



Available online at www.sciencedirect.com

ScienceDirect

Journal of the Franklin Institute 358 (2021) 6281–6308

www.elsevier.com/locate/jfranklin



Novel model reference adaptive control laws for improved transient dynamics and guaranteed saturation constraints

Robert B. Anderson, Julius A. Marshall, Andrea L'Afflitto*

Department of Industrial and Systems Engineering, Virginia Tech, Blacksburg, VA 24061-6100, USA

Received 24 August 2020; received in revised form 12 April 2021; accepted 14 June 2021

Available online 19 June 2021

Abstract

In classical model reference adaptive control (MRAC), the adaptive rates must be tuned to meet multiple competing objectives. Large adaptive rates guarantee rapid convergence of the trajectory tracking error to zero. However, large adaptive rates may also induce saturation of the actuators and excessive overshoots of the closed-loop system's trajectory tracking error. Conversely, low adaptive rates may produce unsatisfactory trajectory tracking performances. To overcome these limitations, in the classical MRAC framework, the adaptive rates must be tuned through an iterative process. Alternative approaches require to modify the plant's reference model or the reference command input. This paper presents the first MRAC laws for nonlinear dynamical systems affected by matched and parametric uncertainties that constrain both the closed-loop system's trajectory tracking error and the control input at all times within user-defined bounds, and enforce a user-defined rate of convergence on the trajectory tracking error. By applying the proposed MRAC laws, the adaptive rates can be set arbitrarily large and both the plant's reference model and the reference command input can be chosen arbitrarily. The user-defined rate of convergence of the closed-loop plant's trajectory is enforced by introducing a user-defined auxiliary reference model, which converges to the trajectory tracking error obtained by applying the classical MRAC laws before its transient dynamics has decayed, and steering the trajectory tracking error to the auxiliary reference model at a rate of convergence that is higher than the rate of convergence of the plant's reference model. The ability of the proposed MRAC laws to prescribe the performance of the closed-loop system's trajectory tracking error and control input is guaranteed by barrier Lyapunov

* Corresponding author.

E-mail addresses: andersonb@vt.edu (R.B. Anderson), mjulius@vt.edu (J.A. Marshall), a.lafflitto@vt.edu (A. L'Afflitto).

functions. Numerical simulations illustrate both the applicability of our theoretical results and their effectiveness compared to other techniques such as prescribed performance control, which allows to constrain both the rate of convergence and the maximum overshoot on the trajectory tracking error of uncertain systems.

© 2021 The Franklin Institute. Published by Elsevier Ltd. All rights reserved.

1. Introduction

In the presence of matched and parametric uncertainties, classical model reference adaptive control (MRAC) guarantees both uniform asymptotic convergence of the closed-loop system's trajectory tracking error and boundedness of both the trajectory tracking error and the adaptive gains [1, Ch. 9]. Several MRAC laws have been proposed to guarantee uniform ultimate boundedness of the closed-loop system's trajectory tracking error in the presence of unmatched uncertainties [1, Ch. 10],[2–4]. Because of its relative ease of implementation, MRAC has been applied to solve control problems in multiple fields, including aerospace [1], biomedical [5], mechanical [6], and electrical engineering [7], to name a few.

The structure of a classical MRAC law is fixed and, given a reference model, the user can tune the adaptive rates to meet some design specifications on the closed-loop system's performance. These specifications include, for example, guaranteeing high responsiveness of the feedback control law to large trajectory tracking errors, bounding the maximum overshoot of the trajectory tracking error, and constraining the \mathcal{L}_∞ -norm of the control input. Large adaptive rates guarantee high responsiveness of the feedback control law to large trajectory tracking errors. However, in the transient phase, large adaptive rates may induce large and rapid excursions of both the control inputs and the trajectory tracking error [1, p. 389]. Both the trajectory tracking error's maximum overshoot and the \mathcal{L}_∞ -norm of the control input can only be estimated in a conservative manner [8,9], and adaptive rates can only be tuned *a posteriori* through an iterative process to prevent both the closed-loop plant's trajectory from exceeding its operative range and the control input from saturating the plant's actuators. As an alternative to tuning the adaptive rates, the user could tune the zeroth-order term in the underlying Lyapunov equation. However, it is not common practice to meet user-defined levels of performance of the closed-loop system by tuning this parameter, since direct correlations between the zeroth-order term in the underlying Lyapunov equation, the trajectory tracking error, and the control input have not been found yet [10]. The structure of classical MRAC does not allow to tune the rate of convergence of the closed-loop system's trajectory tracking error and impose that the closed-loop plant's trajectory reaches the reference model's trajectory before the reference model's transient dynamics has decayed [1, pp. 389–390].

The main results of this paper are *MRAC laws for prescribed performance*. These MRAC laws guarantee asymptotic convergence to zero of the trajectory tracking error and uniform boundedness of both the adaptive gains and the trajectory tracking error. These MRAC laws also allow to impose *a priori* a user-defined rate of convergence on the closed-loop system's trajectory tracking error, and enforce user-defined constraints on both the trajectory tracking error and the control input at all times. Remarkably, the proposed MRAC laws allow the user to set the adaptive rates arbitrarily large and choose the reference command input arbitrarily, and do not require to modify the reference model. To the authors' best knowledge, this is

the first result of this kind within the MRAC framework. The absence of an MRAC law that combines all these properties has been highlighted in several recent works such as [8,11,12].

The adaptive control framework that most closely matches the features of the proposed MRAC laws is *prescribed performance control* [13,14]. This technique applies to uncertain affine in the control dynamical systems and allows to impose both a user-defined rate of convergence and a user-defined bound on the maximum overshoot of the closed-loop system's trajectory tracking error by transforming the constrained system into an unconstrained one. Although prescribed performance control applies to a larger class of dynamical systems than the proposed MRAC laws, it is unable to impose explicitly user-defined constraints on the control input, which is a distinctive feature of the proposed MRAC laws. Furthermore, the proposed MRAC laws for prescribed performance allow to impose not only the maximum overshoot of the closed-loop system's trajectory tracking error, but a larger class of constraints on the trajectory tracking error.

The proposed MRAC laws for prescribed performance are direct adaptive laws, since the adaptive gains are not enforced to converge to the corresponding unknown, true values [1, Ch. 7]. Control techniques that require convergence of the adaptive gains to their respective true values, are useful in problems in which the user needs to characterize online the plant's dynamics or identify faults and failures. Examples of MRAC techniques that guarantee convergence of the adaptive gains to their respective true values include introducing an identification error in the adaptive laws as in *combined direct-indirect MRAC* [15,16], or employing some parameter identification technique such as adaptive observers [17] or least-squares estimation methods [18, Ch. 6], [19, Ch. 4]. Worthy of mention is also *concurrent learning adaptive control* [20–22], whereby appending recorded data of output and state estimate vectors to the baseline adaptive law, exponential convergence of the adaptive gains to their ideal values is guaranteed without requiring persistence of excitation of the input functions, and the closed-loop system's trajectory tracking error converges faster than by applying classical MRAC [23]. *Hybrid MRAC for unmatched uncertainties* [24] provides an improved extension to concurrent learning adaptive control as it applies also to plants with unmatched uncertainties that lie outside the space spanned by the control input. As discussed in [25,26], potential drawbacks of the concurrent learning adaptive control framework include the need to implement both a singular value maximization technique to ensure sufficient richness of the recorded data and a fixed-point smoothing technique to estimate the state derivatives, which in turn are required for estimating the uncertainties. *Composite learning adaptive control* [25] guarantees exponential parameter convergence to the ideal values and practical exponential stability of the trajectory tracking error without the need of a singular value maximization technique or a fixed-point smoothing technique to compute the time derivatives of the state vector. Although the proposed MRAC laws are not designed for online parameter estimation, they guarantee user-defined levels of performance at all times despite modeling uncertainties, do not require to augment the regressor vector, and do not require to introduce identification error dynamics.

The proposed MRAC laws for prescribed performance are attained in three consecutive steps. Firstly, we modify the classical MRAC law by augmenting the vector of feedback variables with a user-defined vector function. Similarly to the classical MRAC law, the proposed adaptive control law guarantees uniform boundedness of the closed-loop system's trajectory tracking error, uniform boundedness of the adaptive gains, and uniform asymptotic convergence of the closed-loop system's trajectory tracking error. However, a difference with the classical MRAC law is that the user is now able to arbitrarily choose part of the feedback con-

trol law and meet additional design specifications. For instance, the authors of [12] recently applied this approach to create an anti-windup system for the MRAC framework.

As a second step toward the proposed MRAC laws for prescribed performance, we introduce a new MRAC framework that we named *two-layer MRAC*. Applying the two-layer MRAC framework, the user introduces an auxiliary reference model in addition to the plant's reference model that characterizes the classical MRAC framework. This auxiliary reference model is designed to converge to the closed-loop system's trajectory tracking error obtained by applying the classical MRAC framework before the transient dynamics of the plant's reference model has decayed, and the two-layer MRAC law steers the closed-loop system's trajectory tracking error to the auxiliary reference model at a rate of convergence that is higher than the rate of convergence of the plant's reference model. Thus, two-layer MRAC guarantees convergence of the closed-loop plant's trajectory to the plant's reference model before its transient dynamics has decayed.

