Differential Games, Asymptotic Stabilization, and Robust Optimal Control of Nonlinear Systems

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Abstract— **In this paper, we develop a unified framework to solve the two-players zero-sum differential game problem over the infinite time horizon. Asymptotic stability of the closed-loop nonlinear system is guaranteed by means of a Lyapunov function that can clearly be seen to be the solution to the steady-state form of the Hamilton-Jacobi-Isaacs equation, and hence, guaranteeing both asymptotic stability and the existence of a saddle point for the system's performance measure. The overall framework provides the foundation for extending optimal linear-quadratic controller synthesis to differential games involving nonlinear dynamical systems with nonlinear-nonquadratic performance measures. Connections to optimal linear and nonlinear regulation for linear and nonlinear dynamical systems with quadratic and nonlinear-nonquadratic cost functionals in the presence of exogenous disturbances are also provided.**

I. Introduction

The seminal work by Isaacs [1] commenced a systematic study of games within the framework of optimal control theory. Differential games, that is, games which dynamics is governed by ordinary differential equations, have proven their relevance by successfully modeling countless applications ranging from aerospace engineering [2] to marine engineering [3], and communication networks [4]. Furthermore, several variations of the differential game problem have been investigated, such as games involving two [1] or more players [5], a single [1] or multiple performance measures [6], and various forms of collaboration among players [7].

Two-players zero-sum differential games are characterized by two players, that is, two control inputs, which are generally named *pursuer* and *evader*, that concurrently strive to minimize or maximize a given performance measure, respectively. This type of games is usually cast over a *finite* time interval and ends when the system trajectory meets some specified condition, such as crossing a given manifold. The study of zero-sum differential games over the *infinite* time interval has received considerably less attention and has been mostly explored for linear dynamical systems with quadratic performance measures [8] and to establish connections with the classic $\mathcal{H}_\infty$ control theory [9]–[13]. Connections between differential game theory and the disturbance rejection problem for nonlinear dynamical systems have been partly discussed in [14], [15].

In this paper, we address the two-players zero-sum differential game problem for nonlinear dynamical systems with nonlinear-nonquadratic performance measures over the infinite time horizon. Specifically, we provide a framework for designing the pursuer's and evader's nonlinear state feedback control laws, which guarantee asymptotic stability of the closed-loop dynamical system and the existence of a saddle point for the system's performance measure. Remarkably, if the end-of-game condition is given by the convergence of the system trajectory to an equilibrium point, then closed-loop asymptotic stability is a key feature to guarantee that this condition is permanently enforced. The framework presented in this paper is also suitable to address problems in which the differential game ends when the system state trajectory enters a *given* neighborhood of an equilibrium point within some time interval that is finite and not assigned *a priori*. No collaboration between the pursuer and the evader to achieve closed-loop asymptotic stability is assumed in this paper. Indeed, the pursuer's control policy is designed to guarantee closed-loop stability with respect to a class of evader's admissible controls, some of which may lead to system instability if applied in conjunction with other pursuer's admissible controls.

In [16], the current status of continuous-time, nonlinear nonquadratic optimal control problems was presented in a tutorial manner. The underlying ideas of the results in [16] are based on the fact that a steady-state solution of the Hamilton-Jacobi-Bellman equation is a Lyapunov function for the nonlinear system and thus guaranteeing both stability and optimality [16], [17]. One of the main contributions of this paper is extending the framework presented in [16] to address two-players zero-sum differential games involving nonlinear dynamical systems with nonlinear-nonquadratic performance measures. Specifically, we prove that if there exists a Lyapunov function that satisfies the steady-state form of the Hamilton-Jacobi-Isaacs equation for the controlled system, then there exists a solution of the differential game on the infinite horizon, that is, there exist pursuer's and evader's control policies that guarantee both asymptotic stability of the closed-loop dynamical system and the existence of a saddle point for the system's performance measure. In this case, we provide an explicit closed-form expression for the performance measure evaluated at the saddle point and characterize the evader's and pursuer's control policies needed to verify the saddle point condition.

Another key point of this paper is the following. If the pursuer's control law guarantees *convergence* of the closed-loop system to an equilibrium point, although the evader applies some admissible control such that the saddle point condition is not satisfied, then we provide a closed-form analytical expression for the best worst-case system's performance measure. Therefore, regarding the evader as an exogenous disturbance, we provide a solution of the optimal control problem for nonlinear dynamical systems with nonlinear-nonquadratic performance measures in the presence of undesired external inputs. The authors in [18] provide a solution of the optimal nonlinear robust control problem using a different approach. Specifically, they achieve the same results as in this paper by minimizing a *derived* performance measure that serves as an upper bound to a nonlinear-nonquadratic cost functional, and hence provide the best worst-case system performance over the class of admissible input disturbances.

In the second part of this paper, we specialize our results to differential games involving affine in the controls dynamical

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systems with quadratic in the controls performance measures. In this case, we provide an *explicit* characterization of the evader's and pursuer's controls that guarantee the existence of a saddle point and global asymptotic stability of the closed-loop system. In the study of the robust control problem for linear and nonlinear dynamical systems, a key issue is to ensure that the state-feedback control law that guarantees disturbance rejection also guarantees asymptotic stability of the undisturbed closed-loop system [14], [18], [19]. If we consider the evader's control as an exogenous disturbance, in this paper we provide sufficient conditions for the pursuer's optimal control law to guarantee asymptotic stability of the closed-loop system in the absence of disturbing inputs. We also specialize our results to linear dynamical systems with quadratic performance measure and provide clear connections with the classic $\mathcal{H}_\infty$ [19] and the $\mathcal{H}_2/\mathcal{H}_\infty$ control theories [18], [20]. A numerical example illustrates the features and the applicability of the theoretical results proven.

