution of this three-vehicle differential game problem proven in [56] overlaps with the numerical solutions presented herein.

Computing a solution of the Hamilton–Jacobi–Isaacs equation, i.e. finding $V(x)$, $x \in \mathscr{D}$, $\phi(x)$, and $\psi(x)$ that verify (38) is usually a computationally expensive task [65–67]. The advantage of our numerical approach to differential game problems is that the pursuer's and evader's feedback control laws $\phi(\cdot)$ and $\psi(\cdot)$ are computed only once for a given spatial resolution and yield for any initial conditions $x(0)$ and radius of capture r_{c}; integrating the equations of motion while consulting look-up tables to determine the ideal inputs is a simpler task that can be performed in real time.

5.2 Biological pest control via importation

In order to expedite pollination, some farmers release a foreign insect species in their fields. Unfortunately, this new species is devastating crops in neighbouring fields and the farmers decide to exterminate this pest by importing its natural enemy. Specifically, farmers would like to find a strategy such that eventually predators and prey will disappear and no foreign species is left.

For this problem, the predator–prey interaction is modelled by the Lotka–Volterra equations [13, p. 198]

$$\dot{x}_1(t) = x_1(t) - x_1(t)x_2(t) + u(t), \quad x_1(0) = x_{10}, \quad t \geq 0, \tag{87}$$

$$\dot{x}_2(t) = -x_2(t) + x_1(t)x_2(t) + w(t), \quad x_2(0) = x_{20}, \tag{88}$$

where x_1 denotes the number of preys, x_2 denotes the number of predators, u denotes the rate at which additional preys are released in the environment, and w denotes the rate at which predators are released. Note that (87) and (88) can be cast in the form of (59) with $n = 2$, $m_1 = 1$, $m_2 = 1$, $f(x) = [x_1 - x_1x_2, \ -x_2 + x_1x_2]^{\mathrm{T}}$, $G_u(x) = [1, 0]^{\mathrm{T}}$, and $G_w(x) = [0, 1]^{\mathrm{T}}$, where $x = [x_1, x_2]^{\mathrm{T}}$.

In this example, we apply the converse approach to differential games developed in Theorem 3. Specifically, we find $u = \phi(x)$ and $w = \psi(x)$ such that the dynamical system (87) and (88) is asymptotically stable. To this goal, let

$$V(x) = px^{\mathrm{T}}x, \tag{89}$$

where $p > 0$, $L(x, u, w) = L_1(x) + L_{2u}(x)u + L_{2w}(x)w + u^{\mathrm{T}}u - w^{\mathrm{T}}w$, and

$$L_{2u}(x) = x_1[1 - 2(p - 1 + x_2)], \tag{90}$$

$$L_{2w}(x) = x_2[1 + 2(p - 1 + x_1)]\,. \tag{91}$$

In this case, (61), (62), and (64)–(66) are trivially satisfied and since

$$\begin{aligned} \frac{\partial V(x)}{\partial x} f(x) &- \frac{1}{2}\frac{\partial V(x)}{\partial x}\big[G_u(x)R_{2u}^{-1}(x)L_{2u}^{\mathrm{T}}(x) \\ &+ G_w(x)R_{2w}^{-1}(x)L_{2w}^{\mathrm{T}}(x)\big] \\ &- \frac{1}{2}\frac{\partial V(x)}{\partial x}\big[G_u(x)R_{2u}^{-1}(x)G_u^{\mathrm{T}}(x) \\ &+ G_w(x)R_{2w}^{-1}(x)G_w^{\mathrm{T}}(x)\big]\left[\frac{\partial V(x)}{\partial x}\right]^{\mathrm{T}} \\ &= -px^{\mathrm{T}}x \\ &< 0, \quad x \in \mathbb{R}^n \setminus \{0\}\,. \end{aligned} \tag{92}$$

Equation (63) is satisfied.