Within the MRAC framework, several approaches have been proposed to set explicitly the rate of convergence of the trajectory tracking error. These approaches modify the reference model's dynamics by adding an error feedback term [27–29], use barrier Lyapunov functions [8,9,30], or involve estimators of the plant's nonlinearities [10,31–33]. Adding an error feedback term in the reference model's dynamics alters the original reference model and deprives the users of their prerogative of designing arbitrarily the reference model and the reference trajectory. Using barrier Lyapunov functions to bound the trajectory tracking error, the norm of the control input grows asymptotically as the trajectory tracking error approaches the boundary of the constraint set. Finally, designing estimators of the plant nonlinearities may result in a complex task for problems of practical interest. Two-layer MRAC allows to impose a user-defined rate of convergence to the closed-loop plant's trajectory without any constraints on the adaptive rates, without modifying the user-defined reference model's dynamics, without employing barrier Lyapunov functions, and without using estimators of the plant's nonlinearities. Notably, also concurrent learning adaptive control [20–22], hybrid MRAC for unmatched uncertainties [24], and model reference composite learning control [26] allow to set the exponential rate of convergence of the trajectory tracking error dynamics by tuning the adaptive rates and the zero-th order term in the underlying Lyapunov equation. The proposed two-layer MRAC allows to explicitly set the rate of convergence for arbitrary large adaptive rates and without needing to tune the zero-th order term in the Lyapunov equation.

As a third step toward the proposed MRAC laws for prescribed performance, we present an original MRAC law that allows to constrain both the trajectory tracking error and the control input at all time. This result is achieved by employing barrier Lyapunov functions to enforce constraints on both the closed-loop system's trajectory tracking error and the control input and hence, guarantee that the constraints on the trajectory tracking error do not necessarily imply larger control efforts. Also this MRAC law comprises a user-defined vector function, which is eventually used to implement our original two-layer MRAC framework, enforce a user-defined rate of convergence on the closed-loop plant's trajectory, and achieve the proposed MRAC laws for prescribed performance.

Existing results on the synthesis of MRAC laws that account explicitly for saturation constraints involve linear dynamical systems [34–38] or feedback-linearizable plants [39,40]. In some cases, such as [38], also the reference model is partly modified to prevent saturation of the control input. Remarkably, the proposed MRAC laws for prescribed performance allow to set arbitrarily large adaptive gains, do not require to modify the reference model, and apply to nonlinear plants that are not necessarily feedback-linearizable. More frequently, constraints

on the control input are enforced implicitly by employing the projection operator [41,42] or barrier Lyapunov functions to bound the adaptive gains [9]. However, if the adaptive gains are bounded within user-defined constraint sets, then uniform ultimate boundedness of the closed-loop system’s trajectory tracking error is guaranteed, but not its asymptotic convergence [8].

Numerical examples show the applicability of our original results. Special emphasis is given to comparing the performance of the proposed MRAC laws for prescribed performance to the performance of classical prescribed performance control [13,14]. Our simulations highlight how both the proposed MRAC laws and classical prescribed performance control are able to impose a user-defined rate of convergence on the closed-loop plant’s trajectory and a user-defined maximum overshoot on the closed-loop system’s trajectory tracking error. However, despite the proposed MRAC laws, classical prescribed performance control does not verify the user-defined saturation constraints.

2. Notation, definitions, and mathematical preliminaries

In this section, we establish notation, definitions, and review some basic results. Throughout this paper, we use two types of mathematical statements, namely existential and universal statements. *Existential statements* are in the form: the condition C is verified for some $x \in \mathcal{X}$. *Universal statements* are in the form: the condition C holds for all $x \in \mathcal{X}$; for universal statements, we generally omit the words “for all” and write: C holds, $x \in \mathcal{X}$. If the universal statements C_1, \dots, C_n hold over the same set \mathcal{X} , then we write: C_1 holds, $x \in \mathcal{X}$, C_2 holds, \dots , and C_n holds.

Let \mathbb{N} denote the set of positive integers, \mathbb{Z} denote the set of integers, \mathbb{R} denote the set of real numbers, \mathbb{C} the set of complex numbers, \mathbb{R}^n the set of $n \times 1$ real column vectors, $\mathbb{R}^{n \times m}$ the set of $n \times m$ real matrices, and $\mathcal{B}_\varepsilon(x)$ the open ball centered at $x \in \mathbb{R}^n$ with radius ε . The interior of the set $\mathcal{E} \subset \mathbb{R}^n$ is denoted by $\overset{\circ}{\mathcal{E}}$ and the boundary of \mathcal{E} is denoted by $\partial\mathcal{E}$. The real part of $z \in \mathbb{C}$ is denoted by $\text{Re}(z)$. The zero vector in \mathbb{R}^n is denoted by 0_n or 0 , the zero matrix in $\mathbb{R}^{n \times m}$ is denoted by $0_{n \times m}$ or 0 , and the identity matrix in $\mathbb{R}^{n \times n}$ is denoted by I_n or I . The block-diagonal matrix formed by $M_i \in \mathbb{R}^{n_i \times n_i}$, $i = 1, \dots, p$, is denoted by $M = \text{blockdiag}(M_1, \dots, M_p)$. The transpose of $B \in \mathbb{R}^{n \times m}$ is denoted by B^T , the trace of $A \in \mathbb{R}^{n \times n}$ is denoted by $\text{tr}(A)$, the eigenvalues of A with maximum and minimum real parts are denoted by $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$, respectively. We write $\|\cdot\|$ for the Euclidean vector norm and the corresponding equi-induced matrix norm, and $\|\cdot\|_F$ for the Frobenius matrix norm [43, pp. 18–19].

Next, we recall the notion of rate of convergence [44] of asymptotically stable, linear, time-invariant dynamical systems on \mathbb{R}^n , whose equilibrium point is at the origin [43, Def. 2.47].

Definition 2.1. Let $x : [t_0, \infty) \rightarrow \mathbb{R}^n$ denote the flow of a linear, time-invariant, uncontrolled, asymptotically stable dynamical system on \mathbb{R}^n , whose equilibrium point is at the origin. Then, the rate of convergence of $x(\cdot)$ is defined as

$$\alpha_{\max}(x(\cdot)) \triangleq \sup_{x_0 \in \mathbb{R}^n \setminus \{0\}} \liminf_{t \rightarrow \infty} \frac{\log \|x(t)\|}{t_0 - t}. \tag{1}$$

It follows from Definition 2.1 that the rate of convergence of a linear dynamical system is the supremum over all $\alpha > 0$ such that $\lim_{t \rightarrow \infty} e^{\alpha t} \|x(t)\| = 0$ [44, p. 55]. Consider the linear, time-invariant, controlled dynamical system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(t_0) = x_0, \quad t \geq t_0, \tag{2}$$

where $x(t) \in \mathbb{R}^n$, $t \geq t_0$, $u(t) \in \mathbb{R}^m$ is bounded, $A \in \mathbb{R}^{n \times n}$ is Hurwitz, and $B \in \mathbb{R}^{n \times m}$. The next result, whose proof is given in the Appendix, provides both the rate of convergence of Eq. (2) and an upper bound on the solution of Eq. (2).

Proposition 2.1. *Consider the linear, time-invariant, dynamical system (2). It holds that $\alpha_{\max}(x(\cdot)) = -\text{Re}(\lambda_{\max}(A))$ and there exist $\xi : [t_0, \infty) \rightarrow \mathbb{R}$, $\gamma_1 \geq 1$, and $\gamma_2 \in (0, 1)$ such that*

$$\|x(t)\| \leq \gamma_1 \|x_0\| e^{\xi(t-t_0)} - \frac{\gamma_1 \|B\|}{\gamma_2 \text{Re}(\lambda_{\max}(A))} \sup_{t \in [t_0, \infty)} \|u(t)\|, \quad t \geq t_0, \tag{3}$$

and $\lim_{t \rightarrow \infty} (\xi(t) - \text{Re}(\lambda_{\max}(A))) = 0^+$.

3. Motivations for a novel MRAC framework

In order to motivate this work, it is worthwhile to recall the classical formulation of the MRAC architecture and highlight some of its critical aspects. Specifically, consider the nonlinear *plant’s dynamics*

$$\dot{x}(t) = Ax(t) + B\Lambda[u(t) + \Theta^T \Phi(t, x(t))], \quad x(t_0) = x_0, \quad t \geq t_0, \tag{4}$$

where $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $t \geq t_0 \geq 0$, denotes the *plant’s trajectory*, $0 \in \mathcal{D}$, $u(t) \in \mathbb{R}^m$ denotes the *control input*, $A \in \mathbb{R}^{n \times n}$ is unknown, $B \in \mathbb{R}^{n \times m}$, $\Lambda \in \mathbb{R}^{m \times m}$ is diagonal, positive-definite, and unknown, $\Theta \in \mathbb{R}^{N \times m}$ is unknown, the user-defined *regressor vector* $\Phi : [t_0, \infty) \times \mathcal{D} \rightarrow \mathbb{R}^N$ is jointly continuous in its arguments, $\Phi(\cdot, x)$ is bounded in t over $[t_0, \infty)$, and $\Phi(t, \cdot)$ is locally Lipschitz continuous in x uniformly in t on compact subsets of $[t_0, \infty)$. Both A and Λ capture *parametric uncertainties*, and $\Theta^T \Phi(t, x)$, $(t, x) \in [t_0, \infty) \times \mathcal{D}$, captures *matched uncertainties*; the choice of the regressor vector $\Phi(\cdot, \cdot)$ to parameterize uncertainties is usually based on some prior knowledge of the plant dynamics [1, Ch. 9]. The diagonal matrix Λ captures faults and failures in the control system and uncertainties in the control gains [1, p. 281]. The range space of B , that is, the space spanned by $\{b_1, \dots, b_m\}$, where $b_i \in \mathbb{R}^n$ denotes the i th column of B , is assumed to be known. Since $B\Lambda = [\lambda_1 b_1, \dots, \lambda_m b_m]$, where $\lambda_i > 0$, $i = 1, \dots, m$, denotes the i th element on the diagonal of Λ , $\|b_i\|$ can be considered as unknown and λ_i can be considered as an estimate of $\|b_i\|$. In problems of practical interest, the assumption that the span of $\{b_1, \dots, b_m\}$ is known can be verified since the directions in the state-space directly affected by the control input are usually known. We assume that the pair $(A, B\Lambda)$ is controllable; in problems of practical interest, this hypothesis can be verified since the structure of A is usually known [1, p. 281].