Due to space limitations, we omit all the proofs in this paper, which can be deduced from the results presented in [21]. In [21], the authors address the differential game problem over the infinite-time horizon considering continuous, but not continuously differentiable, Lyapunov functions and viscosity solutions of the Hamilton-Jacobi-Isaacs equation.

II. Notation, Definitions, and Mathematical Preliminaries

In this section, we establish notation, definitions, and review some basic results. Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}^n$ denote the set of $n \times 1$ real column vectors, $\mathbb{R}^{n\times m}$ denote the set of $n \times m$ real matrices, and $\mathbb{C}$ denote the set of complex numbers. We write $\|\cdot\|$ for the Euclidean vector norm, $V'(x) \triangleq \frac{\partial V(x)}{\partial x}$ for the Fréchet derivative of V at x, I_n or I for the $n \times n$ identity matrix, $0_{n\times m}$ or 0 for the zero $n \times m$ matrix, and A^{T} for the transpose of the matrix A. Given $f : X \times Y \to \mathbb{R}$, where $X \subseteq \mathbb{R}^{m_1}$ and $Y \subseteq \mathbb{R}^{m_2}$, we define

$$\arg\min_{(x,y)\in(X,Y)}\max f(x,y) \triangleq \{(x^*,y^*) \in (X,Y) : f(x^*,y) \le f(x^*,y^*) \le f(x,y^*), \forall (x,y) \in X \times Y\}$$

and

$$\min_{(x,y)\in(X,Y)}\max f(x,y) \triangleq f(x^*,y^*),$$

where $(x^*,y^*) \in \arg\min\max_{(x,y)\in(X,Y)} f(x,y)$. If $(x^*,y^*) \in \arg\min\max_{(x,y)\in(X,Y)} f(x,y)$, then we say that (x^*,y^*) is a *saddle point* for $f(\cdot,\cdot)$ on $X \times Y$.

In this paper, we consider controlled nonlinear dynamical systems of the form

$$\dot{x}(t) = F(x(t),u(t),w(t)), \qquad x(0) = x_0, \qquad t \ge 0, \quad (1)$$

where $F : \mathcal{D} \times U \times W \to \mathbb{R}^n$ is locally Lipschitz continuous in x, u, and w, $\mathcal{D}$ is an open set with $0 \in \mathcal{D} \subseteq \mathbb{R}^n$, $U \subseteq \mathbb{R}^{m_1}$ with $0 \in U$, $W \subseteq \mathbb{R}^{m_2}$ with $0 \in W$, and $F(0,0,0) = 0$. The controls $u(\cdot)$ and $w(\cdot)$ in (1) are restricted to the class of *admissible* controls consisting of continuous functions $u : [0,\infty) \to U$ and $w : [0,\infty) \to W$, and we assume that $x(t) \in \mathcal{D}$, $t \ge 0$, for all admissible controls $u(\cdot)$ and $w(\cdot)$.

Continuous functions $\phi : \mathcal{D} \to U$ and $\psi : \mathcal{D} \to W$ satisfying $\phi(0) = 0$ and $\psi(0) = 0$ are called *control laws*. If $u(t) = \phi(x(t))$, $t \ge 0$, and $w(t) = \psi(x(t))$, where $\phi(\cdot)$ and $\psi(\cdot)$ are control laws and $x(t)$ satisfies (1), then we call $u(\cdot)$ and $w(\cdot)$ *feedback control laws*. Given control laws $\phi(\cdot)$ and $\psi(\cdot)$, and feedback control laws $u(t) = \phi(x(t))$, $t \ge 0$, and $w(t) = \psi(x(t))$, the *closed-loop system* (1) is given by

$$\dot{x}(t) = F(x(t),\phi(x(t)),\psi(x(t))), \quad x(0) = x_0, \quad t \ge 0. \quad (2)$$

Next, we introduce the notion of asymptotically stabilizing feedback control laws. To this goal, consider the controlled nonlinear dynamical system (1) and define the set of regulation controllers

$$\mathcal{S}(x_0) \triangleq \{(u(\cdot),w(\cdot)) : u(\cdot) \text{ and } w(\cdot) \text{ are admissible and } x(\cdot) \text{ given by (1) satisfies } x(t) \to 0 \text{ as } t \to \infty\}.$$

In addition, given the control law $\psi(\cdot)$, let $\mathcal{S}_\psi(x_0) \triangleq \{u(\cdot) : (u(\cdot),\psi(x(\cdot))) \in \mathcal{S}(x_0)\}$ and given the control law $\phi(\cdot)$, let $\mathcal{S}_\phi(x_0) \triangleq \{w(\cdot) : (\phi(x(\cdot)),w(\cdot)) \in \mathcal{S}(x_0)\}$.

Definition 2.1: Consider the controlled dynamical system given by (1). The feedback control law $u(\cdot) = \phi(x(\cdot))$ is *asymptotically stabilizing* if the closed-loop system (2) is asymptotically stable for all admissible controls $w(\cdot) \in \mathcal{S}_\phi(x_0)$. Furthermore, the feedback control law $u(\cdot) = \phi(x(\cdot))$ is *globally asymptotically stabilizing* if the closed-loop system (2) is globally asymptotically stable for all admissible controls $w(\cdot) \in \mathcal{S}_\phi(x_0)$.

