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Differential games, finite-time partial-state stabilisation of nonlinear dynamical systems, and optimal robust control

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ABSTRACT

Two-player zero-sum differential games are addressed within the framework of state-feedback finite-time partial-state stabilisation of nonlinear dynamical systems. Specifically, finite-time partial-state stability of the closed-loop system is guaranteed by means of a Lyapunov function, which we prove to be the value of the game. This Lyapunov function verifies a partial differential equation that corresponds to a steady-state form of the Hamilton–Jacobi–Isaacs equation, and hence guarantees both finite-time stability with respect to part of the system state and the existence of a saddle point for the system's performance measure. Connections to optimal regulation for nonlinear dynamical systems with nonlinear-nonquadratic cost functionals in the presence of exogenous disturbances and parameter uncertainties are also provided. Furthermore, we develop feedback controllers for affine nonlinear systems extending an inverse optimality framework tailored to the finite-time partial-state stabilisation problem. Finally, two illustrative numerical examples show the applicability of the results proven.

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1. Introduction

A large class of dynamical processes is modelled by ordinary differential equations endowed with multiple control inputs, some of which, usually named *evaders*, strive to maximise a given performance measure, whereas some others, usually named *pursuers*, concurrently try to minimise this performance measure. Differential game theory (Isaacs, 1999) provides the theoretical framework needed to solve these problems, where applications range from aerospace engineering (Isaacs, 1999; Shima & Golan, 2006) to marine engineering (Weekly, Tinka, Anderson, & Bayen, 2014), communication networks (Alpcan & Basar, 2005), electrical engineering (Ekneligoda & Weaver, 2014), and economics (Dockner, 2000). The study of zero-sum differential games and their relation with stabilisation problems has been mostly explored for linear dynamical systems with quadratic performance measures (Zhukovskiy, 2003) and to establish connections with the classic $\mathcal{H}_\infty$ control theory (Basar & Olsder, 1998; Doyle, Glover, Khargonekar, & Francis, 1989; Jacobson, 1977; Limebeer, Anderson, Khargonekar, & Green, 1992; Mageirou, 1976). Connections between differential game theory and the disturbance rejection problem for nonlinear dynamical systems have been partly discussed in Ball and Helton (1989) and Basar and Bernhard (2000).

In this paper, we address the two-player zero-sum differential game problem for nonlinear dynamical systems with nonlinear-nonquadratic performance measures over the finite-time horizon. Specifically, we provide a framework for designing the pursuer's and evader's state-feedback control laws, which guarantee partial-state Lyapunov stability of the closed-loop dynamical system, convergence of part of the closed-loop trajectory to an equilibrium point in finite time, and the existence of a saddle point for the system's performance measure. Remarkably, if the end-of-game condition is given by the convergence to an equilibrium point of the trajectory of a subset of the system state variables, then finite-time partial-state stability of the closed-loop system is a key feature to guarantee that this condition is permanently enforced. The framework presented in this paper is also suitable to address problems in which the differential game ends when part of the system state trajectory enters a *given* neighbourhood of an equilibrium point within some time interval that is finite and not assigned a priori. Possible applications for this framework include games of degree (Isaacs, 1999, p.12) such as the game of two cars (Isaacs, 1999, pp. 237–244), whereby the pursuer's goal is to reach some neighbourhood of the evader irrespectively of the angle between the agents' velocity vectors. The pursuer and the evader are not assumed to collaborate to achieve closed-loop finite-time partial-state

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stability. Indeed, the pursuer's control policy is designed to guarantee closed-loop stability with respect to a class of evader's admissible controls, some of which may lead to system instability if applied in conjunction with other pursuer's admissible controls.

Partial-state stabilisation, that is, the problem of stabilising a dynamical system with respect to a subset of the system state variables, arises in many engineering applications (Lum, Bernstein, & Coppola, 1995; Vorotnikov, 1998). For example, in the control of rotating machinery with mass imbalance, spin stabilisation about a non-principal axis of inertia requires motion stabilisation with respect to a subspace instead of the origin (Lum et al., 1995). In general, the need to consider partial stability arises in control problems involving equilibrium coordinates as well as a manifold of coordinates that is closed but *not* compact. The optimal finite-time partial-state stabilisation problem has been addressed in Haddad and L'Afflitto (2015).

Finite-time stabilisation of second-order systems was considered by Haimo (1986) and Bhat and Bernstein (1998), whereas Hong (2002) and Hong, Huang, and Xu (2001) consider finite-time stabilisation of higher order systems as well as finite-time stabilisation using output feedback. Design of globally strongly stabilising continuous controllers for linear and nonlinear systems using the theory of homogeneous systems was studied by Qian and Lin (2001) and Bhat and Bernstein (2005). Finite-time partial stabilisation of chained systems are considered by Jammazi (2008, 2010), whereas finite-time partial stabilisability using continuous and discontinuous homogeneous state feedback controllers is considered in Jammazi (2014). Discontinuous finite-time stabilising feedback controllers have also been developed in the literature (Fuller, 1966; Ryan, 1979, 1991). Alternatively, sliding-mode (typically discontinuous) control design has also been used to guarantee finite-time convergence and more recently finite-time stability (see Bernuau, Efimov, Perruquetti, & Polyakov, 2014, and the numerous references therein). However, for practical implementation, discontinuous feedback controllers can lead to chattering due to system uncertainty or measurement noise, and hence may excite unmodelled high-frequency system dynamics.

Bernstein (1993) provides a framework to solve the state-feedback continuous-time, nonlinear nonquadratic optimal control problems over the infinite-time horizon. The underlying ideas of the results of Bernstein (1993) are based on the fact that the steady-state solution of the Hamilton–Jacobi–Bellman equation is a Lyapunov function for the nonlinear system and thus guaranteeing both stability and optimality (Bernstein, 1993; Haddad & Chellaboina, 2008). One of the main contributions of this paper is extending the framework presented by Bernstein (1993) and Haddad and L'Afflitto (2015) to address two-player zero-sum differential games involving nonlinear dynamical systems with nonlinear-nonquadratic performance measures. Specifically, we prove that if there exists a Lyapunov function that satisfies a partial differential equation corresponding to a steady-state form of the Hamilton–Jacobi–Isaacs equation for the controlled system, then there exist pursuer's and evader's state feedback control policies that guarantee finite-time partial-state stability of the closed-loop dynamical system and the existence of a saddle point for the system's performance measure. In this case, we prove that the Lyapunov function certifying partial-state finite-time stability of the closed-loop system is the value of the game, we provide an explicit closed-form expression for the value of the game at the saddle point, and we characterise the corresponding evader's and pursuer's control policies.

Another key point of this paper is that if the evader applies a control policy for which the pursuer's controller guarantees finite-time *convergence* of part the closed-loop system to an equilibrium point, then we provide a closed-form analytical expression for the least upper bound on system's performance measure. Therefore, regarding the evader's control policy as an exogenous disturbance, results presented in this paper provide a solution of the optimal control problem for nonlinear dynamical systems with nonlinear-nonquadratic performance measures in the presence of undesired external inputs.

In the second part of this paper, we specialise our results to differential games involving affine in the controls dynamical systems with quadratic in the controls performance measures. In this case, we provide an *explicit* characterisation of the evader's and pursuer's controls for a successful completion of the game and prove that the minimax assumption (Isaacs, 1999, p. 35) is verified. Furthermore, we provide sufficient conditions for the pursuer to guarantee finite-time partial-state stability of the closed-loop system if the evader's control is equal to zero, that is, in the absence of disturbing inputs.

Finally, we explore connections of our approach to the differential game problem with inverse optimal control (Freeman & Kokotovic, 1996; Jacobson, 1977; Molinari, 1973; Moylan & Anderson, 1973; Sepulchre, Jankovic, & Kokotovic, 1997), wherein we parametrise a family of partial-state asymptotically stabilising controllers that guarantee the existence of a saddle point for a derived cost functional. Two numerical examples illustrate the features and the applicability of the theoretical results proven.

The main differences between the results developed in this paper and those by Haddad and L'Afflitto (2015) are two. Namely, this work concerns zero-sum differential games, and hence the dynamical systems considered herein are characterised by two control inputs with competing objectives. Haddad and L'Afflitto (2015), instead, consider optimal state-feedback control problems, and hence the dynamical systems considered therein are characterised by one control input only. Moreover, results developed in Haddad and L'Afflitto (2015) do not account for parameter uncertainty and robustness to exogenous disturbances.

2. Notation, definitions, and mathematical preliminaries

In this section, we establish notation and definitions, and review some basic results. Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}^n$ denote the set of $n \times 1$ real column vectors, and $\mathbb{R}^{n\times m}$ denote the set of $n \times m$ real matrices. We write $\|\cdot\|$ for the Euclidean vector norm, $V'(x) \triangleq \frac{\partial V(x)}{\partial x}$ for the Fréchet derivative of V at x, I_n or I for the $n \times n$ identity matrix, $0_{n\times m}$ or 0 for the zero $n \times m$ matrix, and A^{T} for the transpose of the matrix A. Given $f : X \times Y \to \mathbb{R}$, where $X \subseteq \mathbb{R}^{m_1}$ and $Y \subseteq \mathbb{R}^{m_2}$, we define

$$\begin{aligned}&\arg\operatorname*{minmax}_{(x,y)\in(X,Y)} f(x, y)\\&\quad\triangleq \{(x^*, y^*) \in (X, Y) : f(x^*, y^*) \le f(x, y^*), \forall x \in X, \text{ and}\\&\quad f(x^*, y^*) \ge f(x^*, y), \forall y \in Y\}\end{aligned}$$

and

$$\operatorname*{minmax}_{(x,y)\in(X,Y)} f(x, y) \triangleq f(x^*, y^*),\quad (x^*, y^*) \in \arg\operatorname*{minmax}_{(x,y)\in(X,Y)} f(x, y).$$

If $(x^*, y^*) \in \arg\operatorname{minmax}_{(x, y) \in (X, Y)}f(x, y)$, then x^* minimises the cost function $f(x, y)$ with respect to x, whereas y^* maximises the cost function $f(x, y)$ with respect to y. In this case, we say that (x^*, y^*) is a *saddle point* for $f(\cdot, \cdot)$ on $X \times Y$.

The next result provides a key property of saddle points, which states that the minimum of the maximum of a performance measure is equal to the maximum of the minimum of the same cost index.

Lemma 2.1 (Ball & Helton, 1989)**:** *Consider $f : X \times Y \to \mathbb{R}$, where $X \subseteq \mathbb{R}^{m_1}$ and $Y \subseteq \mathbb{R}^{m_2}$, and let $(x^*, y^*) \in$ arg minmax$_{(x, y) \in (X, Y)}f(x, y)$. Then,*

$$\min_{x\in X}\max_{y\in Y} f(x, y) = \max_{y\in Y}\min_{x\in X} f(x, y),\quad (x, y) \in X \times Y. \tag{1}$$

In this paper, we consider nonlinear dynamical systems of the form

$$\dot{x}_1(t) = f_1(x_1(t), x_2(t)),\qquad x_1(0) = x_{10},\qquad t \in \mathcal{I}_{x_0}, \tag{2}$$

$$\dot{x}_2(t) = f_2(x_1(t), x_2(t)),\qquad x_2(0) = x_{20}, \tag{3}$$

where, for every $t \in \mathcal{I}_{x_0}$, $x_1(t) \in \mathcal{D} \subseteq \mathbb{R}^{n_1}$ and $x_2(t) \in \mathbb{R}^{n_2}$, $\mathcal{I}_{x_0} \subset \mathbb{R}$ is the maximal interval of existence of a solution $x(t) \triangleq [x_1^{\mathrm{T}}(t), x_2^{\mathrm{T}}(t)]^{\mathrm{T}}$ of (2) and (3) with initial condition $x_0 \triangleq [x_{10}^{\mathrm{T}}, x_{20}^{\mathrm{T}}]^{\mathrm{T}}$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $f_1 : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1}$ is such that, for every $(x_1, x_2) \in \mathcal{D} \times \mathbb{R}^{n_2}$, $f_1(0, x_2) = 0$ and $f_1(\cdot, \cdot)$ is jointly continuous in x_1 and x_2, and $f_2 : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$ is such that, for every $(x_1, x_2) \in \mathcal{D} \times \mathbb{R}^{n_2}$, $f_2(\cdot, \cdot)$ is jointly continuous in x_1 and x_2. A continuously differentiable function $x : \mathcal{I}_{x_0} \to \mathcal{D} \times \mathbb{R}^{n_2}$ is said to be a *solution* of (2) and (3) on the interval $\mathcal{I}_{x_0} \subset \mathbb{R}$ if $x(\cdot) = [x_1^{\mathrm{T}}(\cdot), x_2^{\mathrm{T}}(\cdot)]^{\mathrm{T}}$ satisfies (2) and (3) for all $t \in \mathcal{I}_{x_0}$. If $x(\cdot) = [x_1^{\mathrm{T}}(\cdot), x_2^{\mathrm{T}}(\cdot)]^{\mathrm{T}}$ is a solution of (2) and (3) on the interval $\mathcal{I}_{x_0} \subset \mathbb{R}$, then $x_1(\cdot)$ is the solution of (2) and $x_2(\cdot)$ is the solution of (3).

The joint continuity of $f(\cdot, \cdot) = [f_1^{\mathrm{T}}(\cdot, \cdot), f_2^{\mathrm{T}}(\cdot, \cdot)]^{\mathrm{T}}$ implies that, for every $(x_1, x_2) \in \mathcal{D} \times \mathbb{R}^{n_2}$, there exists $\tau_0 < 0 < \tau_1$ and a solution $[x_1^{\mathrm{T}}(\cdot), x_2^{\mathrm{T}}(\cdot)]^{\mathrm{T}}$ of (2) and (3) defined on the open interval (τ_0, τ_1) such that $[x_1^{\mathrm{T}}(0), x_2^{\mathrm{T}}(0)]^{\mathrm{T}} = [x_1^{\mathrm{T}}, x_2^{\mathrm{T}}]^{\mathrm{T}}$ (Haddad & Chellaboina, 2008, Theorem 2.2). A solution $t \mapsto [x_1^{\mathrm{T}}(t), x_2^{\mathrm{T}}(t)]^{\mathrm{T}}$ is said to be *right maximally* defined if $[x_1^{\mathrm{T}}, x_2^{\mathrm{T}}]^{\mathrm{T}}$ cannot be extended (either uniquely or nonuniquely) forward in time. We assume that all right maximal solutions to (2) and (3) exist on $[0, \infty)$, and hence we assume that (2) and (3) is *forward complete*. Recall that every bounded solution to (2) and (3) can be extended on a semi-infinite interval $[0, \infty)$ (Haddad & Chellaboina, 2008). That is, if $x : [0, \tau_{x_0}) \to \mathcal{D} \times \mathbb{R}^{n_2}$ is the right maximally defined solution of (2) and (3) such that $x(t) = [x_1^{\mathrm{T}}(t), x_2^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathcal{D}_{\mathrm{c}} \times \mathcal{Q}_{\mathrm{c}}$ for all $t \in [0, \tau_{x_0})$, where $\mathcal{D}_{\mathrm{c}} \subset \mathcal{D}$ and $\mathcal{Q}_{\mathrm{c}} \subset \mathbb{R}^{n_2}$ are compact, then $\tau_{x_0} = \infty$ (Haddad & Chellaboina, 2008, Corollary 2.5).