Consider also the *plant’s reference model*

$$\dot{x}_{\text{ref}}(t) = A_{\text{ref}} x_{\text{ref}}(t) + B_{\text{ref}} r(t), \quad x_{\text{ref}}(t_0) = x_{\text{ref},0}, \quad t \geq t_0, \tag{5}$$

where $x_{\text{ref}}(t) \in \mathbb{R}^n$, $t \geq t_0$, denotes the *reference trajectory*, the *reference command input* $r(t) \in \mathbb{R}^m$ is continuous and bounded, $A_{\text{ref}} \in \mathbb{R}^{n \times n}$ is Hurwitz, and $B_{\text{ref}} \in \mathbb{R}^{n \times m}$, and assume there exist $(K_x, K_r) \in \mathbb{R}^{n \times m} \times \mathbb{R}^{m \times m}$ such that the *matching conditions*

$$A_{\text{ref}} = A + B\Lambda K_x^T, \tag{6}$$

$$B_{\text{ref}} = B\Lambda K_r^T, \tag{7}$$

are verified. Since A_{ref} , B , and B_{ref} are known and the structures of A and Λ are known, in numerous problems of practical interest it is possible to prove the existence of K_x and K_r

that verify the matching conditions (6) and (7), respectively [1, p. 282]. To appreciate the generality of the classical MRAC framework, it is worthwhile to recall that for all multi-body mechanical systems affected by modeling uncertainties, there exist baseline controllers that exploit the plant’s passivity properties, reduce the plant’s equations of motion to the same form as Eq. (4), and verify the matching conditions (6) and (7) [45, Ch. 8], [46].

It follows from Eqs. (4)–(7) that

$$\dot{e}(t) = A_{\text{ref}}e(t) + B\Lambda[u(t) - K_x^T x(t) - K_r^T r(t) + \Theta^T \Phi(t, x(t))], \quad e(t_0) = x_0 - x_{\text{ref},0}, \quad t \geq t_0, \tag{8}$$

where $e(t) \triangleq x(t) - x_{\text{ref}}(t)$ denotes the trajectory tracking error. Therefore, the matching conditions imply that if A , Λ , and Θ were known and

$$u(t) = K_x^T x(t) + K_r^T r(t) - \Theta^T \Phi(t, x(t)), \quad t \geq t_0, \tag{9}$$

then $\lim_{t \rightarrow \infty} e(t) = 0$ [1, pp. 281–282].

Consider the symmetric, positive-definite solution $P \in \mathbb{R}^{n \times n}$ of the Lyapunov equation

$$0 = A_{\text{ref}}^T P + P A_{\text{ref}} + Q, \tag{10}$$

where the zeroth-order term $Q \in \mathbb{R}^{n \times n}$ is a user-defined symmetric, positive-definite matrix, and consider the feedback control law

$$\phi(\pi(t, x, r), \hat{K}) = \hat{K}^T \pi(t, x, r), \quad (t, x, r, \hat{K}) \in [t_0, \infty) \times \mathcal{D} \times \mathbb{R}^m \times \mathbb{R}^{(n+m+N) \times m}, \tag{11}$$

where the adaptive gain matrix $\hat{K}(\cdot)$ verifies the adaptive law

$$\dot{\hat{K}}(t) = -\Gamma \pi(t, x(t), r(t))e^T(t)PB, \quad \hat{K}(t_0) = \hat{K}_0, \quad t \geq t_0, \tag{12}$$

the user-defined adaptive rate matrices $\Gamma_x \in \mathbb{R}^{n \times n}$, $\Gamma_r \in \mathbb{R}^{m \times m}$, and $\Gamma_\Theta \in \mathbb{R}^{N \times N}$ are symmetric and positive-definite, $\Gamma \triangleq \text{blockdiag}(\Gamma_x, \Gamma_r, \Gamma_\Theta)$, and

$$\pi(t, x, r) \triangleq [x^T, r^T, -\Phi^T(t, x)]^T, \quad (t, x, r) \in [t_0, \infty) \times \mathcal{D} \times \mathbb{R}^m, \tag{13}$$

denotes the vector of feedback variables. If $u(t) = \phi(\pi(t, x(t), r(t)), \hat{K}(t))$, $t \geq t_0$, then it follows from Eqs. (4)–(7) that the closed-loop trajectory error dynamics is given by

$$\dot{e}(t) = A_{\text{ref}}e(t) + B\Lambda\Delta K^T(t)\pi(t), \quad e(t_0) = x_0 - x_{\text{ref},0}, \quad t \geq t_0, \tag{14}$$

where $\Delta K(t) \triangleq \hat{K}(t) - K$ and $K \triangleq [K_x^T, K_r^T, \Theta^T]^T$. The next result illustrates the classical MRAC architecture.

Theorem 3.1 ([1, Th. 9.2]). *Consider the plant’s dynamics (4), the plant’s reference model (5), the feedback control law (11), and the adaptive law (12). If the matching conditions (6) and (7) are verified, then both the trajectory tracking error $e(\cdot)$ and the adaptive gain matrix $\hat{K}(\cdot)$ are uniformly bounded, and $e(t) \rightarrow 0$ as $t \rightarrow \infty$ uniformly in $t_0 \in [0, \infty)$.*

Theorem 3.1 proves that applying the feedback control law (11) and the adaptive law (12), the closed-loop plant’s trajectory $x(\cdot)$ converges asymptotically to the reference trajectory $x_{\text{ref}}(\cdot)$. Fig. 1 provides a schematic representation of the MRAC control architecture outlined by Theorem 3.1.

In the following, we highlight two critical aspects of the classical MRAC framework. The proof of Theorem 3.1 [1, pp. 281–285] shows that both $x(\cdot)$ and $\Phi(\cdot, x(\cdot))$ are bounded in t over $[t_0, \infty)$ and since $r(\cdot)$ is bounded in t by assumption, the vector of state variables

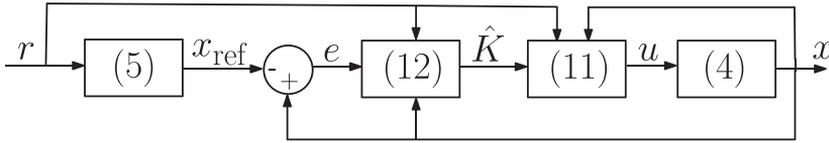


Fig. 1. Schematic representation of the classical MRAC architecture to steer the state $x(\cdot)$ of the nonlinear plant (4) toward the state $x_{\text{ref}}(\cdot)$ of the plant’s reference model (5). Applying the control law (11) and the adaptive law (12), Theorem 3.1 guarantees uniform boundedness of both the closed-loop plant’s trajectory error $e(\cdot)$ and the adaptive gain matrix $\hat{K}(\cdot)$. Moreover, Theorem 3.1 guarantees uniform asymptotic convergence of $e(\cdot)$ to zero.

$\pi(\cdot, x(\cdot), r(\cdot))$ is bounded in t over $[t_0, \infty)$. Therefore, $\pi(\cdot, x(\cdot), r(\cdot))$ can be considered as a bounded input for Eq. (14), and it follows from Proposition 2.1, Eqs. (14), and (5) that $\alpha_{\text{max}}(e(\cdot)) = -\text{Re}(\lambda_{\text{max}}(A_{\text{ref}}))$ and $\alpha_{\text{max}}(x_{\text{ref}}(\cdot)) = -\text{Re}(\lambda_{\text{max}}(A_{\text{ref}}))$. However, it would be desirable to design an MRAC law so that

$$\alpha_{\text{max}}(e(\cdot)) \geq \alpha, \tag{15}$$

where $\alpha > -\text{Re}(\lambda_{\text{min}}(A_{\text{ref}})) \geq -\text{Re}(\lambda_{\text{max}}(A_{\text{ref}}))$ is a user-defined parameter [1, pp. 389–390]. The two-layer MRAC proposed in this paper allows to guarantee this design specification.

Classical MRAC is affected by an additional limitation. Specifically, it is common practice to choose the adaptive rate matrix Γ in Eqs. (12) so that $\text{Re}(\lambda_{\text{min}}(\Gamma))$ is large and hence, the adaptive gain $\hat{K}(\cdot)$ responds rapidly to large values of the norm of the tracking error $\|e(\cdot)\|$. However, it follows from Eq. (11) that large values of $\|\hat{K}(\cdot)\|$ may imply larger values of the norm of $\|u(\cdot)\|$ and hence, the plant’s actuators may saturate in the presence of large trajectory tracking errors. Moreover, it follows from Eq. (14) that larger values of $\|\hat{K}(\cdot)\|$ may produce larger values of $\|e(\cdot)\|$ and hence, the plant’s trajectory may exceed its operative range. In this paper, we provide a novel MRAC framework to constrain both the control input and the trajectory tracking error at all time, while the adaptive rates are set arbitrarily large.