III. Lyapunov Functions and Differential Games

In this section, we use the framework developed in Lemma 2.1 of [16] to obtain a characterization of asymptotically stabilizing feedback control laws that provide a solution of differential games involving nonlinear dynamical systems of the form (1). Specifically, sufficient conditions for the existence of a saddle point are given in a form that corresponds to a steady-state version of the Hamilton-Jacobi-Isaacs equation.

Next, we present a main theorem characterizing feedback controllers that guarantee asymptotic closed-loop stability of (1), and minimize with respect to $u(\cdot)$ and maximize with respect to $w(\cdot)$ a nonlinear-nonquadratic performance functional. For the statement of this result, let $L : \mathcal{D} \times U \times W \to \mathbb{R}$ be jointly continuous in x, u, and w.

Theorem 3.1: Consider the controlled nonlinear dynamical system (1) with

$$J(x_0,u(\cdot),w(\cdot)) \triangleq \int_0^\infty L(x(t),u(t),w(t))\mathrm{d}t, \qquad (3)$$

where $u(\cdot)$ and $w(\cdot)$ are admissible controls. Assume that there exist a continuously differentiable function $V : \mathcal{D} \to \mathbb{R}$ and control laws $\phi : \mathcal{D} \to U$ and $\psi : \mathcal{D} \to W$ such that

$$V(0) = 0, \qquad (4)$$

$$V(x) > 0, \qquad x \in \mathcal{D}\backslash\{0\}, \qquad (5)$$

$$V'(x)F(x,\phi(x),\psi(x)) < 0, \qquad x \in \mathcal{D}, \qquad (6)$$

$$\phi(0) = 0, \qquad (7)$$

$$\psi(0) = 0, \qquad (8)$$

$$L(x,\phi(x),\psi(x)) + V'(x)F(x,\phi(x),\psi(x)) = 0, \qquad x \in \mathcal{D}, \qquad (9)$$

$$L(x,u,\psi(x)) + V'(x)F(x,u,\psi(x)) \ge 0, \qquad (x,u) \in \mathcal{D} \times U, \qquad (10)$$

$$L(x,\phi(x),w) + V'(x)F(x,\phi(x),w) \le 0, \qquad (x,w) \in \mathcal{D} \times W. \qquad (11)$$

Then with the feedback controls $u = \phi(x)$ and $w = \psi(x)$, the closed-loop system given by (2) is asymptotically and there exists a neighborhood $\mathcal{D}_0 \subseteq \mathcal{D}$ of $x = 0$ such that

$$J(x_0, \phi(x(\cdot)), \psi(x(\cdot))) = V(x_0), \qquad x_0 \in \mathcal{D}_0. \quad (12)$$

In addition, if $x_0 \in \mathcal{D}_0$, then

$$J(x_0, \phi(x(\cdot)), \psi(x(\cdot))) = \underset{(u(\cdot),w(\cdot))\in\mathcal{S}_\psi(x_0)\times\mathcal{S}_\phi(x_0)}{\min\max} J(x_0, u(\cdot), w(\cdot)) \quad (13)$$

and

$$J(x_0, \phi(x(\cdot)), w(\cdot)) \leq V(x_0), \qquad w(\cdot) \in \mathcal{S}_\phi(x_0). \quad (14)$$

Finally, if $\mathcal{D} = \mathbb{R}^n$, $U = \mathbb{R}^{m_1}$, $W = \mathbb{R}^{m_2}$, and

$$V(x) \to \infty, \qquad \|x\| \to \infty, \quad (15)$$

then the closed-loop system (2) is globally asymptotically stable.

Theorem 3.1 provides sufficient conditions to solve differential games involving the nonlinear controlled dynamical system (1) with performance measure (3), which termination occurs if $x \to 0$ as $t \to \infty$. Specifically, (9) is the steady-state, Hamilton-Jacobi-Isaacs equation and (9)–(11) guarantee that the saddle point condition (13) is satisfied. Given the control laws $\phi(\cdot)$ and $\psi(\cdot)$, it holds that $\mathcal{S}_\psi(x_0) \times \mathcal{S}_\phi(x_0) \subseteq \mathcal{S}(x_0)$, and restricting our minimization problem to $(u(\cdot), w(\cdot)) \in \mathcal{S}(x_0)$, that is, inputs corresponding to null convergent solutions, can be interpreted as incorporating a system detectability condition through the cost. However, it is important to note that an explicit characterization of $\mathcal{S}(x_0)$, $\mathcal{S}_\psi(x_0)$, and $\mathcal{S}_\phi(x_0)$ is not required.

The feedback control laws $u = \phi(x)$ and $w = \psi(x)$ are independent of the initial condition x_0 and, using (9)–(11), are given by

$$\begin{bmatrix}\phi(x)\\ \psi(x)\end{bmatrix} \in \underset{(u(\cdot),w(\cdot))\in\mathcal{S}_\psi(x_0)\times\mathcal{S}_\phi(x_0)}{\arg\min\max} \Big[L(x, u, w) + V'(x)F(x, u, w)\Big]. \quad (16)$$

It follows from Theorem 3.1 that the pair of control laws $(\phi(\cdot), \psi(\cdot))$ guarantees asymptotic stability of the closed-loop system. However, $\psi(\cdot)$ may be destabilizing in the sense that, given an admissible control $u(\cdot) \notin \mathcal{S}_\psi(x_0)$, the solution $x(t) = 0$, $t \geq 0$, of the nonlinear differential equation

$$\dot{x}(t) = F(x(t), u(t), \psi(x(t))), \quad x(0) = x_0, \quad t \geq 0, \quad (17)$$

is not asymptotically stable and could possibly be unstable.