We assume that the nonlinear dynamical system given by (2) and (3) possesses unique solutions in forward time for all initial conditions except possibly at $x_1 = 0$ in the following sense. For every $(x_1, x_2) \in \mathcal{D}\backslash\{0\} \times \mathbb{R}^{n_2}$, there exists $\tau_x > 0$, where $x = [x_1^{\mathrm{T}}, x_2^{\mathrm{T}}]^{\mathrm{T}}$, such that, if $y_{\mathrm{I}} : [0, \tau_1) \to \mathcal{D} \times \mathbb{R}^{n_2}$ and $y_{\mathrm{II}} : [0, \tau_2) \to \mathcal{D} \times \mathbb{R}^{n_2}$ are two solutions of (2) and (3) with $y_{\mathrm{I}}(0) = y_{\mathrm{II}}(0) = x$, then $\tau_x \le \min\{\tau_1, \tau_2\}$ and $y_{\mathrm{I}}(t) = y_{\mathrm{II}}(t)$ for all $t \in [0, \tau_x)$. Without loss of generality, we assume that, for every (x_1, x_2), τ_x is chosen to be the largest such number in $\overline{\mathbb{R}}_+$. In this case, given $x = [x_1^{\mathrm{T}}, x_2^{\mathrm{T}}]^{\mathrm{T}} \in \mathcal{D} \times \mathbb{R}^{n_2}$, we denote by the continuously differentiable map $s^x(\cdot) \triangleq s(\cdot, x_1, x_2)$ the *trajectory* or the unique *solution curve* of (2) and (3) on $[0, \tau_x)$ satisfying $s(0, x_1, x_2) = [x_1^{\mathrm{T}}, x_2^{\mathrm{T}}]^{\mathrm{T}}$ and we denote by $s_1^x(\cdot)$ the *partial trajectory* or the unique *solution curve* of (2) on $[0, \tau_x)$. Sufficient conditions for forward uniqueness in the absence of Lipschitz continuity are given by Agarwal and Lakshmikantham (1993), Filippov (1988, Section 10), Kawski (1989), and Yoshizawa (1966, Section 1). Finally, we assume that given a continuously differentiable function $x_1 : [0, \infty) \to \mathbb{R}^{n_1}$, the solution $x_2(t)$, $t \geq 0$, to (3) is unique.

In the following, we recall three forms of finite-time partial-state stability.

Definition 2.1 (Haddad & Chellaboina, 2008, Definition 4.1)**:** The nonlinear dynamical system (2) and (3) is *finite-time stable with respect to* x_1 if there exist an open neighbourhood $\mathcal{D}_0 \subseteq \mathcal{D}$ of $x_1 = 0$ and a function $T : \mathcal{D}_0 \setminus \{0\} \times \mathbb{R}^{n_2} \to (0, \infty)$, called the *settling-time function*, such that the following statements hold:

- (i) *Finite-time partial convergence.* For every $(x_{10}, x_{20}) \in \mathcal{D}_0 \setminus \{0\} \times \mathbb{R}^{n_2}$, $s^{x_0}(t)$ is defined on $[0, T(x_{10}, x_{20}))$, where $x_0 = [x_{10}^{\mathrm{T}}, x_{20}^{\mathrm{T}}]^{\mathrm{T}}$, $s_1^{x_0}(t) \in \mathcal{D}_0 \setminus \{0\}$ for all $t \in [0, T(x_{10}, x_{20}))$, and $s_1^{x_0}(t) \to 0$ as $t \to T(x_{10}, x_{20})$.
- (ii) *Partial Lyapunov stability.* For every $\epsilon > 0$ and $x_{20} \in \mathbb{R}^{n_2}$, there exists $\delta = \delta(\epsilon, x_{20}) > 0$ such that $\mathcal{B}_\delta(0) \subset \mathcal{D}_0$ and, for every $x_{10} \in \mathcal{B}_\delta(0) \setminus \{0\}$, $s_1^{x_0}(t) \in \mathcal{B}_\varepsilon(0)$ for all $t \in [0, T(x_{10}, x_{20}))$.

The nonlinear dynamical system (2) and (3) is *finite-time stable with respect to* x_1 *uniformly in* x_{20} if (2) and (3) is finite-time stable with respect to x_1 and the following statement holds:

- (iii) *Partial uniform Lyapunov stability.* For every $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that $\mathcal{B}_\delta(0) \subset \mathcal{D}_0$ and, for every $x_{10} \in \mathcal{B}_\delta(0) \setminus \{0\}$, $s_1^{x_0}(t) \in \mathcal{B}_\varepsilon(0)$ for all $t \in [0, T(x_{10}, x_{20}))$ and for all $x_{20} \in \mathbb{R}^{n_2}$.

The nonlinear dynamical system (2) and (3) is *strongly finite-time stable with respect to* x_1 *uniformly in* x_{20}, if (2) and (3) is finite-time stable with respect to x_1 uniformly in x_{20} and the following statement holds:

- (iv) *Finite-time partial uniform convergence.* For every $(x_{10}, x_{20}) \in \mathcal{D}_0 \setminus \{0\} \times \mathbb{R}^{n_2}$, $s^{x_0}(t)$ is defined on $[0, T(x_{10}, x_{20}))$, $s_1^{x_0}(t) \in \mathcal{D}_0 \setminus \{0\}$ for all $t \in [0, T(x_{10}, x_{20}))$, and $s_1^{x_0}(t) \to 0$ as $t \to T(x_{10}, x_{20})$ uniformly in x_{20} for all $x_{20} \in \mathbb{R}^{n_2}$.

The nonlinear dynamical system (2) and (3) is *globally finite-time stable with respect to* x_1 (respectively, *globally finite-time stable with respect to* x_1 *uniformly in* x_{20} or *globally strongly finite-time stable with respect to* x_1 *uniformly in* x_{20}) if it is finite-time stable with respect to x_1 (respectively, finite-time stable with respect to x_1 uniformly in x_{20} or strongly finite-time stable with respect to x_1 uniformly in x_{20}) with $\mathcal{D}_0 = \mathbb{R}^{n_1}$.

It is important to note that there is a key difference between the partial stability definitions given in Definition 2.1 and the definitions of partial stability given by Vorotnikov (1998). In particular, the partial stability definitions given by Vorotnikov (1998) require that both the initial conditions x_{10} and x_{20} lie in a neighbourhood of the origin, whereas in Definition 2.1, x_{20} can be arbitrary. Furthermore, in our formulation, we require the weaker partial equilibrium condition $f_1(0, x_2) = 0$ for every $x_2 \in \mathbb{R}^{n_2}$, whereas Vorotnikov (1998) requires the stronger equilibrium condition $f_1(0, 0) = 0$ and $f_2(0, 0) = 0$.

Next, we provide sufficient conditions for partial stability of the nonlinear dynamical system given by (2) and (3). For the statement of the following result, define

$$\dot{V}(x_1, x_2) \triangleq V'(x_1, x_2) f(x_1, x_2),$$

where $f(x_1, x_2) \triangleq [f_1^{\mathrm{T}}(x_1, x_2), f_2^{\mathrm{T}}(x_1, x_2)]^{\mathrm{T}}$, for a continuously differentiable function $V : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$.

Theorem 2.1 (Haddad & L'Afflitto, 2015)**:** *Consider the nonlinear dynamical system* $\mathcal{G}$ *given by* (2) *and* (3). *Then, the following statements hold:*

- (*i*) *If there exist a continuously differentiable function* $V : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$, *class* $\mathcal{K}$ *functions* $\alpha(\cdot)$ *and* $\beta(\cdot)$, *a real number* $\theta \in (0, 1)$, $k > 0$, *and an open neighbourhood* $\mathcal{M} \subseteq \mathcal{D}$ *of* $x_1 = 0$, *such that*

$$\alpha(\|x_1\|) \leq V(x_1, x_2) \leq \beta(\|x_1\|), \quad (x_1, x_2) \in \mathcal{M} \times \mathbb{R}^{n_2}, \quad (4)$$

$$\dot{V}(x_1, x_2) \leq -k(V(x_1, x_2))^{\theta}, \quad (x_1, x_2) \in \mathcal{M} \times \mathbb{R}^{n_2}, \quad (5)$$

then $\mathcal{G}$ *is strongly finite-time stable with respect to* x_1 *uniformly in* x_{20}. *Moreover, there exist a neighbourhood* $\mathcal{D}_0$ *of* $x_1 = 0$ *and a settling-time function* $T : \mathcal{D}_0 \times \mathbb{R}^{n_2} \to [0, \infty)$ *such that*

$$T(x_{10}, x_{20}) \leq \frac{(V(x_{10}, x_{20}))^{1-\theta}}{k(1-\theta)}, \quad (x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}, \quad (6)$$

and $T(\cdot, \cdot)$ *is jointly continuous on* $\mathcal{D}_0 \times \mathbb{R}^{n_2}$.
- (*ii*) *If* $\mathcal{M} = \mathcal{D} = \mathbb{R}^{n_1}$ *and there exist a continuously differentiable function* $V : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$, *class* $\mathcal{K}_\infty$ *functions* $\alpha(\cdot)$ *and* $\beta(\cdot)$, *and a real number* $\theta \in (0, 1)$ *such that* (4) *and* (5) *hold, then* $\mathcal{G}$ *is globally strongly finite-time stable with respect to* x_1 *uniformly in* x_{20}. *Moreover, there exists a settling-time function* $T : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to [0, \infty)$ *such that*

(6) holds with $\mathcal{D}_0 = \mathbb{R}^{n_1}$ and $T(\cdot, \cdot)$ is jointly continuous on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$.

In the following, we recall a result that provides connections between Lyapunov functions and nonquadratic cost evaluation. Specifically, we consider the problem of evaluating a nonlinear-nonquadratic performance measure that depends on the solution of the nonlinear dynamical system given by (2) and (3). In particular, the next theorem shows that the nonlinear-nonquadratic performance measure

$$J(x_{10}, x_{20}) \triangleq \int_0^\infty L(x_1(t), x_2(t))\mathrm{d}t, \tag{7}$$

where $L : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$ is jointly continuous in x_1 and x_2, and $x_1(t)$ and $x_2(t)$, $t \geq 0$, satisfy (2) and (3), can be quantified, so long as (2) and (3) are related to an underlying Lyapunov function that is positive definite and decrescent with respect to x_1 and proves finite-time stability of (2) and (3) with respect to x_1 uniformly in x_{20}.

Theorem 2.2 (Haddad & L'Afflitto, 2015)**:** *Consider the nonlinear dynamical system $\mathcal{G}$ given by (2) and (3) with performance measure (7). Assume that there exists a continuously differentiable function $V : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$, class $\mathcal{K}$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, $k > 0$, a real number $\theta \in (0, 1)$, and an open neighbourhood $\mathcal{M} \subseteq \mathcal{D}$ of $x_1 = 0$ such that*

$$\alpha(\|x_1\|) \leq V(x_1, x_2) \leq \beta(\|x_1\|), \quad (x_1, x_2) \in \mathcal{M} \times \mathbb{R}^{n_2}, \tag{8}$$

$$\dot{V}(x_1, x_2) \leq -k(V(x_1, x_2))^{\theta}, \quad (x_1, x_2) \in \mathcal{M} \times \mathbb{R}^{n_2}, \tag{9}$$

$$L(x_1, x_2) + V'(x_1, x_2) f(x_1, x_2) = 0, \quad (x_1, x_2) \in \mathcal{M} \times \mathbb{R}^{n_2}. \tag{10}$$

Then, the nonlinear dynamical system $\mathcal{G}$ is strongly finite-time stable with respect to x_1 uniformly in x_{20} and there exist a neighbourhood $\mathcal{D}_0 \subseteq \mathcal{M}$ of $x_1 = 0$ and a settling-time function $T : \mathcal{D}_0 \times \mathbb{R}^{n_2} \to [0, \infty)$, jointly continuous on $\mathcal{D}_0 \times \mathbb{R}^{n_2}$, such that

$$T(x_{10}, x_{20}) \leq \frac{(V(x_{10}, x_{20}))^{1-\theta}}{k(1-\theta)}, \quad (x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}. \tag{11}$$

In addition, for all $(x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}$,

$$J(x_{10}, x_{20}) = V(x_{10}, x_{20}). \tag{12}$$

Finally, if $\mathcal{M} = \mathcal{D} = \mathbb{R}^{n_1}$ and the functions $\alpha(\cdot)$ and $\beta(\cdot)$ satisfying (8) are class $\mathcal{K}_\infty$, then $\mathcal{G}$ is globally strongly finite-time stable with respect to x_1 uniformly in x_{20}.

In this paper, we consider controlled nonlinear dynamical systems of the form

$$\dot{x}_1(t) = F_1(x_1(t), x_2(t), u(t), w(t)), \quad x_1(0) = x_{10}, \quad t \geq 0, \tag{13}$$

$$\dot{x}_2(t) = F_2(x_1(t), x_2(t), u(t), w(t)), \quad x_2(0) = x_{20}, \tag{14}$$

where, for every $t \geq 0$, $x_1(t) \in \mathcal{D} \subseteq \mathbb{R}^{n_1}$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $x_2(t) \in \mathbb{R}^{n_2}$, $u(t) \in U \subseteq \mathbb{R}^{m_1}$ with $0 \in U$, $w(t) \in W \subseteq \mathbb{R}^{m_2}$ with $0 \in W$, $F_1 : \mathcal{D} \times \mathbb{R}^{n_2} \times U \times W \to \mathbb{R}^{n_1}$ and $F_2 : \mathcal{D} \times \mathbb{R}^{n_2} \times U \times W \to \mathbb{R}^{n_2}$ are jointly continuous in x_1, x_2, u, and w, and $F_1(0, x_2, 0, 0) = 0$ for every $x_2 \in \mathbb{R}^{n_2}$. The controls $u(\cdot)$ and $w(\cdot)$ in (13) and (14) are restricted to the class of *admissible* controls consisting of almost everywhere continuous functions $u(\cdot)$ and $w(\cdot)$ such that $u(t) \in U$, $t \geq 0$, and $w(t) \in W$, respectively.