4. A Novel MRAC Law

In this section, we present a variation of classical MRAC, wherein the vector of feedback variables is augmented by a vector function that can be designed to meet user-defined design specifications, in addition to the uniform boundedness of the adaptive gains and the uniform asymptotic convergence of the trajectory tracking error dynamics guaranteed by Theorem 3.1. For the statement of the next theorem, consider the feedback control law

$$\begin{aligned} \phi(\pi(t, x, r), g(t, x, e, r), \hat{K}, \hat{K}_g) &= \hat{K}^T \pi(t, x, r) + \hat{K}_g^T g(t, x, e, r), \\ (t, x, e, r, \hat{K}, \hat{K}_g) &\in [t_0, \infty) \times \mathcal{D} \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{(n+m+N) \times m} \times \mathbb{R}^{p \times m}, \end{aligned} \tag{16}$$

where the vector of feedback variables $\pi(\cdot, \cdot, \cdot)$ is given by Eq. (13), $g : [t_0, \infty) \times \mathcal{D} \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^p$ is jointly continuous in its arguments, $g(t, \cdot, \cdot, r)$ is locally Lipschitz continuous in (x, e) uniformly in (t, r) on compact subsets of $[t_0, \infty) \times \mathbb{R}^m$, $g(\cdot, x, e, r)$ is bounded for all $t \in [t_0, \infty)$, $\hat{K}(\cdot)$ verifies the adaptive law (12), $\hat{K}_g(\cdot)$ verifies the adaptive law

$$\dot{\hat{K}}_g(t) = -\Gamma_g g(t, x(t), e(t), r(t))e^T(t)PB, \quad \hat{K}_g(t_0) = \hat{K}_{g,0}, \quad t \geq t_0, \tag{17}$$

the user-defined adaptive rate matrix $\Gamma_g \in \mathbb{R}^{p \times p}$ is symmetric and positive-definite, P denotes the symmetric, positive-definite solution of Eq. (10), $x(\cdot)$ denotes the solution of Eq. (4) with

$$u(t) = \phi(\pi(t, x(t), r(t)), g(t, x(t), e(t), r(t)), \hat{K}(t), \hat{K}_g(t)), \tag{18}$$

$e(\cdot)$ denotes the solution of

$$\begin{aligned} \dot{e}(t) &= A_{\text{ref}}e(t) + B\Lambda \left[\Delta K^T(t)\pi(t, x(t), r(t)) + \hat{K}_g^T(t)g(t, x(t), e(t), r(t)) \right], \\ e(t_0) &= x_0 - x_{\text{ref},0}, \quad t \geq t_0, \end{aligned} \tag{19}$$

$\Delta K(t) = \hat{K}(t) - K$, $K = [K_x^T, K_r^T, \Theta^T]^T$, and $(K_x, K_r) \in \mathbb{R}^{n \times m} \times \mathbb{R}^{m \times m}$ verify the matching conditions (6) and (7).

Theorem 4.1. Consider the plant's dynamics (4), the plant's reference model (5), the feedback control law (16), and the adaptive laws (12) and (17). If the matching conditions (6) and (7) are verified, then the trajectory tracking error $e(\cdot)$ and the adaptive gain matrices $\hat{K}(\cdot)$ and $\hat{K}_g(\cdot)$ are uniformly bounded and $e(t) \rightarrow 0$ as $t \rightarrow \infty$ uniformly in $t_0 \in [0, \infty)$.

Theorem 4.1, whose proof is provided in the Appendix, gives the user ample discretion in the control design process by means of the vector function $g(\cdot, \cdot, \cdot, \cdot)$, which can be exploited to meet specifications that could not be enforced directly on the feedback control law by classical MRAC. For instance, setting $p = n$ and $g(t, x, e, r) = K_p e$, $(t, x, e, r) \in [t_0, \infty) \times \mathcal{D} \times \mathbb{R}^n \times \mathbb{R}^m$, where $K_p \in \mathbb{R}^{n \times n}$ is symmetric and positive-definite, introduces a term in Eq. (16) that is proportional to the trajectory tracking error and hence, allows stronger corrective actions and smaller oscillations of the trajectory tracking error; it follows from (12) that in classical MRAC, this effect can be achieved only indirectly by increasing the adaptive rates' norm. The framework proposed in [12], which presents an anti-windup system for the plant (4) with $\Lambda = I$ and $\Theta = 0$, can be deduced from Theorem 4.1 by setting $g(t, x, e, r) = \text{sat}(\hat{K}^T(t)\pi(t, x(t), r(t))) - \hat{K}^T(t)\pi(t, x(t), r(t))$, $(t, x, e, r) \in [t_0, \infty) \times \mathcal{D} \times \mathbb{R}^n \times \mathbb{R}^m$, where $\text{sat}(\cdot)$ denotes the saturation function [47, p. 19], and relaxing the continuity conditions on $g(\cdot, \cdot, \cdot, \cdot)$.

The continuity conditions on $g(\cdot, \cdot, \cdot, \cdot)$ are sufficient to guarantee the existence of a unique solution of the dynamical system (19), (12), and (17) [47, Th. 3.1], and the boundedness of $g(\cdot, x, e, r)$ for all $t \in [t_0, \infty)$ is needed to guarantee that the solution $x(\cdot)$ of Eq. (4) with control input (18) does not diverge in time. Comparing the proposed control law (16) with the classical MRAC law (11), it is apparent that the proposed framework involves an additional set of $p \times m$ ordinary differential equations, namely Eq. (17). If $g(t, x, e, r) = 0$, $(t, x, e, r) \in [t_0, \infty) \times \mathcal{D} \times \mathbb{R}^n \times \mathbb{R}^m$, then Theorem 4.1 specializes to Theorem 3.1.

5. Two-layer model reference adaptive control

5.1. Theoretical formulation

In this section, we present a novel adaptive control framework, which we name *two-layer MRAC*. According to this framework, the user introduces an auxiliary reference model in addition to the plant's reference model (5). If this auxiliary reference model is designed to converge to the closed-loop system's trajectory tracking error obtained by applying the classical MRAC framework before the transient dynamics of the plant's reference model has decayed, then the two-layer MRAC law steers the closed-loop system's trajectory tracking error to the auxiliary reference model at a rate of convergence that is higher than the rate

of convergence of the plant’s reference model. Thus, two-layer MRAC addresses one of the drawbacks of MRAC highlighted in Section 3, namely imposing arbitrary rates of convergence on the trajectory tracking error, while setting arbitrarily large adaptive rates and choosing the plant’s reference model arbitrarily.

Consider the plant (4), the plant’s reference model (5), the vector of feedback variables $\pi(\cdot, \cdot, \cdot)$ given by Eq. (13), and the reference model for the trajectory tracking error’s transient dynamics

$$\dot{e}_{\text{ref,transient}}(t) = A_{\text{transient}} e_{\text{ref,transient}}(t), \quad e_{\text{ref,transient}}(t_0) = x_0 - x_{\text{ref},0}, \quad t \geq t_0, \quad (20)$$

where $A_{\text{transient}} \in \mathbb{R}^{n \times n}$ is Hurwitz. Consider also the auxiliary reference model

$$\dot{\varepsilon}(t) = A_{\text{transient}} \varepsilon(t) + B \Lambda \Delta K^T(t) \pi(t, x(t), r(t)) + B \Lambda \Delta K_g^T(t) e(t), \quad \varepsilon(t_0) = 0, \quad t \geq t_0, \quad (21)$$

where $\Delta K(t) \triangleq \hat{K}(t) - K$, $\Delta K_g(t) \triangleq \hat{K}_g(t) - K_e$, $K = [K_x^T, K_r^T, \Theta^T]^T$, $(K_x, K_r, K_e) \in \mathbb{R}^{n \times m} \times \mathbb{R}^{m \times m} \times \mathbb{R}^{n \times m}$, verify the matching conditions (6), (7), and

$$A_{\text{transient}} = A_{\text{ref}} + B \Lambda K_e^T, \quad (22)$$

$\hat{K} : [t_0, \infty) \rightarrow \mathbb{R}^{(n+m+N) \times m}$ and $\hat{K}_g : [t_0, \infty) \rightarrow \mathbb{R}^{n \times m}$ are such that

$$\dot{\hat{K}}(t) = -\Gamma \pi(t, x(t), r(t)) \varepsilon^T(t) P_{\text{transient}} B, \quad \hat{K}(t_0) = \hat{K}_0, \quad t \geq t_0, \quad (23)$$

$$\dot{\hat{K}}_g(t) = -\Gamma_g e(t) \varepsilon^T(t) P_{\text{transient}} B, \quad \hat{K}_g(t_0) = \hat{K}_{g,0}, \quad (24)$$

$\Gamma \in \mathbb{R}^{(n+m+N) \times (n+m+N)}$ and $\Gamma_g \in \mathbb{R}^{n \times n}$ are symmetric and positive-definite, $P_{\text{transient}} \in \mathbb{R}^{n \times n}$ is the symmetric, positive-definite solution of the algebraic Lyapunov equation

$$0 = A_{\text{transient}}^T P_{\text{transient}} + P_{\text{transient}} A_{\text{transient}} + Q_{\text{transient}}, \quad (25)$$

and $Q_{\text{transient}} \in \mathbb{R}^{n \times n}$ is symmetric and positive-definite. If the matching conditions (6), (7), and (22) are verified, then the trajectory tracking error $e(\cdot)$ can be computed equivalently either as $e(t) = e_{\text{ref,transient}}(t) + \varepsilon(t)$, $t \geq t_0$, where $e_{\text{ref,transient}}(\cdot)$ verifies Eq. (20) and $\varepsilon(\cdot)$ verifies Eq. (21), or as the solution of Eq. (19) with $p = n$ and $g(t, x, e, r) = e$, $(t, x, e, r) \in [t_0, \infty) \times \mathcal{D} \times \mathbb{R}^n \times \mathbb{R}^m$. The next result, which introduces the two-layer MRAC framework, leverages this equivalence and the analysis of the solution of Eq. (21) to prove uniform asymptotic convergence of $e(\cdot)$ to zero.