If we consider the input $w(\cdot)$ in (1) as a disturbance, then the framework developed in Theorem 3.1 provides an analytical expression for the best worst-case systems performance measure over a class of non-disruptive exogenous disturbances $w(\cdot) \in \mathcal{S}_\phi(x_0)$. Specifically, it follows from (12)–(14) that

$$V(x_0) = J(x_0, \phi(x(\cdot)), \psi(x(\cdot))) \geq J(x_0, \phi(x(\cdot)), w(\cdot)), \qquad x_0 \in \mathcal{D}_0, \quad (18)$$

for all admissible inputs $w(\cdot)$ such that $\lim_{t\to\infty} x(t) = 0$, where $x(\cdot)$ is the solution of (1) with $u = \phi(x)$.

It is important to note that if the conditions of Theorem 3.1 are satisfied, then the equilibrium point $x = 0$ of the closed-loop dynamical system is asymptotically stable. Hence, for every $l > 0$, there exists $\hat{t}_\mathrm{f} \geq 0$ such that if $t > \hat{t}_\mathrm{f}$, then $\|x(t)\| < l$, where $x(\cdot)$ denotes the solution of (2). Now, consider a game involving the nonlinear dynamical system (1) and the performance measure (3), which terminal condition is given by $\|x(t_\mathrm{f})\| = l$ for some $t_\mathrm{f} \geq 0$ that is finite and not specified *a priori*. Then Theorem 3.1 provides sufficient conditions to find state-feedback control laws that solve this game. Consequently, the framework developed in Theorem 3.1 is suitable to address games of degree [1, p. 12], such as the *homicidal chauffeur game* [1, pp. 232-237], which terminate when the system trajectory intercepts a given neighborhood of the origin at a *finite time* not given *a priori*.

Remark 3.1: Setting $m_1 = m$ and $m_2 = 0$, the nonlinear controlled dynamical system given by (1) reduces to

$$\dot{x}(t) = F(x(t), u(t)), \qquad x(0) = x_0, \qquad t \geq 0. \quad (19)$$

In this case, the conditions of Theorem 3.1 reduce to the conditions of Theorem 3.1 of [16] characterizing the classical optimal control problem for time-invariant systems on the infinite time interval.

IV. Affine Dynamical Systems, Lyapunov Functions, and Differential Games

In this section, we specialize the results of Section III to nonlinear affine dynamical systems of the form

$$\dot{x}(t) = f(x(t)) + G_u(x(t))u(t) + G_w(x(t))w(t), \qquad x(0) = x_0, \qquad t \geq 0, \quad (20)$$

where, for every $t \geq 0$, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^{m_1}$, $w(t) \in \mathbb{R}^{m_2}$, and we consider performance integrands $L(x, u, w)$ of the form

$$L(x, u, w) = L_1(x) + L_{2u}(x)u + L_{2w}(x)w + u^\mathrm{T} R_{2u}(x)u + w^\mathrm{T} R_{2w}(x)w \quad (21)$$

for all $(x, u, w) \in \mathbb{R}^n \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$, where $L_1 : \mathbb{R}^n \to \mathbb{R}$, $L_{2u} : \mathbb{R}^n \to \mathbb{R}^{1\times m_1}$, $L_{2w} : \mathbb{R}^n \to \mathbb{R}^{1\times m_2}$, $R_{2u} : \mathbb{R}^n \to \mathbb{R}^{m_1\times m_1}$, and $R_{2w} : \mathbb{R}^n \to \mathbb{R}^{m_2\times m_2}$ are continuous on $\mathbb{R}^n$, so that (3) becomes

$$J(x_0, u(\cdot), w(\cdot)) = \int_0^\infty \big[L_1(x(t)) + L_{2u}(x(t))u(t) + L_{2w}(x(t))w(t) + u^\mathrm{T}(t)R_{2u}(x(t))u(t) + w^\mathrm{T}(t)R_{2w}(x(t))w(t)\big]\mathrm{d}t. \quad (22)$$

We assume that $f : \mathbb{R}^n \to \mathbb{R}^n$, $G_u : \mathbb{R}^n \to \mathbb{R}^{n\times m_1}$, and $G_w : \mathbb{R}^n \to \mathbb{R}^{n\times m_2}$ are such that $f(0) = 0$ and $f(\cdot)$, $G_u(\cdot)$, and $G_w(\cdot)$ are locally Lipschitz continuous in x. Furthermore, we assume that $R_{2u}(x) > 0$, $x \in \mathbb{R}^n\backslash\{0\}$, and $R_{2w}(x) < 0$.

Next, we specialize Theorem 3.1 to nonlinear affine dynamical systems with quadratic in the controls performance measures. Specifically, the next result provides an *explicit* characterization of *globally* asymptotically stabilizing state-feedback controls, which solve differential games involving dynamical systems of the form (20) and performance measures of the form (22).