Almost everywhere continuous functions $\phi : \mathcal{D} \times \mathbb{R}^{n_2} \to U$ and $\psi : \mathcal{D} \times \mathbb{R}^{n_2} \to W$ satisfying $\phi(0, x_2) = 0$ almost everywhere for $x_2 \in \mathbb{R}^{n_2}$, and $\psi(0, x_2) = 0$ almost everywhere for $x_2 \in \mathbb{R}^{n_2}$ are called *control laws*. If $u(t) = \phi(x_1(t), x_2(t))$, $t \geq 0$, and $w(t) = \psi(x_1(t), x_2(t))$, where $\phi(\cdot, \cdot)$ and $\psi(\cdot, \cdot)$ are control laws and $x_1(t)$ and $x_2(t)$ satisfy (13) and (14), respectively, then we call $u(\cdot)$ and $w(\cdot)$ *feedback control laws*. Given control laws $\phi(\cdot, \cdot)$ and $\psi(\cdot, \cdot)$, and feedback control laws $u(t) = \phi(x_1(t), x_2(t))$, $t \geq 0$, and $w(t) = \psi(x_1(t), x_2(t))$, the *closed-loop system* (13) and (14) is given by

$$\dot{x}_1(t) = F_1(x_1(t), x_2(t), \phi(x_1(t), x_2(t)), \psi(x_1(t), x_2(t))), \quad x_1(0) = x_{10}, \quad t \geq 0, \tag{15}$$

$$\dot{x}_2(t) = F_2(x_1(t), x_2(t), \phi(x_1(t), x_2(t)), \psi(x_1(t), x_2(t))), \quad x_2(0) = x_{20}. \tag{16}$$

Next, we introduce the notion of finite-time partial-state stabilising feedback control laws. To this goal, consider the controlled nonlinear dynamical system (13) and (14) and define the set of regulation controllers

$$\mathcal{S}(x_{10}, x_{20}) \triangleq \{(u(\cdot), w(\cdot)) : u(\cdot) \text{ and } w(\cdot) \text{are admissible and } x_1(\cdot) \text{ given by (13) satisfies } x_1(t) \to 0 \text{ as } t \to T(x_{10}, x_{20})\}.$$

In addition, given the control law $\psi(\cdot, \cdot)$, let

$$\mathcal{S}_\psi(x_{10}, x_{20}) \triangleq \{u(\cdot) : (u(\cdot), \psi(x_1(\cdot), x_2(\cdot))) \in \mathcal{S}(x_{10}, x_{20})\}$$

and given the control law $\phi(\cdot,\cdot)$, let

$$\mathcal{S}_\phi(x_{10},x_{20}) \triangleq \{w(\cdot) : (\phi(x_1(\cdot),x_2(\cdot)), w(\cdot)) \in \mathcal{S}(x_{10},x_{20})\}.$$

Remarkably, since finite-time partial-state convergence is a stronger condition than asymptotic partial-state convergence, $\mathcal{S}(x_{10},x_{20})$ includes the set of all partial-state null asymptotically convergent controllers.

Definition 2.2: Consider the controlled dynamical system given by (13) and (14). The feedback control law $u(\cdot)=\phi(x_1(\cdot),x_2(\cdot))$ is *strongly finite-time stabilising with respect to* x_1 *uniformly in* x_{20} if the closed-loop system (15) and (16) is strongly finite-time stable with respect to x_1 uniformly in x_{20} for all admissible controls $w(\cdot)\in\mathcal{S}_\phi(x_{10},x_{20})$. Furthermore, the feedback control law $u(\cdot)=\phi(x_1(\cdot),x_2(\cdot))$ is *globally strongly finite-time stabilising with respect to* x_1 *uniformly in* x_{20} if the closed-loop system (15) and (16) is globally strongly finite-time stable with respect to x_1 uniformly in x_{20} for all admissible controls $w(\cdot)\in\mathcal{S}_\phi(x_{10},x_{20})$.

3. Lyapunov functions and differential games

In this section, we use the framework developed in Theorem 2.2 to obtain a characterisation of finite-time partial-state stabilising feedback control laws that provide a solution of differential games involving nonlinear dynamical systems of the form (13) and (14). Specifically, sufficient conditions for the existence of a saddle point are given in a form that corresponds to a steady-state version of the Hamilton–Jacobi–Isaacs equation.

Next, we present a main theorem characterising feedback controllers that guarantee closed-loop, finite-time, partial-state stabilisation of (13) and (14), and minimise with respect to $u(\cdot)$ and maximise with respect to $w(\cdot)$ a nonlinear-nonquadratic performance functional. For the statement of this result, define $F(x_1,x_2,u,w)\triangleq[F_1^{\mathrm{T}}(x_1,x_2,u,w),\quad F_2^{\mathrm{T}}(x_1,x_2,u,w)]^{\mathrm{T}}$ and let $L:\mathcal{D}\times\mathbb{R}^{n_2}\times U\times W\to\mathbb{R}$ be jointly continuous in x_1, x_2, u, and w.

Theorem 3.1: *Consider the controlled nonlinear dynamical system $\mathcal{G}$ given by* (13) *and* (14) *with*

$$J(x_{10},x_{20},u(\cdot),w(\cdot)) \triangleq \int_0^\infty L(x_1(t),x_2(t),u(t),w(t))\mathrm{d}t, \tag{17}$$

where $u(\cdot)$ and $w(\cdot)$ are admissible controls. Assume that there exists a continuously differentiable function $V:\mathcal{D}\times\mathbb{R}^{n_2}\to\mathbb{R}$, class $\mathcal{K}$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, $k>0$, a real number $\theta\in(0,1)$, an open neighbouhood $\mathcal{M}\subseteq\mathcal{D}$ of $x_1=0$, and control laws $\phi:\mathcal{D}\times\mathbb{R}^{n_2}\to U$ and $\psi:\mathcal{D}\times\mathbb{R}^{n_2}\to W$ such that

$$\alpha(\|x_1\|)\le V(x_1,x_2)\le\beta(\|x_1\|),\qquad (x_1,x_2)\in\mathcal{M}\times\mathbb{R}^{n_2}, \tag{18}$$

$$V'(x_1,x_2)F(x_1,x_2,\phi(x_1,x_2),\psi(x_1,x_2))\le -k(V(x_1,x_2))^\theta,\qquad (x_1,x_2)\in\mathcal{M}\times\mathbb{R}^{n_2}, \tag{19}$$

$$\phi(0,x_2)=0,\qquad x_2\in\mathbb{R}^{n_2}, \tag{20}$$

$$\psi(0,x_2)=0,\qquad x_2\in\mathbb{R}^{n_2}, \tag{21}$$

$$\begin{aligned} L(x_1,x_2,\phi(x_1,x_2),\psi(x_1,x_2))+V'(x_1,x_2)&\\ F(x_1,x_2,\phi(x_1,x_2),\psi(x_1,x_2))&=0,\\ (x_1,x_2)\in\mathcal{M}\times\mathbb{R}^{n_2},& \end{aligned} \tag{22}$$

$$\begin{aligned} L(x_1,x_2,u,\psi(x_1,x_2))+V'(x_1,x_2)&\\ F(x_1,x_2,u,\psi(x_1,x_2))&\ge 0,\\ (x_1,x_2,u)\in\mathcal{M}\times\mathbb{R}^{n_2}\times U,& \end{aligned} \tag{23}$$

$$\begin{aligned} L(x_1,x_2,\phi(x_1,x_2),w)+V'(x_1,x_2)&\\ F(x_1,x_2,\phi(x_1,x_2),w)&\le 0,\\ (x_1,x_2,w)\in\mathcal{M}\times\mathbb{R}^{n_2}\times W.& \end{aligned} \tag{24}$$

Then, with the feedback controls $u=\phi(x_1,x_2)$ and $w=\psi(x_1,x_2)$, the closed-loop system given by (15) *and* (16) *is strongly finite-time stable with respect to x_1 uniformly in x_{20} and there exists a neighbourhood $\mathcal{D}_0\subseteq\mathcal{M}$ of $x_1=0$ and a settling-time function $T:\mathcal{D}_0\times\mathbb{R}^{n_2}\to[0,\infty)$, jointly continuous on $\mathcal{D}_0\times\mathbb{R}^{n_2}$, such that*

$$T(x_{10},x_{20})\le\frac{(V(x_{10},x_{20}))^{1-\theta}}{k(1-\theta)},\qquad (x_{10},x_{20})\in\mathcal{D}_0\times\mathbb{R}^{n_2}, \tag{25}$$

and

$$J(x_{10},x_{20},\phi(x_1(\cdot),x_2(\cdot)),\psi(x_1(\cdot),x_2(\cdot)))=V(x_{10},x_{20}),\qquad (x_{10},x_{20})\in\mathcal{D}_0\times\mathbb{R}^{n_2}. \tag{26}$$

In addition, if $(x_{10},x_{20})\in\mathcal{D}_0\times\mathbb{R}^{n_2}$, then

$$\begin{aligned} &J(x_{10},x_{20},\phi(x_1(\cdot),x_2(\cdot)),\psi(x_1(\cdot),x_2(\cdot)))\\ &=\min_{(u(\cdot),w(\cdot))\in\mathcal{S}_\psi(x_{10},x_{20})\times\mathcal{S}_\phi(x_{10},x_{20})}\max J(x_{10},x_{20},u(\cdot),w(\cdot)) \end{aligned} \tag{27}$$

and

$$J(x_{10},x_{20},\phi(x_1(\cdot),x_2(\cdot)),w(\cdot))\le V(x_{10},x_{20}),\qquad w(\cdot)\in\mathcal{S}_\phi(x_{10},x_{20}). \tag{28}$$

Finally, if $\mathcal{M}=\mathcal{D}=\mathbb{R}^{n_1}$, $U=\mathbb{R}^{m_1}$, $W=\mathbb{R}^{m_2}$, and the functions $\alpha(\cdot)$ and $\beta(\cdot)$ satisfying (18) *are class $\mathcal{K}_\infty$, then* *the closed-loop system* (15) *and* (16) *is globally strongly finite-time stable with respect to* x_1 *uniformly in* x_{20}.

Proof: Local and global strong finite-time stability with respect to x_1 uniformly in x_{20} are a direct consequence of (18) and (19) by applying Theorem 2.1 to the closed-loop system given by (15) and (16). Moreover, it follows from Theorem 2.1 that there exists a neighbourhood $\mathcal{D}_0 \subseteq \mathcal{M}$ of $x_1 = 0$ and a settling-time function $T : \mathcal{D}_0 \times \mathbb{R}^{n_2} \to [0, \infty)$, jointly continuous on $\mathcal{D}_0 \times \mathbb{R}^{n_2}$, such that $x_1(t) \to 0$ as $t \to T(x_{10}, x_{20})$ for all initial conditions $(x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}$. Condition (25) is a restatement of (11) and, using (22), condition (26) is a restatement of (12) as applied to the closed-loop system.

Next, let $(x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}$, $u(\cdot)$ and $w(\cdot)$ be admissible controls, and $x_1(t)$, $t \geq 0$, and $x_2(t)$ be solutions of (13) and (14). Then, it follows that

$$\begin{aligned} 0 = &-\dot{V}(x_1(t), x_2(t)) + V'(x_1(t), x_2(t)) \\ &F(x_1(t), x_2(t), u(t), w(t)), \qquad t \geq 0. \end{aligned} \tag{29}$$

Hence,

$$\begin{aligned} &L(x_1(t), x_2(t), u(t), w(t)) \\ &= -\dot{V}(x_1(t), x_2(t)) + L(x_1(t), x_2(t), u(t), w(t)) \\ &\quad + V'(x_1(t), x_2(t))F(x_1(t), x_2(t), u(t), w(t)), \quad t \geq 0. \end{aligned} \tag{30}$$

Now, it follows from (18) that

$$\begin{aligned} 0 = &\lim_{t\to T(x_{10},x_{20})} \alpha(\|x_1(t)\|) \leq \lim_{t\to\infty} V(x_1(t), x_2(t)) \\ \leq &\lim_{t\to T(x_{10},x_{20})} \beta(\|x_1(t)\|) = 0, \end{aligned} \tag{31}$$

for every $u(\cdot) \in \mathcal{S}_\psi(x_0)$, which implies that

$$\lim_{t\to\infty} V(x_1(t), x_2(t)) = 0. \tag{32}$$

Therefore, it follows from (30), (32), and (23) that

$$\begin{aligned} &J(x_{10}, x_{20}, u(\cdot), \psi(x_1(\cdot), x_2(\cdot))) \\ &= \int_0^\infty L(x_1(t), x_2(t), u(t), \psi(x_1(t), x_2(t)))\mathrm{d}t \\ &= \int_0^\infty -\dot{V}(x_1(t), x_2(t))\mathrm{d}t \\ &\quad + \int_0^\infty L(x_1(t), x_2(t), u(t), \psi(x_1(t), x_2(t)))\mathrm{d}t \\ &\quad + \int_0^\infty V'(x_1, x_2)F(x_1(t), x_2(t), u(t), \psi(x_1(t), x_2(t)))\mathrm{d}t \\ &\geq \int_0^\infty -\dot{V}(x_1(t), x_2(t))\mathrm{d}t \\ &= -\lim_{t\to\infty} V(x_1(t), x_2(t)) + V(x_{10}, x_{20}) \\ &= J(x_{10}, x_{20}, \phi(x_1(\cdot), x_2(\cdot)), \psi(x_1(\cdot), x_2(\cdot))). \end{aligned} \tag{33}$$

Similarly, since (31) and (32) are satisfied for every $w(\cdot) \in \mathcal{S}_\phi(x_0)$, it follows from (30), (32), and (24) that

$$\begin{aligned} &J(x_{10}, x_{20}, \phi(x_1(\cdot), x_2(\cdot)), w(\cdot)) \\ &= \int_0^\infty L(x_1(t), x_2(t), \phi(x_1(t), x_2(t)), w(t))\mathrm{d}t \\ &= \int_0^\infty -\dot{V}(x_1(t), x_2(t))\mathrm{d}t \\ &\quad + \int_0^\infty L(x_1(t), x_2(t), \phi(x_1(t), x_2(t)), w(t))\mathrm{d}t \\ &\quad + \int_0^\infty V'(x_1, x_2)F(x_1(t), x_2(t), \phi(x_1(t), x_2(t)), w(t))\mathrm{d}t \\ &\leq \int_0^\infty -\dot{V}(x_1(t), x_2(t))\mathrm{d}t \\ &= -\lim_{t\to\infty} V(x_1(t), x_2(t)) + V(x_{10}, x_{20}) \\ &= J(x_{10}, x_{20}, \phi(x_1(\cdot), x_2(\cdot)), \psi(x_1(\cdot), x_2(\cdot))), \end{aligned} \tag{34}$$

and (33) and (34) yield (27).