Lemma 5.1. Consider the reference model for the trajectory tracking error’s transient dynamics (20), the auxiliary reference model (21), and the adaptive laws (23) and (24). If the matching conditions (6), (7), and (22) are verified, then both the tracking errors $\varepsilon(\cdot)$ and $e(\cdot)$ and the adaptive gain matrices $\hat{K}(\cdot)$ and $\hat{K}_g(\cdot)$ are uniformly bounded. Moreover, $\varepsilon(t) \rightarrow 0$ and $e(t) \rightarrow 0$ as $t \rightarrow \infty$ uniformly in $t_0 \in [0, \infty)$.

The proof of Lemma 5.1 is provided in the Appendix. If $p = n$ and $g(t, x, e, r) = e$, $(t, x, e, r) \in [t_0, \infty) \times \mathcal{D} \times \mathbb{R}^n \times \mathbb{R}^m$, then both Theorem 4.1 and Lemma 5.1 prove the uniform asymptotic convergence of $e(\cdot)$ to zero. However, the adaptive laws (12) and (17) used in Theorem 4.1 are different than the adaptive laws (23) and (24) used in Lemma 5.1. Furthermore, Theorem 4.1 proves the asymptotic convergence of $e(\cdot)$ to zero by analyzing the time derivative of a Lyapunov function candidate along the vector field of the closed-loop system’s

trajectory tracking error, whereas **Lemma 5.1** proves the asymptotic convergence of $e(\cdot)$ to zero by analyzing the asymptotic convergence of $\varepsilon(\cdot)$ to zero, that is, the asymptotic convergence of $e(\cdot)$ to $e_{\text{ref,transient}}(\cdot)$, which, in turn, converges to the origin. For having deduced the asymptotic convergence of $e(\cdot)$ to zero indirectly from the analysis of the convergence of $\varepsilon(\cdot)$ to zero, and for having introduced the reference model for the trajectory tracking error’s transient dynamics (20), the auxiliary reference model (21), and the matching condition (22), the framework presented in **Lemma 5.1** has been named *two-layer MRAC*.

The next result shows how **Lemma 5.1** can be applied to set the rate of convergence of the closed-loop system’s trajectory tracking error and guarantee that the trajectory tracking error’s transient dynamics is faster than the transient dynamics of the plant’s reference model.

Lemma 5.2. *Consider the auxiliary reference model (21), the reference model for the trajectory tracking error’s transient dynamics (20), and the adaptive laws (23) and (24). If Eqs. (6), (7), and (22) are verified and $\text{Re}(\lambda_{\max}(A_{\text{transient}})) < \text{Re}(\lambda_{\min}(A_{\text{ref}}))$, then Eq. (15) is verified with $\alpha > -\text{Re}(\lambda_{\min}(A_{\text{ref}})) \geq -\text{Re}(\lambda_{\max}(A_{\text{ref}}))$.*

Lemma 5.2, whose proof is given in the Appendix, shows that if $\text{Re}(\lambda_{\max}(A_{\text{transient}})) < \text{Re}(\lambda_{\min}(A_{\text{ref}}))$, then the two-layer MRAC framework introduced by **Lemma 5.1** allows to impose the user-defined rate of convergence on the trajectory tracking error $e(\cdot)$ that is faster than the rate of convergence of the reference trajectory $x_{\text{ref}}(\cdot)$. These two results are synthesized by the next theorem, which is the main result of this section. For the statement of this result, it is worthwhile to explicitly present the control law

$$\begin{aligned} \phi(\pi(t, x, r), e, \hat{K}, \hat{K}_g) &= \hat{K}^T \pi(t, x, r) + \hat{K}_g^T e, \\ (t, x, e, r, \hat{K}, \hat{K}_g) &\in [t_0, \infty) \times \mathcal{D} \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{(n+m+N) \times m} \times \mathbb{R}^{n \times m}, \end{aligned} \tag{26}$$

that is deduced from **Eq. (16)** with $p = n$ and $g(t, x, e, r) = e$, $(t, x, e, r) \in [t_0, \infty) \times \mathcal{D} \times \mathbb{R}^n \times \mathbb{R}^m$.

Theorem 5.1. *Consider the plant’s dynamics (4), the plant’s reference model (5), the reference model for the trajectory tracking error’s transient dynamics (20), the feedback control law (26), and the adaptive laws (23) and (24). If the matching conditions (6), (7), and (22) are verified and $\text{Re}(\lambda_{\max}(A_{\text{transient}})) < \text{Re}(\lambda_{\min}(A_{\text{ref}}))$, then the trajectory tracking error $e(\cdot)$ and the adaptive gain matrices $\hat{K}(\cdot)$ and $\hat{K}_g(\cdot)$ are uniformly bounded, $e(t) \rightarrow 0$ as $t \rightarrow \infty$ uniformly in $t_0 \in [0, \infty)$, and **Eq.(15)** is verified with $\alpha > -\text{Re}(\lambda_{\min}(A_{\text{ref}}))$.*

Fig. 2 provides a schematic representation of the control architecture outlined by **Theorem 5.1**. It is worthwhile to note that, to apply **Theorem 5.1**, the trajectory tracking error can be computed as $e(t) = x(t) - x_{\text{ref}}(t)$, $t \geq t_0$, and $\varepsilon(t) = e(t) - e_{\text{ref,transient}}(t)$. In addition to uniform boundedness of the closed-loop system’s trajectory tracking error and of the adaptive gains and uniform asymptotic convergence of the trajectory tracking error, which are already guaranteed by the classical MRAC architecture, the control architecture provided by **Theorem 5.1** also allows to enforce the user-defined rate of convergence, while setting the adaptive rates arbitrarily high. Remarkably, this result has been achieved without any constraints on the adaptive rates, without modifying the user-defined reference model’s dynamics, without employing barrier Lyapunov functions, and without using estimators of the plant’s nonlinearities, as required by alternative frameworks such as [8–10,21,22,24,27–32,48] to name a few.

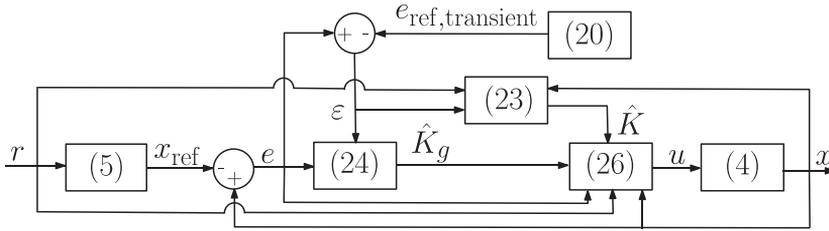


Fig. 2. Schematic representation of the MRAC architecture outlined in Theorem 5.1. Applying the control law (26) and the adaptive laws (23) and (24), Theorem 5.1 guarantees uniform boundedness of both the closed-loop system’s trajectory error $e(\cdot)$ and the adaptive gain matrices $\hat{K}(\cdot)$ and $\hat{K}_g(\cdot)$. Moreover, Theorem 5.1 guarantees uniform asymptotic convergence of $e(\cdot)$ to zero. Lastly, Theorem 5.1 guarantees that $x(\cdot)$ reaches $x_{ref}(\cdot)$ before the transient dynamics of the plant’s reference model (5) has decayed.

5.2. Illustrative numerical example

In this section, we provide a numerical example to illustrate the applicability of Theorem 5.1. The roll dynamics of an unmanned aerial vehicle is captured by [49, pp. 59–64]

$$\begin{aligned} \begin{bmatrix} \dot{\varphi}(t) \\ \dot{\rho}(t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ \theta_1 & \theta_2 \end{bmatrix} \begin{bmatrix} \varphi(t) \\ \rho(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \theta_6 \end{bmatrix} \left(u(t) + \begin{bmatrix} \theta_3 \\ \theta_4 \\ \theta_5 \end{bmatrix}^T \begin{bmatrix} |\varphi(t)| |\rho(t)| \\ |\rho(t)| |\rho(t)| \\ \varphi^3(t) \end{bmatrix} \right), \\ [\varphi(0), \rho(0)]^T &= [\varphi_0, \rho_0]^T, \quad t \geq 0, \end{aligned} \tag{27}$$

where $\varphi(t) \in \mathbb{R}$, $t \geq 0$, denotes the roll angle, $\rho(t) \in \mathbb{R}$ denotes the roll rate, $u(t) \in \mathbb{R}$ denotes the moment needed to deflect the aileron, $\theta_1, \dots, \theta_5 \in \mathbb{R}$, and $\theta_6 > 0$. In particular, both θ_1 and θ_2 capture the effect of the aerodynamic moments acting on the vehicle, and $[\theta_3, \theta_4, \theta_5][|\varphi|\rho, |\rho|\rho, \varphi^3]^T$ captures undesired aerodynamic moments induced by the aileron deflection; we consider $\theta_1, \dots, \theta_6$ as unknown. The nonlinear plant (27) is in the same form as (4) with $n = 2$, $m = 1$, $N = 3$, $\mathcal{D} = \mathbb{R}^n$, $x = [\varphi, \rho]^T$, $A = \begin{bmatrix} 0 & 1 \\ \theta_1 & \theta_2 \end{bmatrix}$, $B = [0, 1]^T$, $\Lambda = \theta_6$, $\Theta = [\theta_3, \theta_4, \theta_5]^T$, $\Phi(t, x) = [|\varphi|\rho, |\rho|\rho, \varphi^3]^T$, $x_0 = [\varphi_0, \rho_0]^T$, and $t_0 = 0$.