Theorem 4.1: Consider the controlled nonlinear affine dynamical system (20) with performance measure (22), where $u(\cdot)$ and $w(\cdot)$ are admissible controls. Assume that there exists a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ such that

$$V(0) = 0, \quad (23)$$

$$V(x) > 0, \qquad x \in \mathbb{R}^n\backslash\{0\}, \quad (24)$$

$$V'(x)f(x)-\frac{1}{2}V'(x)\big[G_u(x)R_{2u}^{-1}(x)L_{2u}^{\mathrm{T}}(x)$$
$$+G_w(x)R_{2w}^{-1}(x)L_{2w}^{\mathrm{T}}(x)\big]-\frac{1}{2}V'(x)\big[G_u(x)R_{2u}^{-1}(x)G_u^{\mathrm{T}}(x)$$
$$+G_w(x)R_{2w}^{-1}(x)G_w^{\mathrm{T}}(x)\big]V'^{\mathrm{T}}(x)<0,\quad x\in\mathbb{R}^n\backslash\{0\},\quad (25)$$
$$L_{2u}(0)=0, \quad (26)$$
$$L_{2w}(0)=0, \quad (27)$$
$$0=L_1(x)+V'(x)f(x)-\frac{1}{4}\left[V'(x)G_u(x)+L_{2u}(x)\right]$$
$$\cdot R_{2u}^{-1}(x)\left[V'(x)G_u(x)+L_{2u}(x)\right]^{\mathrm{T}}$$
$$-\frac{1}{4}\left[V'(x)G_w(x)+L_{2w}(x)\right]$$
$$\cdot R_{2w}^{-1}(x)\left[V'(x)G_w(x)+L_{2w}(x)\right]^{\mathrm{T}},\qquad x\in\mathbb{R}^n,\quad (28)$$
$$V(x)\to\infty,\qquad \|x\|\to\infty. \quad (29)$$

Then, with the feedback controls

$$u=\phi(x)=-\frac{1}{2}R_{2u}^{-1}(x)\left[V'(x)G_u(x)+L_{2u}(x)\right]^{\mathrm{T}},\quad (30)$$
$$w=\psi(x)=-\frac{1}{2}R_{2w}^{-1}(x)\left[V'(x)G_w(x)+L_{2w}(x)\right]^{\mathrm{T}},\quad (31)$$

the closed-loop system

$$\dot{x}(t)=f(x(t))+G_u(x(t))\phi(x(t))+G_w(x(t))\psi(x(t)),$$
$$x(0)=x_0,\qquad t\geq 0,\quad (32)$$

is globally asymptotically stable,

$$J(x_0,\phi(x(\cdot),\psi(x(\cdot))=V(x_0),\qquad x_0\in\mathbb{R}^n,\quad (33)$$

(13) is verified with $x_0\in\mathbb{R}^n$, and

$$J(x_0,\phi(x(\cdot)),w(\cdot))\leq V(x_0),\quad w(\cdot)\in\mathcal{S}_\phi(x_0),\quad x_0\in\mathbb{R}^n. \quad (34)$$

If we regard $w(\cdot)$ in (20) as a disturbance, then Theorem 4.1 explicitly provides a *globally* asymptotically stabilizing control law $u=\phi(x)$ that guarantees disturbance rejection over the class of input disturbances $\mathcal{S}_\phi(x_0)$. In addition, this result provides an analytical closed-form expression for the best worst-case system performance for all $w(\cdot)\in\mathcal{S}_\phi(x_0)$. Therefore, the framework developed in Theorem 4.1 presents a methodology for designing the state-feedback control $u=\phi(x)$ that guarantees robustness and optimal performance over the class of input disturbances $\mathcal{S}_\phi(x_0)$.

A fundamental problem to consider in linear and nonlinear robust control is whether a state-feedback control that guarantees disturbance rejection also guarantees asymptotic stability of the closed-loop dynamical system in absence of input disturbances [14], [18], [19]. The next result provides sufficient conditions for the state-feedback control law (30) to guarantee disturbance rejection for all $w(\cdot)\in\mathcal{S}_\phi(x_0)$ and asymptotic stability of the closed-loop dynamical system for $w=0$.

Theorem 4.2: Consider the controlled nonlinear affine dynamical system (20) with performance measure (22), where $u(\cdot)$ and $w(\cdot)$ are admissible controls. If the conditions of Theorem 4.1 are satisfied with $L_1(x)>0$, $x\in\mathbb{R}^n\backslash\{0\}$, and $L_{2u}(x)=0$, then the zero solution $x(t)\equiv 0$, $t\geq 0$, of (20) with

$$u=\phi(x)=-\frac{1}{2}R_{2u}^{-1}(x)\left[V'(x)G_u(x)+L_{2u}(x)\right]^{\mathrm{T}},\quad (35)$$
$$w=0, \quad (36)$$

is globally asymptotically stable. Furthermore, it holds that

$$J(x_0,\phi(x(\cdot)),0)\leq V(x_0),\qquad x_0\in\mathbb{R}^n,\quad (37)$$

where $V(\cdot)$ satisfies (23)–(29).

Theorem 4.2 establishes a connection between the game-theoretic framework developed in this paper and the dissipativity-based framework developed in [18] to solve the nonlinear disturbance rejection problem, where the authors provide an upper bound on the performance measure (22) with $u=\phi(x)$ and $w=0$.