Finally, it follows from (27) that $(\phi(x_1(\cdot), x_2(\cdot)), \psi(x_1(\cdot), x_2(\cdot)))$ is a saddle point for the performance measure (17) on $\mathcal{S}_\psi(x_{10}, x_{20}) \times \mathcal{S}_\phi(x_{10}, x_{20})$, $(x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}$. Hence, (28) directly follows from Lemma 2.1. ■

Given the control laws $\phi(\cdot, \cdot)$ and $\psi(\cdot, \cdot)$, it holds that $\mathcal{S}_\psi(x_{10}, x_{20}) \times \mathcal{S}_\phi(x_{10}, x_{20}) \subseteq \mathcal{S}(x_{10}, x_{20})$, and restricting our problem to $(u(\cdot), w(\cdot)) \in \mathcal{S}(x_{10}, x_{20})$, that is, inputs corresponding to partial-state null convergent solutions, can be interpreted as incorporating a system detectability condition through the cost. However, it is important to note that an explicit characterisation of $\mathcal{S}(x_{10}, x_{20})$, $\mathcal{S}_\psi(x_{10}, x_{20})$, and $\mathcal{S}_\phi(x_{10}, x_{20})$ is not required.

Equation (22) is the steady-state, Hamilton–Jacobi–Isaacs equation and (22)–(24) guarantee that the saddle point condition (27) is verified. Specifically, it follows from (22)–(24) that the feedback control laws $u = \phi(x_1, x_2)$ and $w = \psi(x_1, x_2)$, which are independent of the initial conditions x_{10} and x_{20}, are given by

$$\begin{aligned} \begin{bmatrix} \phi(x_1, x_2) \\ \psi(x_1, x_2) \end{bmatrix} \in &\operatorname*{argminmax}_{(u(\cdot),w(\cdot))\in\mathcal{S}_\psi(x_{10},x_{20})\times\mathcal{S}_\phi(x_{10},x_{20})} [L(x_1, x_2, u, w) \\ &+ V'(x_1, x_2)F(x_1, x_2, u, w)]. \end{aligned} \tag{35}$$

Moreover, it follows from (22)–(24) that the Lyapunov function $V(\cdot, \cdot)$ is the value of the game.

It follows from Theorem 3.1 that the pair of control laws $(\phi(\cdot, \cdot), \psi(\cdot, \cdot))$ guarantees strong finite-time stability with respect to x_1 uniformly in x_{20} of the closed-loop system. However, the state feedback control law $\psi(\cdot, \cdot)$ may be destabilising in the sense that, given an admissible control $u(\cdot) \notin \mathcal{S}_\psi(x_{10}, x_{20})$, the solution $x_1(t) = 0$, $t \geq 0$, of

$$\begin{aligned}\dot{x}_1(t) &= F_1(x_1(t), x_2(t), u(t), \psi(x_1(t), x_2(t))),\\ &\qquad x_1(0) = x_{10}, \qquad t \geq 0,\\ \dot{x}_2(t) &= F_2(x_1(t), x_2(t), u(t), \psi(x_1(t), x_2(t))),\\ &\qquad x_2(0) = x_{20}, \end{aligned} \tag{36}$$

is *not* finite-time stable with respect to x_1 uniformly in x_{20} or may even be unstable. Furthermore, if we consider the input $w(\cdot)$ in (13) and (14) as a disturbance, then the framework developed in Theorem 3.1 provides an analytical expression for the least upper bound on the performance measure (17) over a class of disturbances $\mathcal{S}_\phi(x_{10}, x_{20})$. Specifically, it follows from (26)–(28) that

$$\begin{aligned}&J(x_{10}, x_{20}, \phi(x_1(\cdot), x_2(\cdot)), w(\cdot))\\ &\quad \leq J(x_{10}, x_{20}, \phi(x_1(\cdot), x_2(\cdot)), \psi(x_1(\cdot), x_2(\cdot)))\\ &\quad = V(x_{10}, x_{20}), \qquad (x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}, \end{aligned} \tag{37}$$

for all admissible inputs $w(\cdot)$ such that $\lim_{t\to T(x_{10},x_{20})} x_1(t) = 0$, where $x_1(\cdot)$ is the solution of (13) with $u = \phi(x_1, x_2)$. Finally, it follows from (22) and (19) that the performance integrand $L(\cdot, \cdot, \cdot, \cdot)$ evaluated along the closed-loop trajectory (15) and (16) is bounded from below by a fractional power of the Lyapunov function $V(\cdot, \cdot)$, which is non-negative and equal to zero if and only if $x_1 = 0$.

Possible applications of Theorem 3.1 involve pursuer–evader differential games, such as the *game of two cars* (Isaacs, 1999, pp. 237–244). Specifically, if the game of two cars evolves on a plane, then the system dynamics is characterised by a state vector comprising three components, namely two components that identify the position of the pursuer with respect to the evader and the angle between the velocity vector of the pursuer and the velocity vector of the evader. If the game terminates when evader is reached by the pursuer, then this end-of-game condition involves only two of the three state vector components and does not depend on the angle between the pursuer's and the evader's velocity vectors. Therefore, Theorem 3.1 allows us solving this game of degree with (17) as performance measure (Isaacs, 1999, p. 12). The next remark highlights additional applications of Theorem 3.1.

Remark 3.1: If the conditions of Theorem 3.1 are satisfied, then the closed-loop dynamical system (15) and (16) is strongly finite-time stable with respect to x_1 uniformly in x_{20}. Hence, for every $l > 0$, there exists $\hat{t}_f \geq 0$ such that if $t > \hat{t}_f$, then $\|x_1(t)\| < l$, where $x_1(\cdot)$ denotes the solution of (15). Now, consider a game involving the nonlinear dynamical system (13) and (14) and the performance measure (17), whose terminal condition is given by $\|x_1(t_f)\| = l$ for some $t_f \geq 0$ that is finite and not specified a priori. Then, Theorem 3.1 provides sufficient conditions to find state-feedback control laws that solve this differential game.

The game of two cars allows us appreciating the implications of this remark. Specifically, if we impose that this game terminates when the pursuer is at a distance $l > 0$ from the evader in finite time, then it follows from Remark 3.1 that Theorem 3.1 is adequate to solve this version of the game of two cars.

Remark 3.2: Setting $m_1 = m$ and $m_2 = 0$, the nonlinear controlled dynamical system given by (13) and (14) reduces to

$$\begin{aligned}\dot{x}_1(t) &= F_1(x_1(t), x_2(t), u(t)), \quad x_1(0) = x_{10}, \quad t \geq 0,\\ \dot{x}_2(t) &= F_2(x_1(t), x_2(t), u(t)), \quad x_2(0) = x_{20}, \end{aligned} \tag{38}$$

and the conditions of Theorem 3.1 reduce to the conditions of Theorem 4.2 of Haddad and L'Afflitto (2015).

4. Affine dynamical systems, Lyapunov functions, and differential games

In this section, we specialise the results of Section 3 to nonlinear affine dynamical systems of the form

$$\begin{aligned}\dot{x}_1(t) = {} & f_1(x_1(t), x_2(t)) + G_{1u}(x_1(t), x_2(t))u(t)\\ & + G_{1w}(x_1(t), x_2(t))w(t), \quad x_1(0) = x_{10}, \quad t \geq 0, \end{aligned} \tag{39}$$

$$\begin{aligned}\dot{x}_2(t) = {} & f_2(x_1(t), x_2(t)) + G_{2u}(x_1(t), x_2(t))u(t)\\ & + G_{2w}(x_1(t), x_2(t))w(t), \quad x_2(0) = x_{20}, \end{aligned} \tag{40}$$

where, for every $t \geq 0$, $x_1(t) \in \mathbb{R}^{n_1}$, $x_2(t) \in \mathbb{R}^{n_2}$, $u(t) \in \mathbb{R}^{m_1}$, $w(t) \in \mathbb{R}^{m_2}$, and $f_1 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1}$, $f_2 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$, $G_{1u} : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1 \times m_1}$, $G_{1w} : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1 \times m_2}$, $G_{2u} : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_2 \times m_1}$, and $G_{2w} : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_2 \times m_2}$ are such that $f_1(0, x_2) = 0$ for all $x_2 \in \mathbb{R}^{n_2}$, $f_1(\cdot, \cdot)$, $f_2(\cdot, \cdot)$, $G_{1u}(\cdot, \cdot)$, $G_{1w}(\cdot, \cdot)$, $G_{2u}(\cdot, \cdot)$, and $G_{2w}(\cdot, \cdot)$ are jointly continuous in x_1 and x_2. We also consider performance integrands $L(x_1, x_2, u, w)$ of the form

$$\begin{aligned}L(x_1, x_2, u, w) = {} & L_1(x_1, x_2) + L_{2u}(x_1, x_2)u\\ & + L_{2w}(x_1, x_2)w + u^{\mathrm{T}}R_{2u}(x_1, x_2)u\\ & + w^{\mathrm{T}}R_{2w}(x_1, x_2)w,\\ & (x_1, x_2, u, w) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}, \end{aligned} \tag{41}$$

where $L_1 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, $L_{2u} : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{1 \times m_1}$, $L_{2w} : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{1 \times m_2}$, $R_{2u} : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{m_1 \times m_1}$, and $R_{2w}:\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to\mathbb{R}^{m_2\times m_2}$ are continuous on $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ and such that $R_{2u}(x_1, x_2)\geq N(x_1)>0$, $(x_1, x_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$, and $R_{2w}(x_1, x_2)\leq -N(x_1)<0$, so that (17) becomes

$$\begin{aligned}J(x_{10}, x_{20}, u(\cdot), w(\cdot)) = \int_0^\infty \big[&L_1(x_1(t), x_2(t))\\ &+L_{2u}(x_1(t), x_2(t))u(t)+L_{2w}(x_1(t), x_2(t))w(t)\\ &+u^{\mathrm{T}}(t)R_{2u}(x_1(t), x_2(t))u(t)\\ &+w^{\mathrm{T}}(t)R_{2w}(x_1(t), x_2(t))w(t)\big]\mathrm{d}t. \qquad (42)\end{aligned}$$

The next result proves that the *minimax assumption* (Isaacs, 1999, p. 35) is verified for differential games involving nonlinear affine dynamical systems with quadratic in the controls performance measures. Specifically, given the nonlinear dynamical system (39) and (40) with performance measure (42), we prove that the minimum with respect to $u\in U$ of the maximum with respect to $w\in W$ of the *Hamiltonian function*

$$\begin{aligned}H(x_1, x_2, \lambda^{\mathrm{T}}, u, w) \triangleq{}& L_1(x_1, x_2)+L_{2u}(x_1, x_2)u\\ &+L_{2w}(x_1, x_2)w+u^{\mathrm{T}}R_{2u}(x_1, x_2)u\\ &+w^{\mathrm{T}}R_{2w}(x_1, x_2)w+\lambda^{\mathrm{T}}[f(x_1, x_2)\\ &+G_u(x_1, x_2)u+G_w(x_1, x_2)w], \end{aligned} \qquad (43)$$

where $\lambda\in\mathbb{R}^{n_1+n_2}$ and

$$\begin{aligned}f(x_1, x_2) &\triangleq [f_1^{\mathrm{T}}(x_1, x_2), f_2^{\mathrm{T}}(x_1, x_2)]^{\mathrm{T}},\\ G_u(x_1, x_2) &\triangleq [G_{1u}^{\mathrm{T}}(x_1, x_2), G_{2u}^{\mathrm{T}}(x_1, x_2)]^{\mathrm{T}},\\ G_w(x_1, x_2) &\triangleq [G_{1w}^{\mathrm{T}}(x_1, x_2), G_{2w}^{\mathrm{T}}(x_1, x_2)]^{\mathrm{T}},\end{aligned}$$

is equal to the maximum with respect to $w\in W$ of the minimum with respect to $u\in U$ of $H(\cdot)$.

Proposition 4.1: *Consider the controlled nonlinear affine dynamical system given by* (39) *and* (40) *with performance measure* (42). *Then,*

$$\min_{u\in U}\max_{w\in W} H(x_1, x_2, \lambda^{\mathrm{T}}, u, w) = \max_{w\in W}\min_{u\in U} H(x_1, x_2, \lambda^{\mathrm{T}}, u, w), \qquad (44)$$

where $H(\cdot)$ is given by (43) *and $\lambda\in\mathbb{R}^{n_1+n_2}$.*

Proof: It holds that $w^*\triangleq\arg\max_{w\in W}H(x_1, x_2, \lambda^{\mathrm{T}}, u, w)$ satisfies

$$\left.\frac{\partial H(x_1, x_2, \lambda^{\mathrm{T}}, u, w)}{\partial w}\right|_{w=w^*} = 0, \qquad (45)$$

which implies that

$$w^* = -\frac{1}{2}R_{2w}^{-1}(x_1, x_2)\left[\lambda^{\mathrm{T}}G_w(x_1, x_2)+L_{2w}(x_1, x_2)\right]^{\mathrm{T}}. \qquad (46)$$

Hence, $u_*\triangleq\arg\min_{u\in U}H(x_1, x_2, \lambda^{\mathrm{T}}, u, w^*)$ is such that

$$\left.\frac{\partial H(x_1, x_2, \lambda^{\mathrm{T}}, u, w^*)}{\partial u}\right|_{u=u_*} = 0, \qquad (47)$$

which implies that

$$u_* = -\frac{1}{2}R_{2u}^{-1}(x_1, x_2)\left[\lambda^{\mathrm{T}}G_u(x_1, x_2)+L_{2u}(x_1, x_2)\right]^{\mathrm{T}}. \qquad (48)$$

Similarly, $u^*\triangleq\arg\min_{u\in U}H(x_1, x_2, \lambda, u, w)$ satisfies

$$u^* = -\frac{1}{2}R_{2u}^{-1}(x_1, x_2)\left[\lambda^{\mathrm{T}}G_u(x_1, x_2)+L_{2u}(x_1, x_2)\right]^{\mathrm{T}} \qquad (49)$$

and $w_*\triangleq\arg\max_{w\in W}H(x_1, x_2, \lambda, u^*, w)$ is given by

$$w_* = -\frac{1}{2}R_{2w}^{-1}(x_1, x_2)\left[\lambda^{\mathrm{T}}G_w(x_1, x_2)+L_{2w}(x_1, x_2)\right]^{\mathrm{T}}. \qquad (50)$$

The result now follows noting that $u_*=u^*$ and $w^*=w_*$. ■

Next, we specialise Theorem 3.1 to nonlinear affine dynamical systems with quadratic in the controls performance measures. Specifically, the next result provides an *explicit* characterisation of *globally* strongly finite-time stabilising state-feedback controls, which solve differential games involving dynamical systems of the form (39) and (40) and performance measures of the form (42).