Our goal is to design an MRAC law so that the closed-loop trajectory $x(\cdot)$ eventually tracks the trajectory $x_{ref}(\cdot)$ of the plant’s reference model (5) with reference command input

$$r(t) = 0.2 \left(\frac{t}{2\pi} - \left\lfloor \frac{t}{2\pi} \right\rfloor \right), \quad t \geq 0, \tag{28}$$

capturing a sawtooth function, where $\lfloor t \rfloor \triangleq \max \{m \in \mathbb{Z} : m \leq t\}$ denotes the floor function. Furthermore, we want to guarantee that the trajectory tracking error’s transient dynamics is faster than the transient dynamics of the plant’s reference model. These design specifications can be met by applying Theorem 5.1.

Let $A_{ref} = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}$, $B_{ref} = \begin{bmatrix} 0 \\ k_3 \end{bmatrix}$, and $A_{transient} = \begin{bmatrix} 0 & & 1 \\ -[\sigma \operatorname{Re}(\lambda_{\min}(A_{ref}))]^2 & 2\sigma \operatorname{Re}(\lambda_{\min}(A_{ref})) & \end{bmatrix}$, where $k_1, k_2, k_3 > 0$ and $\sigma > 1$. The matrix A_{ref} is Hurwitz, the pair (A_{ref}, B_{ref}) is controllable, the matching conditions (6), (7), and (22) are verified by $K_x = \begin{bmatrix} \theta_1 - k_1 & \theta_2 - k_2 \\ \theta_6 & \theta_6 \end{bmatrix}^T$, $K_r = \frac{k_3}{\theta_6}$, and $K_e = \theta_6^{-1} [k_1 + k_3, k_2 + k_4]^T$, and $\operatorname{Re}(\lambda_{\max}(A_{transient})) = \sigma \operatorname{Re}(\lambda_{\min}(A_{ref})) < 0$. Therefore, the hypotheses of Theorem 5.1 are verified.

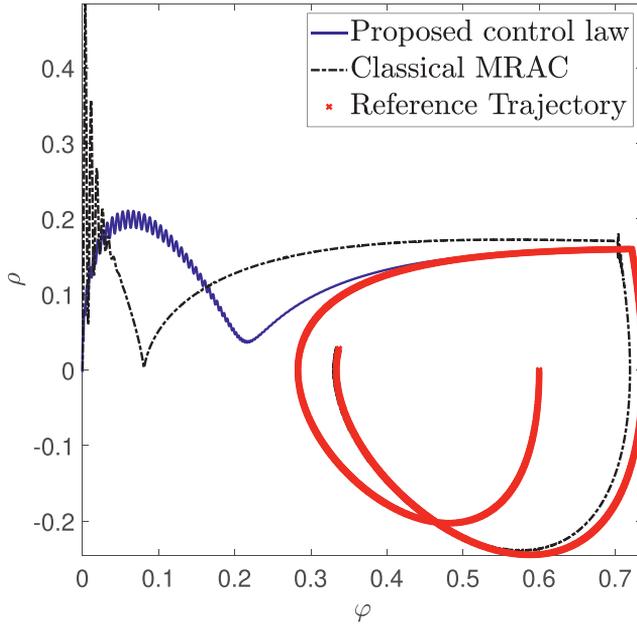


Fig. 3. Closed-loop plant’s trajectory obtained applying the control law (16) with $p = n$ and $g(t, x, e, r) = e$, $(t, x, e, r) \in [t_0, \infty) \times \mathcal{D} \times \mathbb{R}^n \times \mathbb{R}^m$, and the adaptive laws (23) and (24), and closed-loop plant’s trajectory obtained applying the classical MRAC law (11) with adaptive law (12). By applying Theorem 5.1 and setting $\text{Re}(\lambda_{\max}(A_{\text{transient}})) = \sigma \text{Re}(\lambda_{\min}(A_{\text{ref}}))$ with $\sigma = 2$, the closed-loop plant’s trajectory reaches the reference trajectory before the closed-loop plant’s trajectory obtained applying Theorem 3.1.

Let $\theta_1 = -0.018$, $\theta_2 = 0.015$, $\theta_3 = -0.062$, $\theta_4 = 0.009$, $\theta_5 = 0.021$, $\theta_6 = 0.75$, $k_1 = 1.0002$, $k_2 = 1.7218$, $k_3 = 5$, $x_0 = 0$, $x_{\text{ref},0} = [0.6, 0]^T$, $\Gamma = 5 \cdot 10^5 I_6$, $\Gamma_g = 5 \cdot 10^5 I_2$, $Q = I_2$, and $Q_{\text{transient}} = Q$. Fig. 3 shows the solution $x(t)$, $t \in [0, 3\pi]$ s, of Eq. (27) obtained applying the feedback control law (26) and the adaptive laws (23) and (24) with $\sigma = 2$, the solution $x(t)$ of Eq. (27) obtained applying the control law (11) and the classical adaptive law (12), and the solution $x_{\text{ref}}(t)$ of (5). Fig. 4 shows the norm of the trajectory tracking error applying the proposed two-layer model referenced adaptive control framework with $\sigma \in \{2, 5, 10\}$ and the norm of the trajectory tracking error applying the classical MRAC framework. It is apparent that applying the proposed approach, the trajectory tracking error converges to zero faster than applying the classical MRAC framework. Moreover, larger values of σ guarantee faster convergence of the trajectory tracking error to zero, but also imply larger overshoots. Lastly, Fig. 5 shows the control inputs obtained applying the proposed model referenced adaptive control law with $\sigma \in \{2, 5, 10\}$ and applying the classical MRAC framework. Larger values of σ imply larger control inputs and, applying the proposed adaptive control framework with $\sigma < 7.8095$, the absolute value of the maximum control input required by the proposed framework is smaller than the absolute value of the maximum control input required by the classical MRAC framework.

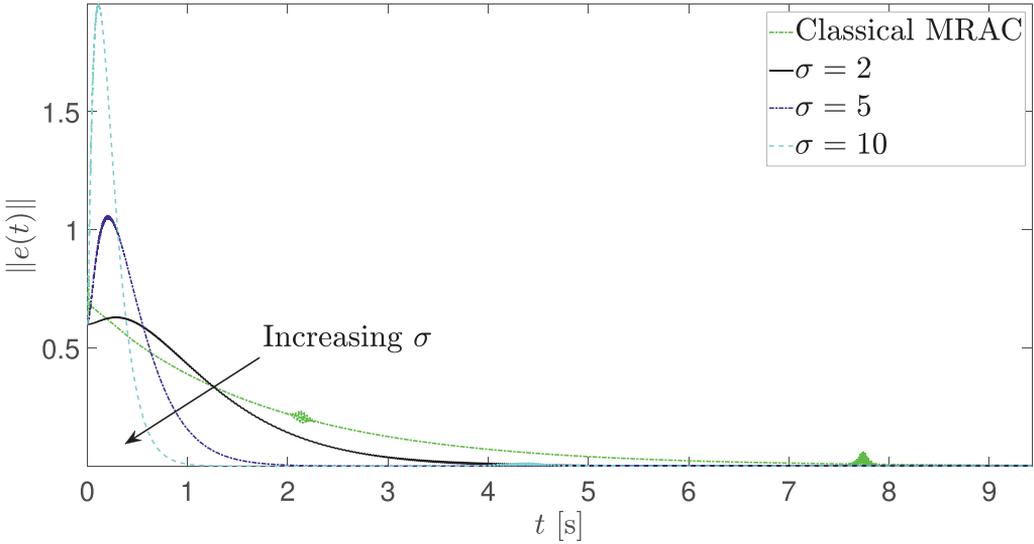


Fig. 4. Norm of the trajectory tracking error obtained by applying the proposed two-layer MRAC framework, and norm of the trajectory tracking error obtained by applying the classical MRAC framework.

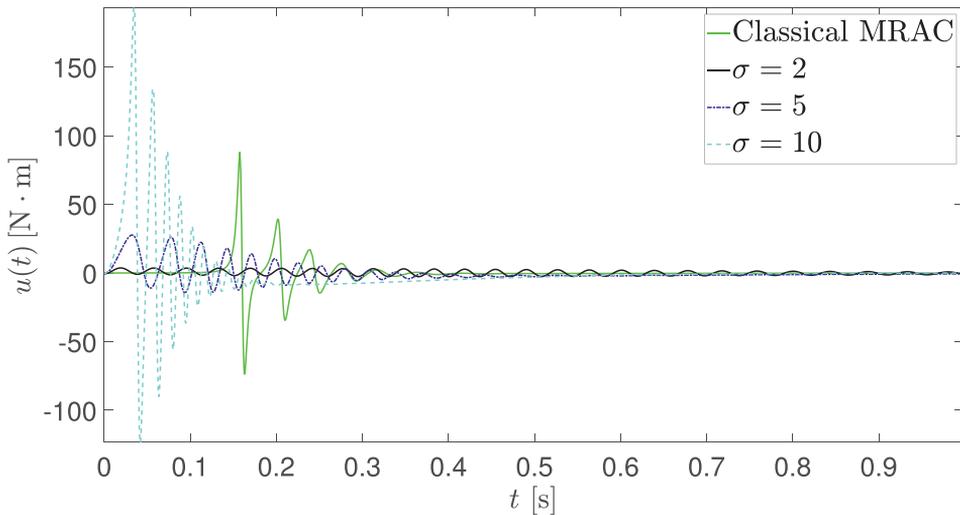


Fig. 5. Control inputs obtained by applying Theorem 5.1 with $\text{Re}(\lambda_{\max}(A_{\text{transient}})) = \sigma \text{Re}(\lambda_{\min}(A_{\text{ref}}))$ and $\sigma \in \{2, 5, 10\}$, and Theorem 3.1. Larger values of σ imply larger control inputs.