Next, we use Theorems 4.1 and 4.2 to address the linear-quadratic differential game problem. Specifically, for the statement of the next result consider the linear time-invariant dynamical system

$$\dot{x}(t)=Ax(t)+B_uu(t)+B_ww(t),\quad x(0)=x_0,\quad t\geq t_0, \quad (38)$$

with performance measure

$$J(x_0,u(\cdot),w(\cdot))=\int_0^\infty\big[x^{\mathrm{T}}(t)R_1x(t)+u^{\mathrm{T}}(t)R_{2u}u(t)$$
$$-\gamma^2w^{\mathrm{T}}(t)w(t)\big]\mathrm{d}t,\quad (39)$$

where $x(t)\in\mathbb{R}^n$, $t\geq 0$, $u(t)\in\mathbb{R}^{m_1}$, $w(t)\in\mathbb{R}^{m_2}$, $A\in\mathbb{R}^{n\times n}$, $B_u\in\mathbb{R}^{n\times m_1}$, $B_w\in\mathbb{R}^{n\times m_2}$, $R_1\in\mathbb{R}^{n\times n}$, $R_{2u}\in\mathbb{R}^{m_1\times m_1}$, and $\gamma>0$. In addition, we assume that $R_1>0$, $R_{2u}>0$, and $B_uR_{2u}^{-1}B_u^{\mathrm{T}}-\gamma^{-2}B_wB_w^{\mathrm{T}}\geq 0$.

Corollary 4.1: Consider the linear time-invariant dynamical system (38) with quadratic performance measure (39), where $u(\cdot)$ and $w(\cdot)$ are admissible controls. If there exists $P\in\mathbb{R}^{n\times n}$ such that $P>0$ and

$$0=A^{\mathrm{T}}P+PA+R_1-PB_uR_{2u}^{-1}B_u^{\mathrm{T}}P+\gamma^{-2}PB_wB_w^{\mathrm{T}}P, \quad (40)$$

then, with the feedback controls

$$u=\phi(x)=-R_{2u}^{-1}B_u^{\mathrm{T}}Px, \quad (41)$$
$$w=\psi(x)=\gamma^{-2}B_w^{\mathrm{T}}Px, \quad (42)$$

the dynamical system (38) is globally asymptotically stable,

$$J(x_0,\phi(x(\cdot)),\psi(x(\cdot)))=x_0^{\mathrm{T}}Px_0,\qquad x_0\in\mathbb{R}^n,\quad (43)$$

(13) is satisfied with $x_0\in\mathbb{R}^n$, and

$$J(x_0,\phi(x(\cdot)),w(\cdot))\leq x_0^{\mathrm{T}}Px_0,\ \ w(\cdot)\in\mathcal{S}_\phi(x_0),\ \ x_0\in\mathbb{R}^n. \quad (44)$$

In addition, the zero solution $x(t)\equiv 0$, $t\geq 0$, of (38) with $u=\phi(x)$ and $w=0$ is globally asymptotically stable and

$$J(x_0,\phi(x(\cdot)),0)\leq J(x_0,\phi(x(\cdot)),\psi(x(\cdot))),\quad x_0\in\mathbb{R}^n. \quad (45)$$

Corollary 4.1 gives sufficient conditions for global asymptotic stability of the linear dynamical system (38) with state feedback control laws (41) and (42). Since the closed-loop linear dynamical system

$$\dot{x}(t)=\left(A-B_uR_{2u}^{-1}B_u^{\mathrm{T}}P+\gamma^{-2}B_wB_w^{\mathrm{T}}P\right)x(t),$$
$$x(0)=x_0,\qquad t\geq 0,\quad (46)$$

is globally asymptotically stable, (46) is globally exponentially stable [22].

Remarkably, if the conditions of Corollary 4.1 are satisfied, then it follows from Theorem 6.3.1 of [19] that

$$|||G(s)|||_\infty\leq\gamma,\qquad s\in\mathbb{C}, \quad (47)$$

where $|||G(s)|||_\infty \triangleq \sup_{\omega\in\mathbb{R}} \sigma_{\max}[G(\jmath\omega)]$ denotes the $\mathcal{H}_\infty$ norm, $\sigma_{\max}[\cdot]$ denotes the maximum singular value,

$$G(s) \triangleq (C - D_u R_{2u}^{-1} B_u^{\mathrm{T}} P)^{\mathrm{T}} (sI_n - A + B_u R_{2u}^{-1} B_u^{\mathrm{T}} P) B_w \tag{48}$$

denotes the closed-loop transfer function of (38) with output

$$z(t) = Cx(t) + D_u u(t),$$

$R_1 = C^{\mathrm{T}}C$, and $R_{2u} = D_u^{\mathrm{T}} D_u$. Furthermore, if the conditions of Corollary 4.1 are satisfied, then it follows from the Bounded Real Lemma [17, Th. 5.15] that $G(s)$, $s \in \mathbb{C}$, is bounded real [17, Def. 5.19] and nonexpansive [17, Def. 5.12].

Lastly, it is important to notice that Corollary 4.1 establishes connections with the mixed-norm $\mathcal{H}_2/\mathcal{H}_\infty$ frameworks developed in Theorem 3.1 of [20] and Corollary 4.1 of [18]. In particular, under the same assumptions as in Corollary 4.1, the authors in [18] prove global exponential stability and nonexpansivity of the linear time-invariant dynamical system (38) with $u = \phi(x)$ and $w = 0$. Moreover, the authors in [18] prove that both (43) and (44) are satisfied.