Theorem 4.1: *Consider the controlled nonlinear affine dynamical system given by* (39) *and* (40) *with performance measure* (42), *where $u(\cdot)$ and $w(\cdot)$ are admissible controls. Assume that there exist a continuously differentiable function $V:\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to\mathbb{R}$, class $\mathcal{K}_\infty$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, $k>0$, and a real number $\theta\in(0, 1)$ such that*

$$\alpha(\|x_1\|)\leq V(x_1, x_2)\leq\beta(\|x_1\|), \qquad (x_1, x_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}, \qquad (51)$$

$$\begin{aligned}&V'(x_1, x_2)f(x_1, x_2)\\ &\quad-\frac{1}{2}V'(x_1, x_2)\big[G_u(x_1, x_2)R_{2u}^{-1}(x_1, x_2)L_{2u}^{\mathrm{T}}(x_1, x_2)\\ &\quad+G_w(x_1, x_2)R_{2w}^{-1}(x_1, x_2)L_{2w}^{\mathrm{T}}(x_1, x_2)\big]\\ &\quad-\frac{1}{2}V'(x_1, x_2)\big[G_u(x_1, x_2)R_{2u}^{-1}(x_1, x_2)G_u^{\mathrm{T}}(x_1, x_2)\\ &\quad+G_w(x_1, x_2)R_{2w}^{-1}(x_1, x_2)G_w^{\mathrm{T}}(x_1, x_2)\big]V'^{\mathrm{T}}(x_1, x_2)\\ &\leq -k(V(x_1, x_2))^\theta, \quad (x_1, x_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}, \qquad (52)\end{aligned}$$

$$L_{2u}(0, x_2)=0, \quad x_2\in\mathbb{R}^{n_2}, \qquad (53)$$

$$L_{2w}(0, x_2) = 0, \quad x_2 \in \mathbb{R}^{n_2}, \tag{54}$$

$$\begin{aligned} 0 = {} & L_1(x_1, x_2) + V'(x_1, x_2) f(x_1, x_2) \\ & - \frac{1}{4}\left[V'(x_1, x_2)G_u(x_1, x_2) + L_{2u}(x_1, x_2)\right] \\ & \cdot R_{2u}^{-1}(x_1, x_2)\left[V'(x_1, x_2)G_u(x_1, x_2) + L_{2u}(x_1, x_2)\right]^{\mathrm{T}} \\ & - \frac{1}{4}\left[V'(x_1, x_2)G_w(x_1, x_2) + L_{2w}(x_1, x_2)\right] \\ & \cdot R_{2w}^{-1}(x_1, x_2)[V'(x_1, x_2)G_w(x_1, x_2) \\ & + L_{2w}(x_1, x_2)]^{\mathrm{T}}, \qquad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}. \end{aligned} \tag{55}$$

Then, with the feedback controls

$$u = \phi(x_1, x_2) = -\frac{1}{2}R_{2u}^{-1}(x_1, x_2)[V'(x_1, x_2)G_u(x_1, x_2) + L_{2u}(x_1, x_2)]^{\mathrm{T}}, \tag{56}$$

$$w = \psi(x_1, x_2) = -\frac{1}{2}R_{2w}^{-1}(x_1, x_2)[V'(x_1, x_2)G_w(x_1, x_2) + L_{2w}(x_1, x_2)]^{\mathrm{T}}, \tag{57}$$

the closed-loop system

$$\begin{aligned} \dot{x}_1(t) = {} & f_1(x_1(t), x_2(t)) + G_{1u}(x_1(t), x_2(t))\phi(x_1(t), x_2(t)) \\ & + G_{1w}(x_1(t), x_2(t))\psi(x_1(t), x_2(t)), \\ & x_1(0) = x_{10}, \qquad t \geq 0, \end{aligned} \tag{58}$$

$$\begin{aligned} \dot{x}_2(t) = {} & f_2(x_1(t), x_2(t)) + G_{2u}(x_1(t), x_2(t))\phi(x_1(t), x_2(t)) \\ & + G_{2w}(x_1(t), x_2(t))\psi(x_1(t), x_2(t)), \\ & x_2(0) = x_{20}, \end{aligned} \tag{59}$$

is globally strongly finite-time stable with respect to x_1 uniformly in x_{20} and there exists a settling-time function $T : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to [0, \infty)$, jointly continuous on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, such that (25) *is satisfied. Moreover,*

$$J(x_{10}, x_{20}, \phi(x_1(\cdot), x_2(\cdot)), \psi(x_1(\cdot), x_2(\cdot))) = V(x_{10}, x_{20}), \quad (x_{10}, x_{20}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \tag{60}$$

(27) *is verified with $(x_{10}, x_{20}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, and*

$$J(x_{10}, x_{20}, \phi(x_1(\cdot), x_2(\cdot)), w(\cdot)) \leq V(x_{10}, x_{20}), \quad w(\cdot) \in \mathcal{S}_\phi(x_{10}, x_{20}), \quad (x_{10}, x_{20}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}. \tag{61}$$

Proof: The result is a consequence of Theorem 3.1 with $\mathcal{D} = \mathbb{R}^{n_1}$, $U = \mathbb{R}^{m_1}$, $W = \mathbb{R}^{m_2}$, $F_1(x_1, x_2, u, w) = f_1(x_1, x_2) + G_{1u}(x_1, x_2)u + G_{1w}(x_1, x_2)w$, $F_2(x_1, x_2, u, w) = f_2(x_1, x_2) + G_{2u}(x_1, x_2)u + G_{2w}(x_1, x_2)w$, and $L(x_1, x_2, u, w) = L_1(x_1, x_2) + L_{2u}(x_1, x_2)u + L_{2w}(x_1, x_2)w + u^{\mathrm{T}}R_{2u}(x_1, x_2)u + w^{\mathrm{T}}R_{2w}(x_1, x_2)w$. Specifically, it follows from (35) and Proposition 4.1 that

$$\begin{aligned} & \left[\phi^{\mathrm{T}}(x_1, x_2),\ \psi^{\mathrm{T}}(x_1, x_2)\right]^{\mathrm{T}} \\ & = \operatorname*{arg\,min}_{u(\cdot)\in\mathcal{S}_\psi(x_{10},x_{20})} \operatorname*{arg\,max}_{w(\cdot)\in\mathcal{S}_\phi(x_{10},x_{20})} H\left(x_1, x_2, V'(x_1, x_2), u, w\right) \\ & = \operatorname*{arg\,max}_{w(\cdot)\in\mathcal{S}_\phi(x_{10},x_{20})} \operatorname*{arg\,min}_{u(\cdot)\in\mathcal{S}_\psi(x_{10},x_{20})} H\left(x_1, x_2, V'(x_1, x_2), u, w\right), \end{aligned} \tag{62}$$

where $H(\cdot)$ is given by (43). Thus, the feedback control laws (56) and (57) follow from (35) by setting

$$\begin{aligned} & \frac{\partial}{\partial[u^{\mathrm{T}} w^{\mathrm{T}}]^{\mathrm{T}}}\Big[L_1(x_1, x_2) + L_{2u}(x_1, x_2)u + L_{2w}(x_1, x_2)w \\ & + u^{\mathrm{T}}R_{2u}(x_1, x_2)u + w^{\mathrm{T}}R_{2w}(x_1, x_2)w \\ & + V'(x_1, x_2)f(x_1, x_2) + V'(x_1, x_2)G_u(x_1, x_2)u \\ & + V'(x_1, x_2)G_w(x_1, x_2)w\Big] = 0. \end{aligned} \tag{63}$$

Now, with $u = \phi(x_1, x_2)$ and $w = \psi(x_1, x_2)$ given by (56) and (57), respectively, conditions (51), (52), and (55) imply (18), (19), and (22), respectively. Next, since $V(\cdot,\cdot)$ is continuously differentiable and, by (51), $V(0, x_2)$ is a local minimum of $V(\cdot,\cdot)$, it follows that $V'(0, x_2) = 0$, $x_2 \in \mathbb{R}^{n_2}$, and hence, it follows from (53), (54), (56), and (57) that $\phi(0, x_2) = 0$ and $\psi(0, x_2) = 0$, $x_2 \in \mathbb{R}^{n_2}$, which imply (20) and (21), respectively. Finally, since

$$\begin{aligned} & L(x_1, x_2, u, \psi(x_1, x_2)) + V'(x_1, x_2)[f(x_1, x_2) \\ & \quad + G_u(x_1, x_2)u + G_w(x_1, x_2)\psi(x_1, x_2)] \\ & = L(x_1, x_2, u, \psi(x_1, x_2)) + V'(x_1, x_2)[f(x_1, x_2) \\ & \quad + G_u(x_1, x_2)u \\ & \quad + G_w(x_1, x_2)\psi(x_1, x_2)] \\ & \quad - L(x_1, x_2, \phi(x_1, x_2), \psi(x_1, x_2)) \\ & \quad - V'(x_1, x_2)[f(x_1, x_2) + G_u(x_1, x_2)\phi(x_1, x_2) \\ & \quad + G_w(x_1, x_2)\psi(x_1, x_2)] \\ & = [u - \phi(x_1, x_2)]^{\mathrm{T}} R_{2u}(x_1, x_2)\, [u - \phi(x_1, x_2)] \\ & \geq 0, \qquad (x_1, x_2, u) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{m_1}, \end{aligned} \tag{64}$$

and

$$\begin{aligned} & L(x_1, x_2, \phi(x_1, x_2), w) + V'(x_1, x_2)[f(x_1, x_2) \\ & \quad + G_u(x_1, x_2)\phi(x_1, x_2) + G_w(x_1, x_2)w] \\ & = L(x_1, x_2, \phi(x_1, x_2), w) + V'(x_1, x_2)[f(x_1, x_2) \\ & \quad + G_u(x_1, x_2)\phi(x_1, x_2) \\ & \quad + G_w(x_1, x_2)w] - L(x_1, x_2, \phi(x_1, x_2), \psi(x_1, x_2)) \end{aligned}$$

$$
\begin{aligned}
&- V'(x_1, x_2)[f(x_1, x_2) + G_u(x_1, x_2)\phi(x_1, x_2) \\
&\quad + G_w(x_1, x_2)\psi(x_1, x_2)] \\
&= [w - \psi(x_1, x_2)]^{\mathrm{T}} R_{2w}(x_1, x_2)\,[w - \psi(x_1, x_2)] \\
&\le 0, \qquad (x_1, x_2, w) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{m_2}, \qquad (65)
\end{aligned}
$$

conditions (23) and (24) hold. The result now follows as a direct consequence of Theorem 3.1. ■

Regarding $w(\cdot)$ in (39) and (40) as a disturbance, Theorem 4.1 provides a *globally* strongly finite-time partial-state stabilising control law $u = \phi(x_1, x_2)$ that ensures disturbance rejection over the class of disturbance inputs $\mathcal{S}_\phi(x_{10}, x_{20})$. In addition, this result provides an analytical closed-form expression for the minimum with respect to $u(\cdot)$ of the performance measure (42) for all $w(\cdot) \in \mathcal{S}_\phi(x_{10}, x_{20})$. Therefore, the framework developed in Theorem 4.1 presents a methodology for designing the state-feedback controls $u = \phi(x_1, x_2)$ that guarantee robustness and optimal performance over the class of disturbance inputs $\mathcal{S}_\phi(x_{10}, x_{20})$.

A relevant problem in linear and nonlinear robust control is whether a state-feedback control that guarantees disturbance rejection also guarantees asymptotic stability of the closed-loop dynamical system in absence of disturbance inputs (Ball & Helton, 1989; Green & Limebeer, 1995; Haddad & Chellaboina, 1998). The next result provides sufficient conditions for the state-feedback control law (56) to guarantee disturbance rejection for all $w(\cdot) \in \mathcal{S}_\phi(x_{10}, x_{20})$ and finite-time partial-state stability of the closed-loop dynamical system for $w = 0$.

Theorem 4.2: *Consider the controlled nonlinear dynamical system* (39) *and* (40) *with performance measure* (42), *where $u(\cdot)$ and $w(\cdot)$ are admissible controls, and let $p > 0$ and $\lambda \in (0, 1)$. If the conditions of Theorem* 4.1 *are satisfied with $L_1(x_1, x_2) \ge p(V(x_1, x_2))^\lambda$, $(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, and $L_{2u}(x_1, x_2) = 0$, then with*

$$
\begin{aligned}
u = \phi(x_1, x_2) &= -\frac{1}{2} R_{2u}^{-1}(x_1, x_2)[V'(x_1, x_2)G_u(x_1, x_2) \\
&\quad + L_{2u}(x_1, x_2)]^{\mathrm{T}}, \qquad (66)
\end{aligned}
$$

$$
w = 0, \qquad (67)
$$

the affine in the controls dynamical system (39) *and* (40) *is globally strongly finite-time stable with respect to x_1 uniformly in x_{20} and there exists a settling-time function $T : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to [0, \infty)$, jointly continuous on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, such that* (25) *is verified. Furthermore, it holds that*

$$
\begin{aligned}
&J(x_{10}, x_{20}, \phi(x_1(\cdot), x_2(\cdot)), 0) \le V(x_{10}, x_{20}), \\
&\qquad\qquad (x_{10}, x_{20}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \qquad (68)
\end{aligned}
$$

where $V(\cdot, \cdot)$ satisfies (51)–(55).