6. MRAC laws in the presence of constraints

6.1. Theoretical formulation

In this section, we design MRAC laws for prescribed performance. These adaptive laws guarantee both uniform asymptotic convergence of the closed-loop system’s trajectory tracking error and uniform boundedness of the trajectory tracking error and of the adaptive rates,

exploit barrier Lyapunov functions to enforce user-defined constraints on both the closed-loop system’s trajectory tracking error and control input, and leverage the proposed two-layer MRAC framework to impose a user-defined rate of convergence on the trajectory tracking error. To present these MRAC laws, consider the plant (4) with $\Lambda = I_m$, the plant’s reference model (5), and the constraint sets

$$\mathcal{E} \triangleq \{e \in \mathbb{R}^n : h_e(e^T M e) \geq 0\}, \tag{29}$$

$$\mathcal{U} \triangleq \{u \in \mathbb{R}^m : h_u(u) \geq 0\}, \tag{30}$$

where $M \in \mathbb{R}^{n \times n}$ is a symmetric and positive-definite solution of the inequality

$$A_{\text{ref}}^T M + M A_{\text{ref}} \leq 0, \tag{31}$$

$h_e : \mathbb{R} \rightarrow \mathbb{R}$, $h_u : \mathbb{R}^m \rightarrow \mathbb{R}$, $h_e(0) > 0$, $h_u(0) > 0$, $h_e(\cdot)$ is continuously differentiable for all $e \in \mathcal{E}$, and $h_u(\cdot)$ is continuously differentiable, bounded on compact subsets of \mathbb{R}^m , and such that $\frac{\partial h_u(u)}{\partial u}$ is bounded over compact subsets of \mathbb{R}^m . For example, if

$$h_e(e^T M e) = \eta_e - e^T e, \quad e \in \mathbb{R}^n, \tag{32}$$

$$h_u(u) = \eta_u - u^T u, \quad u \in \mathbb{R}^m, \tag{33}$$

where $\eta_e, \eta_u > 0$, then Eq. (29) captures a closed ball of radius η_e and Eq. (30) captures saturation constraints on the control input.

Assuming that $\Lambda = I_m$ in (4) is without loss of generality. Indeed, let $\phi(t)$, $t \geq t_0$, denote, for brevity, the feedback control law (16) along the trajectories of the closed-loop system, that is, $\phi(\pi(t, x(t), r(t)))$, $g(t, x(t), e(t), r(t))$, $\hat{K}(t)$, $\hat{K}_g(t)$. Then, $B\Lambda[\phi(t) + \Theta^T \Phi(t, x(t))] = B[\phi(t) + \tilde{\Theta}^T \tilde{\Phi}(t, x(t))]$, $(t, x) \in [t_0, \infty) \times \mathcal{D}$, where $\tilde{\Theta}^T \triangleq [\Lambda \Theta^T, (\Lambda - I_m)]$ and $\tilde{\Phi}(t, x) \triangleq [\Phi^T(t, x), \phi^T(t)]^T$. Therefore, assuming that $\Lambda = I_m$ in Eq. (4) and capturing the plant’s parametric and matched uncertainties by $\tilde{\Theta}^T \tilde{\Phi}(t, x)$, $(t, x) \in [t_0, \infty) \times \mathcal{D}$, is equivalent to capturing the plant dynamics by means of Eq. (4). This equivalence, however, is proven by augmenting the regressor vector and hence, yields at the expense of a larger number of adaptive laws to integrate and a higher computational cost.

For the statement of the results of this section, define $h'_e : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$h'_e(e^T M e) \triangleq \left. \frac{\partial h_e(\beta)}{\partial \beta} \right|_{\beta=e^T M e}, \quad e \in \mathbb{R}^n, \tag{34}$$

let

$$V_e(e) \triangleq h_e^{-1}(e^T M e) e^T P e, \quad e \in \mathring{\mathcal{E}}, \tag{35}$$

$$V_g(t, \hat{K}_g) \triangleq h_u^{-1}(\phi(t)) \text{tr}^{\frac{1}{2}} \left(\hat{K}_g^T \Gamma_g^{-1} \hat{K}_g \right), \quad (t, \hat{K}_g) \in [t_0, \infty) \times \mathbb{R}^{p \times m}, \tag{36}$$

and consider the adaptive laws

$$\dot{\hat{K}}(t) = -\Gamma \frac{\pi(t, x(t), r(t))}{h_e(e^T(t) M e(t))} e^T(t) [P - V_e(e(t)) h'_e(e^T(t) M e(t)) M] B, \quad \hat{K}(t_0) = \hat{K}_0, \quad t \geq t_0, \tag{37}$$

$$\begin{aligned} \dot{\hat{K}}_g(t) = & -\frac{2h_u^2(\phi(t))V_g(t, \hat{K}_g(t))}{h_e(e^T(t)Me(t))} \Gamma_g g(t, x(t), e(t), r(t))e^T(t) [P - V_e(e(t))h'_e(e^T(t)Me(t))M] B \\ & + \frac{\gamma^T(t)h_{du}(t) + \chi}{h_u(\phi(t))} \hat{K}_g(t), \quad \hat{K}_g(t_0) = \hat{K}_{g,0}, \end{aligned} \tag{38}$$

where $\pi(\cdot, \cdot, \cdot)$ is given by Eq. (13), $e(t) \in \mathring{\mathcal{E}}$ denotes the solution of Eq. (19), $g : [t_0, \infty) \times \mathcal{D} \times \mathcal{E} \times \mathbb{R}^m \rightarrow \mathbb{R}^p$ is jointly continuous in its arguments, $g(t, \cdot, \cdot, r)$ is locally Lipschitz continuous in (x, e) uniformly in (t, r) on compact subsets of $[t_0, \infty) \times \mathbb{R}^m$, $g(\cdot, x, e, r)$ is bounded for all $t \in [t_0, \infty)$, both $\Gamma \in \mathbb{R}^{(n+m+N) \times (n+m+N)}$ and $\Gamma_g \in \mathbb{R}^{p \times p}$ are symmetric and positive-definite, $\phi(t) \in \mathring{\mathcal{U}}$, $\hat{K}_g(t_0) \neq 0$, $\gamma(\cdot)$, $h_{du}(\cdot)$, and χ are such that

$$\dot{h}_u(\phi(t)) = \gamma^T(t)h_{du}(t) + \beta(t), \tag{39}$$

$\gamma : [t_0, \infty) \rightarrow \mathbb{R}^{N_r}$ is known, $h_{du} : [t_0, \infty) \rightarrow \mathbb{R}^{N_r}$, $\beta(t) \in [-\beta_{\max}, \beta_{\max}]$ is unknown, $\beta_{\max} \geq 0$, $\gamma^T(t)h_{du}(t)$ is continuous over $[t_0, \infty)$, and $\chi < -\beta_{\max}$. Given the user-defined parameter $N_f \in \mathbb{N}$, the term $\gamma^T(t)h_{du}(t)$, $t \geq t_0$, in Eq. (39) approximates the time derivative of $h_u(\phi(t))$ and can be computed, for instance, using differentiators [50, pp. 235–236]. Several filters, such as the finite impulse response filter [51, Ch. 7] or the fixed-point smoother [52, Ch. 5], can be employed to design differentiators, which guarantee that the approximation error $\beta(\cdot)$ in Eq. (39) is bounded.

Lastly, to state the results of this section, the following two assumptions are made. To formulate these assumptions, recall that the null-controllable region of Eq. (8) associated to the constraint set \mathcal{U} is the set $\mathcal{N}_{\mathcal{U}} \subseteq \mathbb{R}^n$ such that if $e(t_0) \in \mathcal{N}_{\mathcal{U}}$, then there exist both $T > t_0$ and $u : [t_0, T] \rightarrow \mathbb{R}^m$ such that $u(t) \in \mathcal{U}$, $t \in [t_0, T]$, and $e(T) = 0$ [53]. Furthermore, recall that if K_x , K_r , and Θ were known, then the control input given by (9) would guarantee asymptotic convergence of the closed-loop trajectory tracking error.

Assumption 6.1. Consider the open-loop trajectory tracking error dynamics (8) and the constraint sets \mathcal{E} and \mathcal{U} given by Eq. (29) and Eq. (30), respectively. It holds that $\mathcal{E} \subseteq \mathcal{N}_{\mathcal{U}}$.

Assumption 6.2. The control input $u(\cdot)$ given by Eq. (9) is such that $u(t) \in \mathring{\mathcal{U}}$, $t \geq t_0$, and the corresponding trajectory tracking error $e(\cdot)$, which verifies Eq. (14) with $\Delta K(t) = 0$, $t \geq t_0$, is such that $e(t) \in \mathring{\mathcal{E}}$.

The next theorem proves that the MRAC law (16) and the adaptive laws (37) and (38) enforce the user-defined constraints on both the closed-loop system’s trajectory tracking error and the control input given by Eq. (29) and Eq. (30), respectively, and guarantee uniform asymptotic convergence of the closed-loop system’s trajectory tracking error.

Theorem 6.1. Consider the plant’s dynamics (4) with $\Lambda = I_m$, the plant’s reference model (5), the constraint sets (29) and (30), the feedback control law (16), and the adaptive laws (37) and (38). If $(e(t_0), \phi(t_0)) \in \mathring{\mathcal{E}} \times \mathring{\mathcal{U}}$, $h'_e(e^T Me) \leq 0$, $e \in \mathring{\mathcal{E}}$, Assumptions 6.1 and 6.2 are verified, and the matching conditions (6) and (7) are verified, then both the trajectory tracking error $e(\cdot)$ and the adaptive gain matrices $\hat{K}(\cdot)$ and $\hat{K}_g(\cdot)$ are uniformly bounded, $(e(t), \phi(t)) \in \mathring{\mathcal{E}} \times \mathring{\mathcal{U}}$, $t \geq t_0$, and $e(t) \rightarrow 0$ as $t \rightarrow \infty$ uniformly in $t_0 \in [0, \infty)$.