V. Illustrative Numerical Example

In this section, we provide a numerical example to highlight the approach to the differential game problem developed in the paper. Consider the axisymmetric spacecraft given by [23, p.753]

$$\dot\omega_1(t) = I_{23}\omega_3\omega_2(t) + u(t), \quad \omega_1(0) = \omega_{10}, \quad t \ge 0, \tag{49}$$

$$\dot\omega_2(t) = -I_{23}\omega_3\omega_1(t) + w(t), \quad \omega_2(0) = \omega_{20}, \tag{50}$$

where $I_{23} \triangleq (I_2 - I_3)/I_1$, I_1, I_2, and I_3 are the principal moments of inertia of the spacecraft such that $0 < I_1 = I_2 < I_3$, $\omega_1 : [0,\infty) \to \mathbb{R}$, $\omega_2 : [0,\infty) \to \mathbb{R}$, and $\omega_3 \in \mathbb{R}$ denote the components of the angular velocity vector with respect to a given inertial reference frame expressed in a central body reference frame, and $u : [0,\infty) \to \mathbb{R}$ and $w : [0,\infty) \to \mathbb{R}$ are the spacecraft control moments. For this example, we seek state feedback controllers $u = \phi(x)$ and $w = \psi(x)$, where $x = [\omega_1, \omega_2]^{\mathrm{T}}$, such that the performance measure

$$J(x_0, u(\cdot), w(\cdot)) = \int_0^\infty \left[I_{23}^2\left(\omega_1^2(t) - \omega_2^2(t)\right) + 4I_{23}\omega_2(t)w(t) + u^2(t) - w^2(t)\right]\mathrm{d}t, \tag{51}$$

where $x_0 = [\omega_{10}, \omega_{20}]^{\mathrm{T}}$, satisfies (13) and the affine dynamical system given by (49) and (50) is globally asymptotically stable. Minimizing with respect to u and maximizing with respect to w the term $\int_0^\infty \left[u^2(t) - w^2(t)\right]\mathrm{d}t$ in (51) implies minimizing the difference in control effort along two inertia axes. Furthermore, the term $\int_0^\infty \left[\omega_1^2(t) - \omega_2^2(t)\right]\mathrm{d}t$ in (51) captures the difference in kinetic energy due to the angular velocities $\omega_1(\cdot)$ and $\omega_2(\cdot)$.

Note that (49) and (50) with performance measure (51) can be cast in the form of (20) with performance measure (22). In this case, Theorem 4.1 can be applied with $n = 2$, $m_1 = 1$, $m_2 = 1$, $f(x) = \left[I_{23}\omega_3\omega_2, -I_{23}\omega_3\omega_1\right]^{\mathrm{T}}$, $G_u(x) = [1, 0]^{\mathrm{T}}$, $G_w(x) = [0, 1]^{\mathrm{T}}$, $L_1(x) = I_{23}^2\left(\omega_1^2 - \omega_2^2\right)$, $L_{2u}(x) = 0$, $L_{2w}(x) = 4I_{23}\omega_2$, $R_{2u}(x) = 1$, and $R_{2w}(x) = -1$ to characterize the asymptotically stabilizing controllers. Specifically, in this case (28) reduces to

$$0 = L_1(x) + V'(x)f(x) - \frac{1}{4}\left[V'(x)G_u(x)\right] \cdot R_{2u}^{-1}(x)\left[V'(x)G_u(x)\right]^{\mathrm{T}} - \frac{1}{4}\left[V'(x)G_w(x) + L_{2w}(x)\right] \cdot R_{2w}^{-1}(x)\left[V'(x)G_w(x) + L_{2w}(x)\right]^{\mathrm{T}}, \quad x \in \mathbb{R}^n. \tag{52}$$

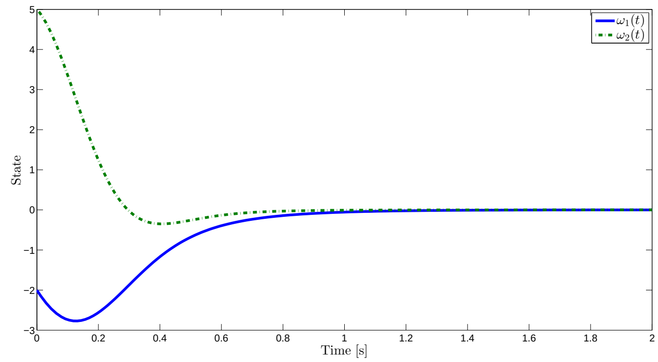

Fig. 1. Closed-loop system trajectories versus time in the presence of disturbances.

Now, choosing

$$V(x) = -I_{23}x^{\mathrm{T}}x, \qquad x \in \mathbb{R}^n, \tag{53}$$

(52) is verified, and (23)–(27) and (29) are satisfied.

Since all of the conditions of Theorem 4.1 hold, it follows that the feedback control laws (30) and (31) given by

$$\phi(x) = -\frac{1}{2}R_{2u}^{-1}(x)\left[G_u^{\mathrm{T}}(x)V'^{\mathrm{T}}(x) + L_{2u}^{\mathrm{T}}(x)\right] = I_{23}\omega_1, \tag{54}$$

$$\psi(x) = -\frac{1}{2}R_{2w}^{-1}(x)\left[G_w^{\mathrm{T}}(x)V'^{\mathrm{T}}(x) + L_{2w}^{\mathrm{T}}(x)\right] = I_{23}\omega_2, \tag{55}$$

guarantee that the dynamical system (49) and (50) is globally asymptotically stable and

$$J(x_0, \phi(x(\cdot)), \psi(x(\cdot))) = -I_{23}x_0^{\mathrm{T}}x_0, \qquad x_0 \in \mathbb{R}^n. \tag{56}$$