Proof: The result follows as a consequence of Theorem 4.1. Specifically, since $L_1(x_1, x_2) \ge p(V(x_1, x_2))^\lambda$, $(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, $R_{2u}(x_1, x_2) \ge N(x_1) > 0$, $R_{2w}(x_1, x_2) \le -N(x_1) < 0$, and $L_{2u}(x_1, x_2) = 0$, it follows from (55) that

$$
\begin{aligned}
&- p(V(x_1, x_2))^\lambda \ge V'(x_1, x_2) f(x_1, x_2) \\
&\quad - \frac{1}{4}\left[V'(x_1, x_2)G_u(x_1, x_2)\right] \\
&\quad \cdot R_{2u}^{-1}(x_1, x_2)\left[V'(x_1, x_2)G_u(x_1, x_2)\right]^{\mathrm{T}} \\
&\ge V'(x_1, x_2) f(x_1, x_2) - \frac{1}{2}\left[V'(x_1, x_2)G_u(x_1, x_2)\right] \\
&\quad \cdot R_{2u}^{-1}(x_1, x_2)\left[V'(x_1, x_2)G_u(x_1, x_2)\right]^{\mathrm{T}}, \\
&\qquad\qquad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}. \qquad (69)
\end{aligned}
$$

Hence, global strong finite-time stability with respect to x_1 uniformly in x_{20} of (39) and (40) with $u = \phi(x_1, x_2)$ and $w = 0$ and the existence of a jointly continuous settling-time function $T(\cdot, \cdot)$ such that (25) is satisfied directly follow from Theorem 5.1 of Haddad and L'Afflitto (2015).

Next, since $x_1(t)$ converges to $x_1 = 0$ in finite time, it holds that $0 \in \mathcal{S}_\phi(x_{10}, x_{20})$. Therefore, since the assumptions of Theorem 4.1 are satisfied, (68) directly follows from (61) with $w = 0$, which concludes the proof. ■

5. Converse differential games

In this section, we extend the notion of *inverse optimal control problem* (Anderson & Moore, 1990; Freeman & Kokotovic, 1996; Jacobson, 1977; Molinari, 1973; Moylan & Anderson, 1973) to address the converse differential game problem (Zhukovskiy, 2003, p. 61) for nonlinear affine in the control dynamical systems. In particular, to avoid the complexity in solving the Hamilton–Jacobi–Isaacs equation (55), we do not attempt to find a saddle point of a given cost functional, but rather we parameterise a family of finite-time partial-state stabilising controllers that verify the saddle point condition for some derived cost functional. The performance integrand is shown to explicitly depend on the nonlinear system dynamics, the Lyapunov function of the closed-loop system, and the feedback control laws, wherein the coupling is introduced via the Hamilton–Jacobi–Isaacs equation. Hence, by varying the parameters in the Lyapunov function and the performance integrand, the proposed framework can be used to characterise a class of finite-time partial-state stabilising controllers that can meet closed-loop system response constraints.

Theorem 5.1: *Consider the controlled nonlinear affine dynamical system given by* (39) *and* (40) *with performance measure* (42), *where $u(\cdot)$ and $w(\cdot)$ are admissible controls. Assume that there exist a continuously differentiable* *function* $V : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, *class* $\mathcal{K}_\infty$ *functions* $\alpha(\cdot)$ *and* $\beta(\cdot)$, $k > 0$, *and a real number* $\theta \in (0, 1)$ *such that*

$$\alpha(\|x_1\|) \le V(x_1, x_2) \le \beta(\|x_1\|), \qquad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \tag{70}$$

$$\begin{aligned} &V'(x_1, x_2) f(x_1, x_2) - \frac{1}{2} V'(x_1, x_2)\big[G_u(x_1, x_2) \\ &\quad R_{2u}^{-1}(x_1, x_2) L_{2u}^{\mathrm{T}}(x_1, x_2) \\ &\quad + G_w(x_1, x_2) R_{2w}^{-1}(x_1, x_2) L_{2w}^{\mathrm{T}}(x_1, x_2)\big] \\ &\quad - \frac{1}{2} V'(x_1, x_2)\big[G_u(x_1, x_2) R_{2u}^{-1}(x_1, x_2) G_u^{\mathrm{T}}(x_1, x_2) \\ &\quad + G_w(x_1, x_2) R_{2w}^{-1}(x_1, x_2) G_w^{\mathrm{T}}(x_1, x_2)\big] V'^{\mathrm{T}}(x_1, x_2) \\ &\le -k(V(x_1, x_2))^\theta, \quad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \end{aligned} \tag{71}$$

$$L_{2u}(0, x_2) = 0, \qquad x_2 \in \mathbb{R}^{n_2}, \tag{72}$$

$$L_{2w}(0, x_2) = 0, \qquad x_2 \in \mathbb{R}^{n_2}. \tag{73}$$

Then, with the feedback controls

$$\begin{aligned} u = \phi(x_1, x_2) = &-\frac{1}{2} R_{2u}^{-1}(x_1, x_2)[V'(x_1, x_2) G_u(x_1, x_2) \\ &+ L_{2u}(x_1, x_2)]^{\mathrm{T}}, \end{aligned} \tag{74}$$

$$\begin{aligned} w = \psi(x_1, x_2) = &-\frac{1}{2} R_{2w}^{-1}(x_1, x_2)[V'(x_1, x_2) G_w(x_1, x_2) \\ &+ L_{2w}(x_1, x_2)]^{\mathrm{T}}, \end{aligned} \tag{75}$$

the closed-loop system given by (58) *and* (59) *is globally strongly finite-time stable with respect to* x_1 *uniformly in* x_{20} *and there exists a settling-time function* $T : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to [0, \infty)$, *jointly continuous on* $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, *such that* (25) *is satisfied. In addition, the performance functional* (42) *with*

$$\begin{aligned} L_1(x_1, x_2) = &\ \phi^{\mathrm{T}}(x_1, x_2) R_{2u}(x_1, x_2) \phi(x_1, x_2) \\ &+ \psi^{\mathrm{T}}(x_1, x_2) R_{2w}(x_1, x_2) \psi(x_1, x_2) \\ &- V'(x_1, x_2) f(x_1, x_2), \end{aligned} \tag{76}$$

is such that

$$\begin{aligned} &J(x_{10}, x_{20}, \phi(x_1(\cdot), x_2(\cdot)), \psi(x_1(\cdot), x_2(\cdot))) = V(x_{10}, x_{20}), \\ &\qquad (x_{10}, x_{20}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \end{aligned} \tag{77}$$

and (27) *and* (61) *are verified with* $(x_{10}, x_{20}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. *Finally, let* $p > 0$ *and* $\lambda \in (0, 1)$. *If*

$$L_1(x_1, x_2) \ge p(V(x_1, x_2))^\lambda, \qquad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \tag{78}$$

and $L_{2u}(x_1, x_2) = 0$, *then the affine in the controls dynamical system given by* (39) *and* (40) *with* $u = \phi(x_1, x_2)$ *and* $w = 0$ *is globally strongly finite-time stable with respect to* x_1 *uniformly in* x_{20}, *and*

$$\begin{aligned} &J(x_{10}, x_{20}, \phi(x_1(\cdot), x_2(\cdot)), 0) \le V(x_{10}, x_{20}), \\ &\qquad (x_{10}, x_{20}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}. \end{aligned} \tag{79}$$

Proof: The proof follows as in the proofs of Theorems 4.1 and 4.2. ■

6. Illustrative numerical examples

In this section, we provide two numerical examples to highlight the direct and converse approaches to the differential game problem developed in the paper.

6.1 Spin stabilisation of a symmetric spacecraft with input disturbance

Consider the equations of motion of a symmetric spacecraft given by Wie (1998, p. 753)

$$\dot{\omega}_1(t) = \alpha_1 u(t), \qquad \omega_1(0) = \omega_{10}, \qquad t \ge 0, \tag{80}$$

$$\dot{\omega}_2(t) = \alpha_1 w(t), \qquad \omega_2(0) = \omega_{20}, \tag{81}$$

$$\dot{\omega}_3(t) = \alpha_2 u(t) + \alpha_3 w(t), \qquad \omega_3(0) = \omega_{30}, \tag{82}$$

where $[\omega_1, \omega_2, \omega_3]^{\mathrm{T}} : [0, \infty) \to \mathbb{R}^3$ denotes the angular velocity vector with respect to a given inertial reference frame expressed in a central body reference frame, $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$, $\alpha_1 > 0$, and $u : [0, \infty) \to \mathbb{R}$ and $w : [0, \infty) \to \mathbb{R}$ are the spacecraft control moments. For this example, we seek state feedback controllers $u = \phi(x_1, x_2)$ and $w = \psi(x_1, x_2)$, where $x_1 = [\omega_1, \omega_2]^{\mathrm{T}}$ and $x_2 = \omega_3$, such that the performance measure

$$\begin{aligned} J(x_{10}, x_{20}, u(\cdot), w(\cdot)) = &\int_0^\infty \big[9\alpha_1^2\big(\omega_1^2(t) - \omega_2^2(t)\big) \\ &+ 2\alpha_1 \|x_1(t)\|^2 (\omega_1(t) u(t) - 5\omega_2(t) w(t)) \\ &+ \|x_1(t)\|^4 \big(u^2(t) - w^2(t)\big)\big] \mathrm{d}t \end{aligned} \tag{83}$$

where $x_{10} = [\omega_{10}, \omega_{20}]^{\mathrm{T}}$ and $x_{20} = \omega_{30}$ satisfy (27) and the affine dynamical system given by (80)–(82) is globally strongly finite-time stable with respect to x_1 uniformly in x_{20}. Minimising with respect to u and maximising with respect to w, the term $\int_0^\infty \big[u^2(t) - w^2(t)\big] \mathrm{d}t$ in (83) implies minimising the difference in control effort along two inertia axes. Furthermore, the term $\int_0^\infty \big[\omega_1^2(t) - \omega_2^2(t)\big] \mathrm{d}t$ in (83) captures the difference in kinetic energy due to the angular velocities $\omega_1(\cdot)$ and $\omega_2(\cdot)$.

Note that (80)–(82) with performance measure (83) can be cast in the form of (39) and (40) with performance measure (42).

In this case, Theorem 4.1 can be applied with $n_1 = 2$, $n_2 = 1$, $m_1 = 1$, $m_2 = 1$, $f(x_1, x_2) = 0$, $G_u(x_1, x_2) = [\alpha_1, 0, \alpha_2]^{\mathrm{T}}$, $G_w(x_1, x_2) = [0, \alpha_1, \alpha_3]^{\mathrm{T}}$, $L_1(x_1, x_2) = 9\alpha_1^2\left(\omega_1^2(t) - \omega_2^2(t)\right)$, $L_{2u}(x_1, x_2) = 2\alpha_1\|x_1\|^2\omega_1$, $L_{2w}(x_1, x_2) = -10\alpha_1\|x_1\|^2\omega_2$, $R_{2u}(x_1, x_2) = \|x_1\|^4$, and $R_{2w}(x_1, x_2) = -\|x_1\|^4$ to characterise the finite-time partial-state stabilising controllers. In this case, (55) is verified by

$$V(x_1, x_2) = \left[x_1^{\mathrm{T}}x_1\right]^2, \qquad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}. \quad (84)$$

Hence, (51) holds with $\alpha(\|x_1\|) = \beta(\|x_1\|) = \|x_1\|^4$. In addition, (53) and (54) are satisfied, and since

$$\begin{aligned} &V'(x_1, x_2)f(x_1, x_2) - \frac{1}{2}V'(x_1, x_2) \\ &\quad\times\big[G_u(x_1, x_2)R_{2u}^{-1}(x_1, x_2)L_{2u}^{\mathrm{T}}(x_1, x_2) \\ &\quad+ G_w(x_1, x_2)R_{2w}^{-1}(x_1, x_2)L_{2w}^{\mathrm{T}}(x_1, x_2)\big] \\ &\quad- \frac{1}{2}V'(x_1, x_2)\big[G_u(x_1, x_2)R_{2u}^{-1}(x_1, x_2)G_u^{\mathrm{T}}(x_1, x_2) \\ &\quad+ G_w(x_1, x_2)R_{2w}^{-1}(x_1, x_2)G_w^{\mathrm{T}}(x_1, x_2)\big]V'^{\mathrm{T}}(x_1, x_2) \\ &= -12\alpha_1^2\|x_1\|^2, \qquad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \end{aligned} \quad (85)$$

(52) is satisfied with $\theta = \frac{1}{2}$ and $k = 12\alpha_1^2$.