Since all of the conditions of Theorem 3 hold, it follows that the feedback control laws

$$\phi(x) = \left(x_2 - \frac{3}{2}\right)x_1, \tag{93}$$

$$\psi(x) = \left(\frac{1}{2} - x_1\right)x_2, \tag{94}$$

which are restatements of (93) and (94), respectively, guarantee that the dynamical system (87) and (88) is globally asymptotically stable. In addition, the cost function (60) with

$$\begin{aligned} L_1(x) = &\left[(2p - 3)x_2 + \frac{9}{4} - 2p\right]x_1^2 \\ &+ (1 - 2p)x_1x_2^2 + \left(2p - \frac{1}{4}\right)x_2^2, \end{aligned} \tag{95}$$

is such that

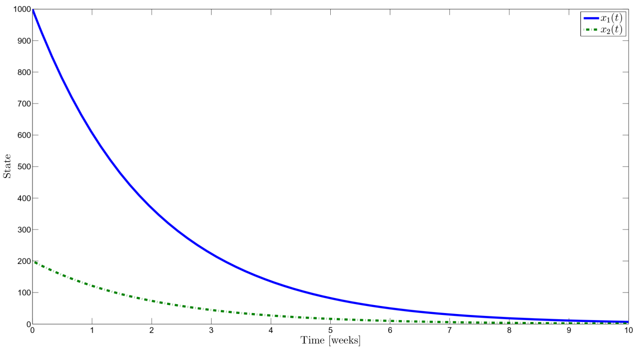

Fig. 3 *Closed-loop system trajectories versus time*

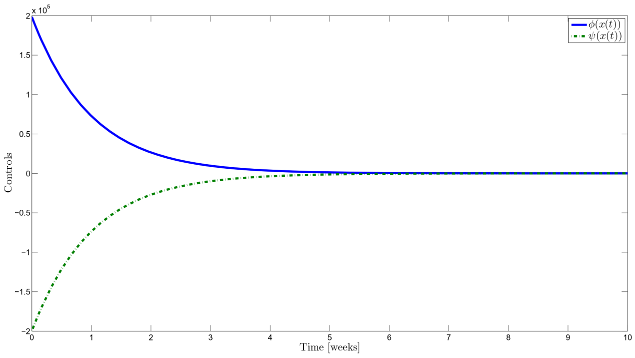

Fig. 4 *Control signal versus time*

$$J(x_0, \phi(x(\cdot)), \psi(x(\cdot))) = \min_{(u(\cdot), w(\cdot)) \in \mathcal{S}_\psi(x_0) \times \mathcal{S}_\phi(x_0)} \max J(x_0, u(\cdot), w(\cdot)) = p x_0^{\mathrm{T}} x_0 \tag{96}$$

for all $x_0 \in \mathbb{R}^n$, and (44) is verified. Note that although (96) explicitly depends on the parameter p, the asymptotically stabilising controllers (93) and (94) do not depend on p.

Let $x_{10} = 10^3$ and $x_{20} = 200$, Fig. 3 shows the number of predators and preys versus time. Note that $x(t) = [x_1(t),\ x_2(t)]^{\mathrm{T}} \to 0$ as $t \to \infty$. Fig. 4 shows the control signals versus time. Finally, $J(x(0), \phi(x(\cdot)), \psi(x(\cdot))) = (10^6 + 4 \times 10^4)p$.

6 Conclusion

In this paper, we proved sufficient conditions to solve a two-player zero-sum differential game problem over the infinite time horizon, whose end-of-game condition is the asymptotic stabilisation of the closed-loop system in spite of the evader's destabilising control action. Specifically, assuming no collaboration between the pursuer and the evader, we proved for the first time that if there exists a *continuous* Lyapunov function that is a viscosity solution of the steady-state Hamilton–Jacobi–Isaacs equation for the controlled system, then there exists a solution of the differential game on the infinite horizon, wherein the end-of-game condition is the asymptotic stabilisation of the closed-loop system. Moreover, we provided an analytic expression for the cost function evaluated at the saddle point and characterised the corresponding pursuer's and the evader's control laws. A key contribution of this paper is that our differential game framework is extended to solve optimal robust control problems involving non-linear dynamical systems with non-linear-non-quadratic cost functions in the presence of exogenous disturbances. Furthermore, the results presented herein provide a generalisation of the classical mixed-norm $\mathcal{H}_2/\mathcal{H}_\infty$ optimal robust control framework.

Our approach involves computing continuous, but not continuously differentiable, solutions of the Hamilton–Jacobi–Isaacs equation, which are known as *viscosity solutions*. Continuously differentiable solutions of the steady-state Hamilton–Jacobi–Isaacs equation can be explicitly found for simpler differential game problems [48], but do not always exist [32]. On the contrary, viscosity solutions of the steady-state Hamilton–Jacobi–Isaacs with Dirichlet boundary conditions always exist [38], but cannot be expressed in analytical form for problems of practical interest. For this reason, we extended a well-known inverse optimality framework to address differential games involving affine in the control dynamical systems with non-linear-non-quadratic cost functions and linear dynamical systems with polynomial performance integrands.

7 Acknowledgments

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