Now, suppose the thruster delivering the control moment w is defective and

$$w = \left[\varepsilon - \frac{1}{I_{23}}\delta(t)\right]\psi(x), \qquad (t,x) \in [0,\infty) \times \mathbb{R}^n, \tag{57}$$

where $\varepsilon > 0$ is arbitrarily small and $\delta : [0,\infty) \to [0,\infty)$ is continuous on the set of positive real numbers. Then the closed-loop dynamical system is given by

$$\dot x(t) = f(x) + G_u(x(t))\phi(x(t)) + G_w(x(t)) \cdot \left[\varepsilon - \frac{1}{I_{23}}\delta(t)\right]\psi(x(t)), \quad x(0) = x_0, \quad t \ge 0, \tag{58}$$

and the radially unbounded decrescent Lyapunov function (53) is such that

$$\dot V(t,x) = -2I_{23}^2\omega_1^2 - 2\varepsilon I_{23}^2\omega_2^2 + 2I_{23}\omega_2^2\delta(t) \le -2I_{23}^2\omega_1^2 - 2\varepsilon I_{23}^2\omega_2^2, \quad (t,x) \in [0,\infty) \times \mathbb{R}^n. \tag{59}$$

Since $\omega_1^2 + \varepsilon\omega_2^2$ is a class $\mathcal{K}_\infty$ function on $\mathbb{R}^n$, it follows from Theorem 4.6 of [17] that the closed-loop nonlinear dynamical system (58) is globally uniformly asymptotically stable, which implies that $\lim_{t\to\infty} x(t) = 0$. Hence, the input

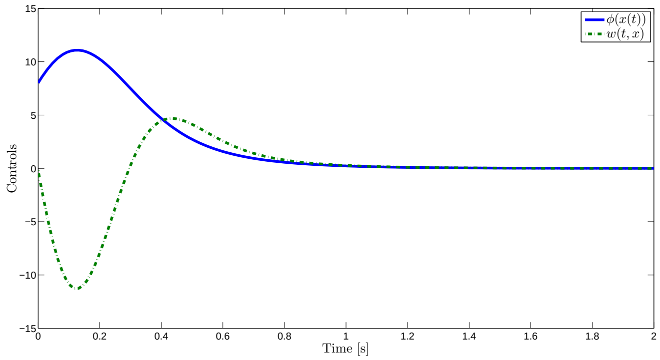

Fig. 2. Control signal and disturbance input versus time.

function (57) is such that $w(\cdot) \in \mathcal{S}_\phi(x_0)$ and it follows from Theorem 4.1 that

$$J(x_0, \phi(x(\cdot)), w(\cdot)) \leq -I_{23} x_0^{\mathrm{T}} x_0, \qquad x_0 \in \mathbb{R}^n. \quad (60)$$

Let $I_1 = I_2 = 4\,\mathrm{kg}\cdot\mathrm{m}^2$, $I_3 = 20\,\mathrm{kg}\cdot\mathrm{m}^2$, $\omega_{10} = -2\,\mathrm{Hz}$, $\omega_{20} = 5\,\mathrm{Hz}$, and $\omega_3 = 1\,\mathrm{Hz}$. Next, let $\varepsilon = 10^{-2}$ and $\delta(\cdot)$ be a triangle wave, that is,

$$\delta(t) = 200\left[(-1)^{\lfloor t+\frac{1}{2}\rfloor}\left(t - \left\lfloor t + \frac{1}{2}\right\rfloor\right) + 1\right], \ \ t \geq 0, \quad (61)$$

where $\lfloor\cdot\rfloor$ denotes the floor function. Figure 1 shows the state trajectories of (49) and (50) with $u = \phi(x)$ and $w = \left[\varepsilon - \frac{1}{I_{23}}\delta(t)\right]\psi(x)$ versus time, and Figure 2 shows the corresponding control signal versus time. Note that $x(t) \to 0$ as $t \to \infty$. Finally, $J(x_0, \phi(x(\cdot)), w(\cdot)) \leq J(x_0, \phi(x(\cdot)), \psi(x(\cdot))) = 116\,\mathrm{Hz}^2$, for all $w(\cdot) \in \mathcal{S}_\phi(x_0)$.

VI. Conclusion

In this paper, we provided a systematic framework to solve the two-players zero-sum differential game problem over the infinite time horizon. Specifically, we characterized the pursuer's and the evader's control laws needed to guarantee asymptotic stability of the closed-loop system and verify the saddle point condition for the system performance measure. In addition, we provided an analytic expression for the performance measure evaluated at the saddle point.

A key contribution of this work is that we apply our game-theoretic approach to solve optimal control problems involving nonlinear dynamical systems with nonlinear-nonquadratic performance measures in the presence of exogenous disturbances. Specifically, given a nonlinear dynamical system with nonlinear-nonquadratic performance measure, we provided an explicit expression for the system's minimum performance cost over a set of input disturbances. Furthermore, in the case of affine dynamical systems with quadratic in the controls performance measures, we gave an explicit closed-form expression of the optimal state feedback control law that guarantees disturbance rejection, minimization of the performance measure, and asymptotic stability of the closed-loop system. These results have been specialized to linear dynamical systems with quadratic performance measures resuming classic results from $\mathcal{H}_\infty$ and $\mathcal{H}_2/\mathcal{H}_\infty$ control theories.

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