Since all of the conditions of Theorem 4.1 hold, it follows that the feedback control laws (56) and (57) given by

$$\begin{aligned} \phi(x_1, x_2) &= -\frac{1}{2}R_{2u}^{-1}(x_1, x_2)\big[G_u^{\mathrm{T}}(x_1, x_2)V'^{\mathrm{T}}(x_1, x_2) \\ &\quad+ L_{2u}^{\mathrm{T}}(x_1, x_2)\big] = -3\alpha_1\|x_1\|^{-2}\omega_1, \end{aligned} \quad (86)$$

$$\begin{aligned} \psi(x_1, x_2) &= -\frac{1}{2}R_{2w}^{-1}(x_1, x_2)\big[G_w^{\mathrm{T}}(x_1, x_2)V'^{\mathrm{T}}(x_1, x_2) \\ &\quad+ L_{2w}^{\mathrm{T}}(x_1, x_2)\big] = -3\alpha_1\|x_1\|^{-2}\omega_2, \end{aligned} \quad (87)$$

guarantee that the dynamical system (80)–(82) is globally strongly finite-time stable with respect to x_1 uniformly in x_{20} and there exists a settling-time function $T : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to [0, \infty)$, jointly continuous on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, such that

$$T(x_{10}, x_{20}) \le \frac{1}{6}\alpha_1^{-2}\|x_{10}\|^2, \qquad (x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}. \quad (88)$$

Moreover,

$$\begin{aligned} &J(x_{10}, x_{20}, \phi(\cdot, \cdot), \psi(\cdot, \cdot)) \\ &\quad= \min_{(u(\cdot), w(\cdot)) \in \mathcal{S}_\psi(x_{10}, x_{20}) \times \mathcal{S}_\phi(x_{10}, x_{20})} \max J(x_{10}, x_{20}, u(\cdot), w(\cdot)) \\ &\quad= \left(\omega_{10}^2 + \omega_{20}^2\right)^2, \qquad (x_{10}, x_{20}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}. \end{aligned} \quad (89)$$

Now, suppose the thruster delivering the control moment w is defective and

$$w = -\omega_2^{-1}\delta(t), \qquad (t, x_1, x_2) \in [0, \infty) \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \quad (90)$$

where $\epsilon > 0$ and δ: $[0, \infty) \to [0, \infty)$ is continuous on the set of nonnegative real numbers. Then, the closed-loop dynamical system is given by

$$\dot{\omega}_1(t) = \alpha_1\phi(x_1(t), x_2(t)), \qquad \omega_1(0) = \omega_{10}, \qquad t \ge 0, \quad (91)$$

$$\dot{\omega}_2(t) = \alpha_1\left[-\omega_2^{-1}(t)\delta(t)\right], \qquad \omega_2(0) = \omega_{20}, \quad (92)$$

$$\begin{aligned} \dot{\omega}_3(t) = \alpha_2\phi(x_1(t), x_2(t)) - \alpha_3\omega_2^{-1}(t)\delta(t), \\ \omega_3(0) = \omega_{30}, \end{aligned} \quad (93)$$

which is equivalent to the time-invariant nonlinear dynamical system

$$\dot{\omega}_1(t) = \alpha_1\phi(x_1(t), x_2(t)), \qquad \omega_1(0) = \omega_{10}, \qquad t \ge 0, \quad (94)$$

$$\dot{\omega}_2(t) = -\alpha_1\omega_2^{-1}(t)\delta(\hat{x}_2(t)), \qquad \omega_2(0) = \omega_{20}, \quad (95)$$

$$\begin{aligned} \dot{\omega}_3(t) = \alpha_2\phi(x_1(t), x_2(t)) - \alpha_3\omega_2^{-1}(t)\delta(t), \\ \omega_3(0) = \omega_{30}, \end{aligned} \quad (96)$$

$$\dot{\omega}_4(t) = 1, \qquad \omega_4(0) = 0, \quad (97)$$

where $\hat{x}_2 \triangleq \omega_4$.

In this case, the Lyapunov function (84) is such that

$$\begin{aligned} \dot{V}(x_1, \hat{x}_2) &= V'(x_1, \hat{x}_2)[f(x_1, \hat{x}_2) + G_u(x_1, \hat{x}_2)\phi(x_1, \hat{x}_2) \\ &\quad- \omega_2^{-1}G_w(x_1, \hat{x}_2)\delta(\|\hat{x}_2\|)] \\ &= -12\alpha_1^2\omega_1^2 - 4\alpha_1\delta(\|\hat{x}_2\|)\|x_1\|^2 \\ &\le -4\alpha_1\delta(\|\hat{x}_2\|)\left(V(x_1, \hat{x}_2)\right)^{\frac{1}{2}}, \\ &\qquad (x_1, \hat{x}_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2+1}, \end{aligned} \quad (98)$$

and it follows from Theorem 3.1 of Haddad and L'Afflitto (2015) that the dynamical system (94), (95), and (97) is globally finite-time stable with respect to x_1 uniformly in $\hat{x}_2(0)$, which implies that $\lim_{t\to T(x_{10}, \hat{x}_{20})} x_1(t) = 0$, where

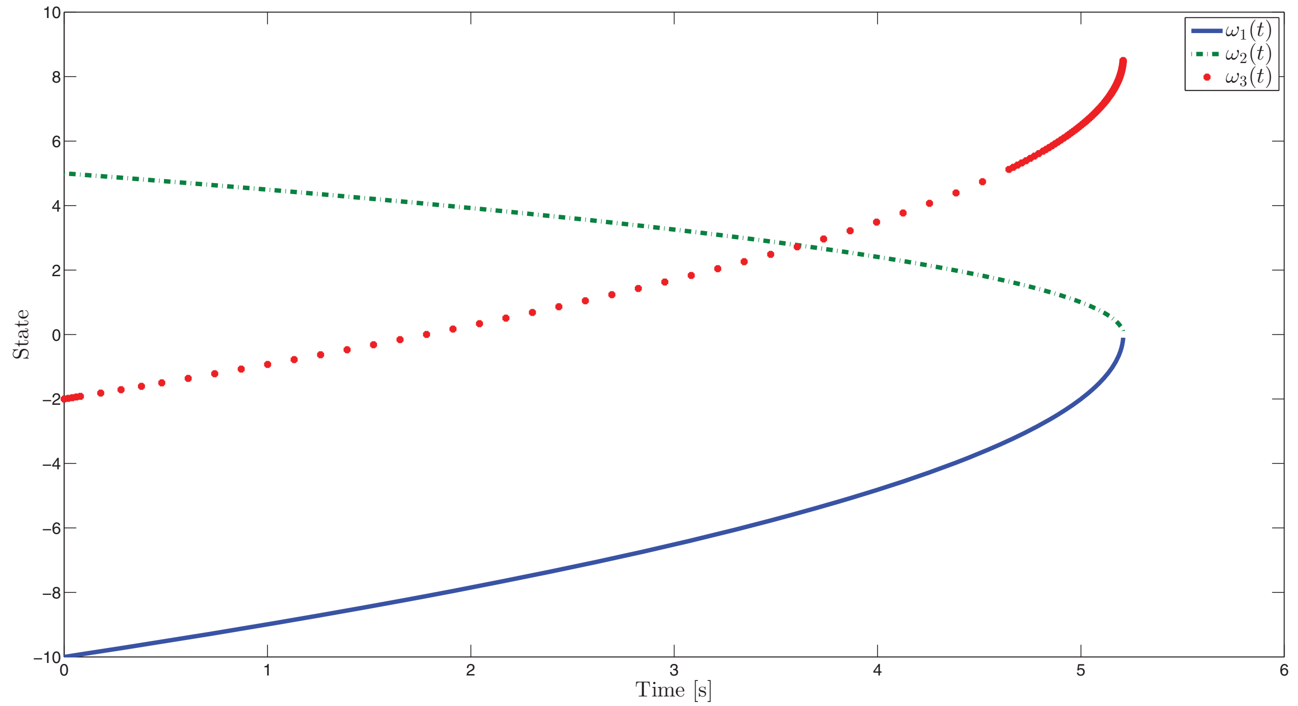

Figure 1. Closed-loop system trajectories versus time.

$T : \mathbb{R}^{n_1} \times \mathbb{R}$ is a jointly continuous settling-time function. Hence, the input function (90) is such that $w(\cdot) \in \mathcal{S}_\phi(x_{10}, x_{20})$ and it follows from Theorem 4.1 that

$$J(x_{10}, x_{20}, \phi(\cdot,\cdot), w(\cdot)) \le J(x_{10}, x_{20}, \phi(\cdot,\cdot), \psi(\cdot,\cdot)) = \|x_{10}\|^2, \quad (x_{10}, x_{20}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}. \tag{99}$$

Let $I_1 = 4\,\mathrm{kg\,m}^2$, $I_2 = 10\,\mathrm{kg\,m}^2$, $I_3 = 20\,\mathrm{kg\,m}^2$, $\omega_{10} = -10\,\mathrm{Hz}$, $\omega_{20} = 5\,\mathrm{Hz}$, and $\omega_{30} = 2\,\mathrm{Hz}$. Figure 1 shows the state trajectories of (80) and (81) with $u = \phi(x_1, x_2)$ and $w = \psi(x_1, x_2)$ versus time, and Figure 2 shows the corresponding control signal versus time. Next, let $\delta(t) = \sin^2(t)$, $t \ge 0$. Figure 3 shows the state trajectories of (80) and (81) with $u = \phi(x_1, x_2)$ and $w = -\omega_2^{-1}\delta(t)$ versus time, and Figure 4 shows the corresponding control signal versus time. Note that $x_1(t) \to 0$ in finite-time both in Figures 1 and 3. Finally, $J(x_{10}, x_{20}, \phi(\cdot,\cdot), \psi(\cdot,\cdot)) = J(x_{10}, x_{20}, \phi(\cdot,\cdot), w(\cdot)) = 15,625\,\mathrm{Hz}^2$, for all $w(\cdot) \in \mathcal{S}_\phi(x_{10}, x_{20})$.

6.2 Spin stabilisation of a spacecraft with parameter uncertainty

Consider the equations of motion of a spacecraft with one axis of symmetry (Wie, 1998, p. 753) given by

$$\dot\omega_1(t) = I_{23}\omega_2(t)\omega_3(t) + \hat\alpha_1 u_1(t), \quad \omega_1(0) = \omega_{10}, \quad t \ge 0, \tag{100}$$

$$\dot\omega_2(t) = -I_{23}\omega_3(t)\omega_1(t) + \hat\alpha_1 u_2(t), \quad \omega_2(0) = \omega_{20}, \tag{101}$$

$$\dot\omega_3(t) = \alpha_3 u_1(t) + \alpha_4 u_2(t), \quad \omega_3(0) = \omega_{30}, \tag{102}$$

where $I_{23} = (I_2 - I_3)/I_1$, I_1, I_2, and I_3 are the principal moments of inertia of the spacecraft such that $0 < I_1 = I_2 < I_3$, α_3 and $\alpha_4 \in \mathbb{R}$, $\hat\alpha_1 > 0$, and u_1 and u_2 are the spacecraft control moments. Haddad and L'Afflitto (2015) prove that the state-feedback control

$$u = \hat\phi(x_1, x_2) = \Big[-\hat\alpha_1^{-1} I_{23}\omega_3\omega_2 - \frac{2}{3}\hat\alpha_1 p^{\frac{2}{3}}\omega_1\|x_1\|^{-\frac{2}{3}}, \ \hat\alpha_1^{-1} I_{23}\omega_3\omega_1 - \frac{2}{3}\hat\alpha_1 p^{\frac{2}{3}}\omega_2\|x_1\|^{-\frac{2}{3}} \Big]^{\mathrm{T}}, \tag{103}$$

where $x_1 = [\omega_1, \omega_2]^{\mathrm{T}}$ and $x_2 = \omega_3$ guarantee global strong finite-time stability of (100)–(102) with respect to x_1 uniformly in $x_{20} = \omega_{30}$ for any $p > 0$. Furthermore, if $u = \hat\phi(x_1, x_2)$, then the performance functional (42) with

$$L_1(x_1, x_2) = \frac{4}{9}\hat\alpha_1^2 p^{\frac{4}{3}}\|x_1\|^{\frac{1}{3}} + \hat\alpha_1^{-2} I_{23}^2\omega_3^2\|x_1\|^2, \tag{104}$$

$$L_{2u}(x_1, x_2) = 2\big[\hat\alpha_1^{-1} I_{23}\omega_3\omega_2 \ -\hat\alpha_1^{-1} I_{23}\omega_3\omega_1\big], \tag{105}$$

$$R_{2u}(x_1, x_2) = 1, \tag{106}$$

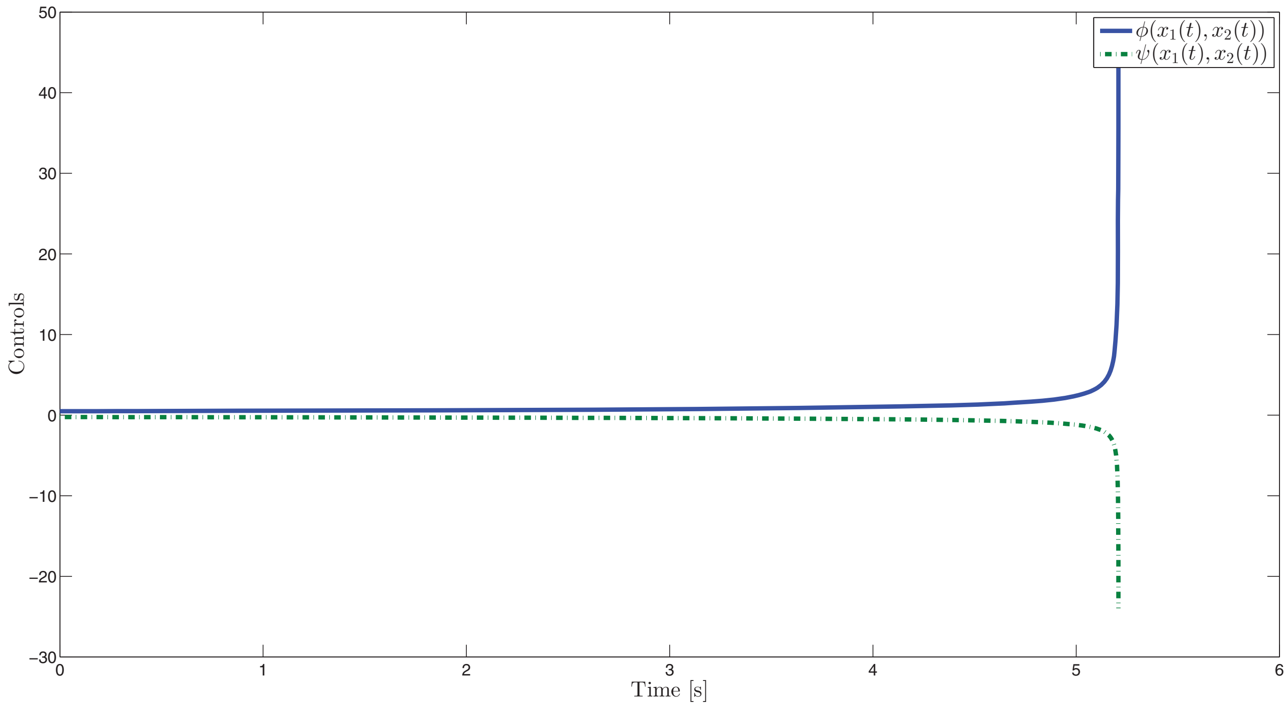

Figure 2. Control signal versus time.

and $w = 0$ is minimised with respect to $u(\cdot)$ in the sense that

$$J(x_{10}, x_{20}, \hat{\phi}(\cdot,\cdot), 0) = \min_{(u(\cdot),0)\in\mathcal{S}(x_{10},x_{20})} J(x_{10}, x_{20}, u(\cdot), 0) = p^{\frac{2}{3}} \left(x_{10}^{\mathrm{T}} x_{10}\right)^{\frac{2}{3}}, \tag{107}$$

for all $(x_{10}, x_{20}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, where $x_{10} = [\omega_{10}, \omega_{20}]^{\mathrm{T}}$.

In this example, we apply the converse differential game framework developed in this paper to analyse the robustness of (103) to parametric uncertainties in $\hat{\alpha}_1$. Specifically, let $\hat{\alpha}_1 = \alpha_1 + w$, where $\alpha_1 > 0$ is an estimate of $\hat{\alpha}_1$ and w is unknown. We apply Theorem 5.1 to find

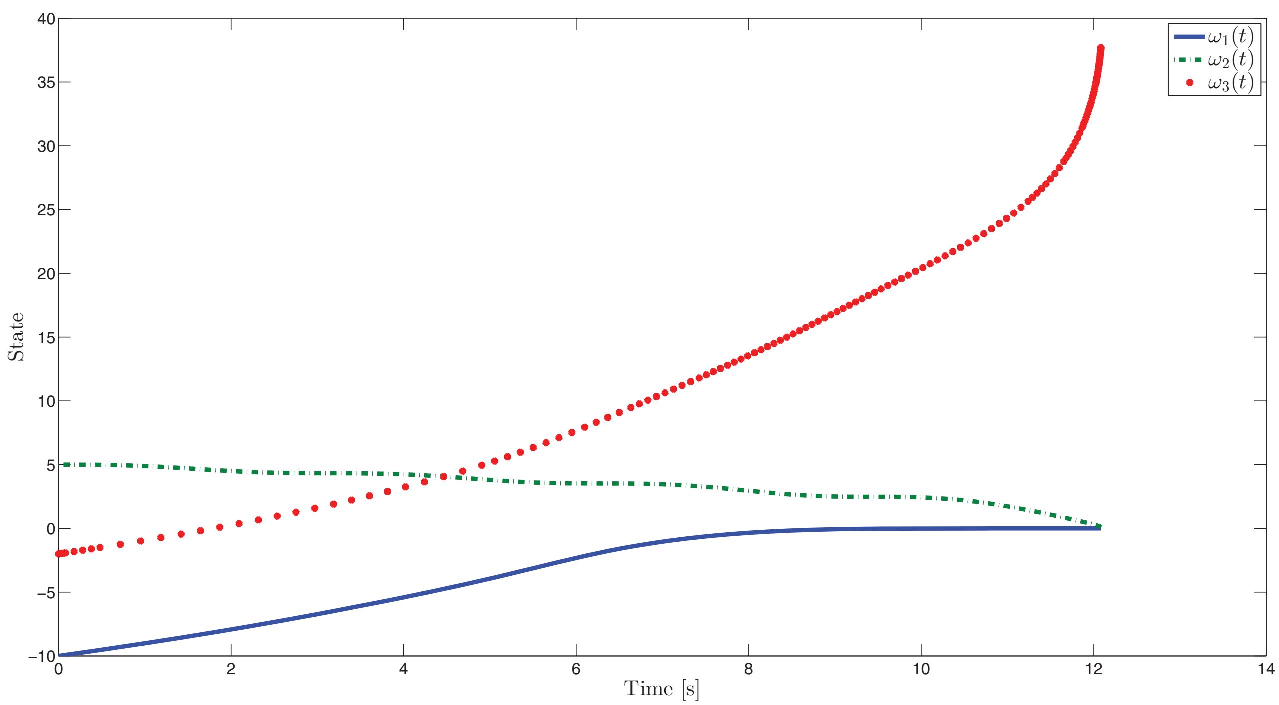

Figure 3. Closed-loop system trajectories versus time in the presence of disturbances.

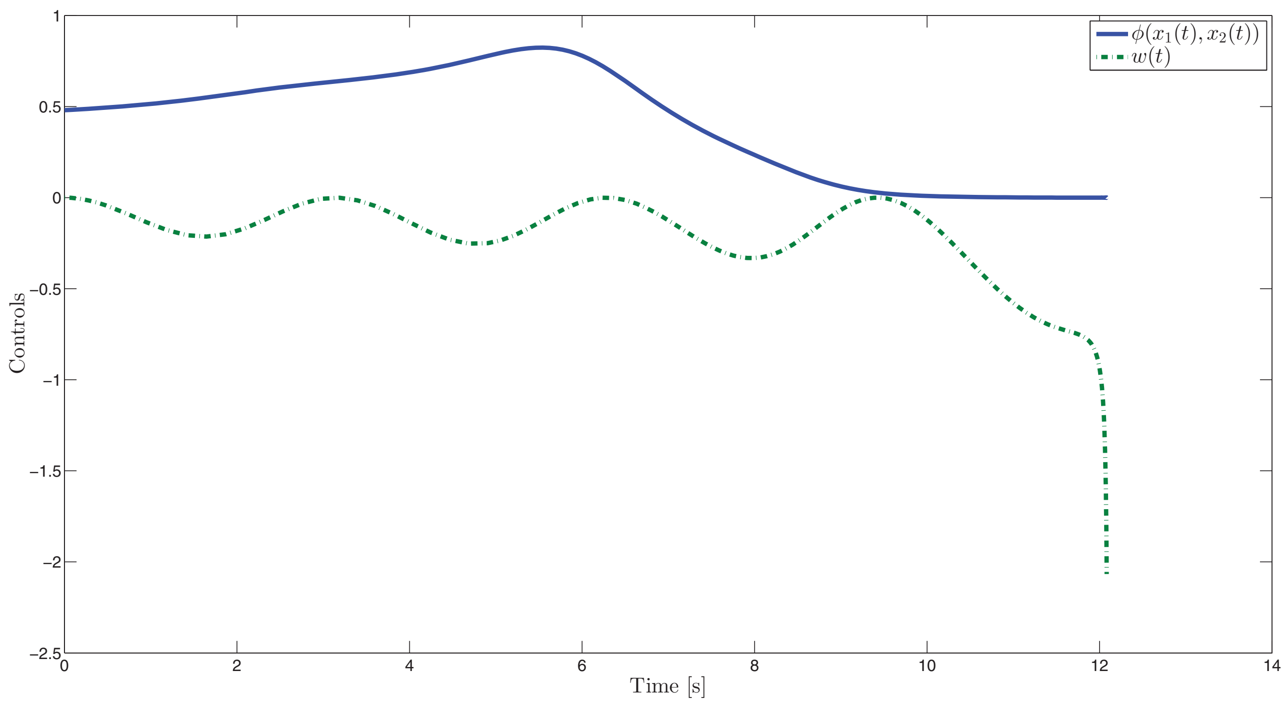

Figure 4. Control signal versus time.

$W \subseteq \mathbb{R}$ such that (100)–(102) with

$$u = \phi(x_1, x_2) = \Big[-\alpha_1^{-1} I_{23}\omega_3\omega_2 - \frac{2}{3}\alpha_1 p^{\frac{2}{3}} \omega_1 \|x_1\|^{-\frac{2}{3}}, \quad \alpha_1^{-1} I_{23}\omega_3\omega_1 - \frac{2}{3}\alpha_1 p^{\frac{2}{3}} \omega_2 \|x_1\|^{-\frac{2}{3}} \Big]^{\mathrm{T}} \tag{108}$$

and $w \in W$ is globally strongly finite-time stable with respect to x_1 uniformly in x_{20}. Furthermore, we parametrise a class of performance measures $J(x_{10}, x_{20}, u(\cdot), w)$ of the form (42) such that

$$J(x_{10}, x_{20}, \phi(\cdot,\cdot), 0) = J(x_{10}, x_{20}, \phi(\cdot,\cdot), w) = p^{\frac{2}{3}} \left(x_{10}^{\mathrm{T}} x_{10}\right)^{\frac{2}{3}}, \quad (x_{10}, x_{20}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \tag{109}$$

for all $w \in W$.

Note that, in the presence of parameter uncertainty, the closed-loop dynamical system (100)–(102) with feedback control (108) can be written in the same form as (39) and (40) with $n_1 = 2$, $n_2 = 1$, $m_1 = 2$, $m_2 = 1$, $f(x_1, x_2) = [I_{23}\omega_2\omega_3, -I_{23}\omega_3\omega_1, 0]^{\mathrm{T}}$, $G_u(x_1, x_2) = \left[\begin{smallmatrix} \alpha_1 & 0 & \alpha_3 \\ 0 & \alpha_1 & \alpha_4 \end{smallmatrix}\right]^{\mathrm{T}}$, $G_w(x_1, x_2) = \left[\phi^{\mathrm{T}}(x_1, x_2), 0\right]^{\mathrm{T}}$, and $u = \phi(x_1, x_2)$. Furthermore, consider the performance measure (42) with $R_{2u}(x_1, x_2)$ given by (106) and

$$L_{2u}(x_1, x_2) = 2\left[\alpha_1^{-1} I_{23}\omega_3\omega_2 \;\; -\alpha_1^{-1} I_{23}\omega_3\omega_1\right], \tag{110}$$

$$L_{2w}(x_1, x_2) = \frac{8}{9}\alpha_1 p^{\frac{4}{3}} \left(\omega_1^2 + \omega_2^2\right)^{\frac{1}{3}}, \tag{111}$$

$$R_{2w}(x_1, x_2) = -1. \tag{112}$$

Remarkably, (108) and (105) are in the same form as (103) and (110), respectively. Let

$$V(x_1, x_2) = p^{\frac{2}{3}} \left(x_1^{\mathrm{T}} x_1\right)^{\frac{2}{3}} \tag{113}$$

and note that (70) is verified with $\alpha(\|x_1\|) = \beta(\|x_1\|) = p^{\frac{2}{3}} \|x_1\|^{\frac{4}{3}}$, (71) is verified with $k = \frac{8}{9}\alpha_1^2 p$ and $\theta = \frac{1}{2}$, and (72) and (73) hold. Thus, it follows from Theorem 5.1 that (108) is a restatement of (74), $\psi(x_1, x_2) = 0$, and

$$J(x_{10}, x_{20}, \phi(\cdot,\cdot), 0) = p^{\frac{2}{3}} \left(\omega_{10}^2 + \omega_{20}^2\right)^{\frac{2}{3}}, \quad (x_{10}, x_{20}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \tag{114}$$

where

$$L_1(x_1, x_2) = \frac{4}{9}\alpha_1^2 p^{\frac{4}{3}} \|x_1\|^{\frac{2}{3}} + \alpha_1^{-2} I_{23}^2 \omega_3^2 \|x_1\|^2. \tag{115}$$

The positive definite, decrescent, radially unbounded Lyapunov function (113) is such that

$$\dot{V}(x_1, x_2, w) = V'(x_1, x_2)[f(x_1, x_2) + G_u(x_1, x_2)\phi(x_1, x_2) + G_w(x_1, x_2)w]$$

$$= -\frac{8}{9}\alpha_1 p^{\frac{4}{3}} (\alpha_1 + w) \|x_1\|^{\frac{2}{3}}$$
$$= -\frac{8}{9} p\alpha_1 (\alpha_1 + w) V^{\frac{1}{2}}(x_1, x_2),$$
$$(x_1, x_2, w) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times W. \quad (116)$$

Thus, it follows from Theorem 2.1 that the nonlinear affine in the control dynamical system (100)–(102) with $u = \phi(x_1, x_2)$ is globally strongly finite-time stable with respect to x_1 uniformly in x_{20} for all $w \in W$, where $W = (-\alpha_1, \infty)$. Moreover, there exists a settling-time function $T : \mathbb{R}^2 \times \mathbb{R} \to [0, \infty)$ such that

$$T(x_{10}, x_{20}) \le \frac{9}{4}\alpha_1^{-1} p^{-\frac{2}{3}} (\alpha_1 + w)^{-1} \|x_{10}\|^{\frac{2}{3}},$$
$$(x_{10}, x_{20}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}. \quad (117)$$

Since (100)–(102) with $u = \phi(x_1, x_2)$ and $w \in W$ is globally strongly finite-time stable with respect to x_1 uniformly in x_{20}, it holds that $\lim_{t\to T(x_{10},x_{20})} x_1(t) = 0$. Consequently, any constant function $w(t) = w$, $w \in W$, is such that $w(\cdot) \in \mathcal{S}_\phi(x_{10}, x_{20})$ and it follows from Theorem 5.1 that (61) is verified, that is,

$$J(x_{10}, x_{20}, \phi(x_1(\cdot), x_2(\cdot)), w) \le p^{\frac{2}{3}} \left(x_{10}^{\mathrm{T}} x_{10}\right)^{\frac{2}{3}},$$
$$w \in (-\alpha_1, \infty), \quad (x_{10}, x_{20}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}. \quad (118)$$

We have therefore proven that the state-feedback control law (108), which is a function of the *estimated* parameter α_1, guarantees global strong finite-time stability with respect to x_1 uniformly in x_{20} of (100)–(102) for all $\hat{\alpha}_1 \in (0, \infty)$. Furthermore, we have provided the least upper bound on the optimal performance measure (107) with $L_1(x_1, x_2)$, $L_{2u}(x_1, x_2)$, $L_{2w}(x_1, x_2)$, $R_{2u}(x_1, x_2)$, and $R_{2w}(x_1, x_2)$ given by (115), (110), (111), (106), and (112), respectively.

7. Conclusion

In this paper, we addressed the problem of finding feedback control laws that solve the two-player zero-sum differential game problem and guarantee finite-time partial-state stability of the closed-loop system. Specifically, we proved sufficient conditions for the existence of pursuer's and evader's state feedback control laws that guarantee partial-state finite-time stability of the closed-loop system and the existence of a saddle point for the system's performance measure.

Our framework also allowed us solving optimal control problems involving nonlinear dynamical systems with nonlinear-nonquadratic performance measures in the presence of exogenous disturbances. Specifically, we provided an explicit expression for the least upper bound on system's optimal performance measure over a set of disturbance inputs. Furthermore, in the case of affine dynamical systems with quadratic in the controls performance measures, we gave an explicit closed-form expression of the optimal state-feedback control laws that guarantee finite-time partial-state stability of the closed-loop system.

Finally, we developed feedback controllers for affine nonlinear dynamical systems extending a well-known inverse optimality framework. The applicability of the theoretical results developed herein is demonstrated by two illustrative numerical examples, which concern the spin stabilisation in finite time of a spacecraft.

Disclosure statement

No potential conflict of interest was reported by the author.

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