The proof of Theorem 6.1 is provided in the Appendix. It is worthwhile to note that if $h_u(u) = 1$, $u \in \mathbb{R}^m$, then $\mathcal{U} = \mathbb{R}^m$ and Assumptions 6.1 and 6.2 are always verified. In this case, by setting $h_{du}(t) = 0$, $t \geq t_0$, and $\chi = 0$, Theorem 6.1 allows to enforce the constraints

on the trajectory tracking error. We also note that if no constraint is imposed, that is, if $h_e(e^T Me) = 1$, $e \in \mathbb{R}^n$, and $h_u(u) = 1$, $u \in \mathbb{R}^m$, then Eq. (37) and (38) reduce to Eqs. (12) and (17), respectively, and Theorem 6.1 reduces to Theorem 4.1. Lastly, we note that the term $\frac{h_u^2(\phi(t))}{h_e(e^T(t)Me(t))}$, $t \geq t_0$, in Eq. (38) captures the effect of competing design objectives namely, constraining the control input by means of $h_u(\cdot)$, which induces smaller values of $\|\dot{\hat{K}}_g(\cdot)\|$, and constraining the trajectory tracking error by means of $h_e(\cdot)$, which induces larger values of $\|\hat{K}_g(\cdot)\|$.

Assumption 6.1 postulates the existence of a control input that meets the user-defined constraints on both the trajectory tracking error and the control input, and Assumption 6.2 postulates that the control input given by Eq. (9) meets the user-defined constraints. If the uncontrolled trajectory tracking error dynamics (8) is unstable, then there may not exist a control input that regulates the trajectory tracking error and meets the constraints captured by \mathcal{U} in Eq. (30) at all times [54, pp. 222–223]. Assumptions 6.1 and 6.2 guarantee that there exist control inputs, including Eq. (9), that meet the user-defined constraints. To verify Assumption 6.1, the null-controllable region of Eq. (8) may be characterized by applying some of the techniques presented in [53,55–57], which leverage the analysis of the basin of attraction of Lyapunov functions and the analysis of time-optimal trajectories for the plant dynamics. The assumption that $h_e(0) > 0$ implies that $0 \in \mathring{\mathcal{E}}$, and for the problems addressed in this paper, the origin is always contained in the null-controllable region of Eq. (8). Indeed, the pair (A, B) is controllable and hence, it follows from the matching condition (6) with $\Lambda = I_m$ that the pair (A_{ref}, B) is controllable, and it follows from Theorem 6.1 of [58] that the null-controllable region $\mathcal{N}_{\mathcal{U}}$ of Eq. (8) associated to \mathcal{U} is an open, connected set containing the origin.

Theorem 6.1 does not require to specify the function $g(\cdot, \cdot, \cdot, \cdot)$ in the control law (16) and in the adaptive law (38). Therefore, $g(\cdot, \cdot, \cdot, \cdot)$ may serve as a design parameter to enforce additional design objectives. The next theorem is the main result of this paper and shows how the two-layer MRAC framework introduced in Section 5 can be used to create MRAC laws for prescribed performance, that is, MRAC laws that guarantee boundedness of the adaptive gains, constrain both the trajectory tracking error and the control input, enforce uniform asymptotic convergence to zero of the trajectory tracking error, and assure that the trajectory tracking error’s rate of convergence is arbitrarily high. For the statement of this result, consider the constraint sets (29) and (30) and the adaptive laws

$$\begin{aligned} \dot{\hat{K}}(t) &= -\Gamma \frac{\pi(t, x(t), r(t))}{h_e(e^T(t)Me(t))} \varepsilon^T(t) \left[P_{\text{transient}} - \tilde{V}_e(e(t), \varepsilon(t)) h'_e(e^T(t)Me(t)) M \right] B, \\ \hat{K}(t_0) &= \hat{K}_0, \quad t \geq t_0, \end{aligned} \tag{40}$$

$$\begin{aligned} \dot{\hat{K}}_g(t) &= -\frac{2h_u^2(\phi(t)) \tilde{V}_g(t, \hat{K}_g(t))}{h_e(e^T(t)Me(t))} \Gamma_g e(t) \varepsilon^T(t) \left[P_{\text{transient}} - \tilde{V}_e(e(t), \varepsilon(t)) h'_e(e^T(t)Me(t)) M \right] B \\ &+ \frac{\gamma^T(t) h_{du}(t) + \chi}{2h_u(\phi(t))} \hat{K}_g(t), \quad \hat{K}_g(t_0) = \hat{K}_{g,0}, \end{aligned} \tag{41}$$

where both $\Gamma \in \mathbb{R}^{(n+m+N) \times (n+m+N)}$ and $\Gamma_g \in \mathbb{R}^{n \times n}$ are symmetric and positive-definite, $\varepsilon(\cdot)$ denotes the solution of (21) with $\Lambda = I_n$, $e(t) \in \mathring{\mathcal{E}}$ denotes the solution of Eq. (19) with $p = n$ and $g(t, x, e, r) = e$, $(t, x, e, r) \in [t_0, \infty) \times \mathcal{D} \times \mathring{\mathcal{E}} \times \mathbb{R}^m$, $P_{\text{transient}} \in \mathbb{R}^{n \times n}$ denotes the symmetric, positive-definite solution of Eq. (25), $\phi(t) \in \mathcal{U}$ denotes, for brevity,

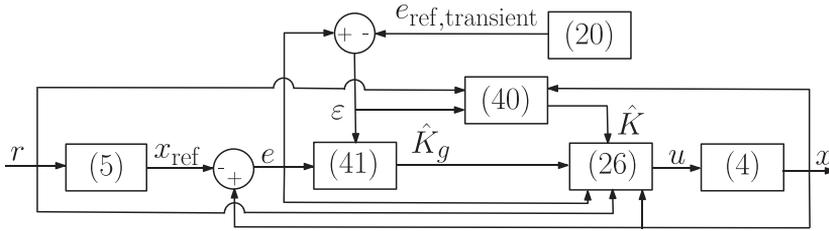


Fig. 6. Schematic representation of the MRAC architecture outlined in Theorem 6.2. Applying the control law (26) and the adaptive laws (40) and (41), Theorem 6.2 guarantees uniform boundedness of the adaptive gain matrices $\hat{K}(\cdot)$ and $\hat{K}_g(\cdot)$. Moreover, Theorem 6.2 guarantees uniform asymptotic convergence of $e(\cdot)$ to zero. Additionally, Theorem 6.2 guarantees that $x(\cdot)$ reaches $x_{ref}(\cdot)$ before the transient dynamics of the plant’s reference model has decayed. Lastly, Theorem 6.2 guarantees that the user-defined constraints on the closed-loop system’s trajectory tracking error and the control input captured by Eq. (29) and Eq. (30), respectively.

the feedback control law (26) along the trajectories of the closed-loop system, that is, $\phi(\pi(t, x(t), r(t)), e(t), \hat{K}(t), \hat{K}_g(t))$,

$$\tilde{V}_e(e, \varepsilon) \triangleq h_e^{-1}(e^T M e) \varepsilon^T P_{\text{transient}} \varepsilon, \quad (e, \varepsilon) \in \mathring{\mathcal{E}} \times \mathbb{R}^n, \tag{42}$$

$$\tilde{V}_g(t, \hat{K}_g) \triangleq h_u^{-1}(\phi(t)) \text{tr}^{\frac{1}{2}}(\Delta K_g^T \Gamma_g^{-1} \Delta K_g), \quad (t, \hat{K}_g) \in [t_0, \infty) \times \mathbb{R}^{n \times m}, \tag{43}$$

$\Delta K_g(t) = \hat{K}_g(t) - K_e$, K_e verifies the matching condition (22) with $\Lambda = I_n$, $\hat{K}_g(t_0) \neq K_e$, $\gamma(\cdot)$ and $h_{du}(\cdot)$ are such that Eq. (39) is verified, and $\chi < -\beta_{\max}$. Lastly, we enunciate the following assumption.

Assumption 6.3. The trajectory $e_{\text{ref,transient}}(\cdot)$ of the reference model for the trajectory tracking error’s transient dynamics (20) is such that $e_{\text{ref,transient}}(t) \in \mathring{\mathcal{E}}, t \geq t_0$.

Theorem 6.2. Consider the plant’s dynamics (4) with $\Lambda = I_m$, the plant’s reference model (5), the reference model for the trajectory tracking error’s transient dynamics (20), the constraint sets (29) and (30), the feedback control law (26), and the adaptive laws (40) and (41). If $(e(t_0), \phi(t_0)) \in \mathring{\mathcal{E}} \times \mathring{\mathcal{U}}, h'_e(e^T M e) \leq 0, e \in \mathring{\mathcal{E}},$ Assumptions 6.1–6.3 are verified, the matching conditions (6), (7), and (22) are verified with $\Lambda = I_m$, and $\text{Re}(\lambda_{\max}(A_{\text{transient}})) < \text{Re}(\lambda_{\min}(A_{\text{ref}}))$, then both the trajectory tracking error $e(\cdot)$ and the adaptive gain matrices $\hat{K}(\cdot)$ and $\hat{K}_g(\cdot)$ are uniformly bounded. Moreover, $(e(t), \phi(t)) \in \mathring{\mathcal{E}} \times \mathring{\mathcal{U}}, t \geq t_0$, and $e(t) \rightarrow 0$ and $\varepsilon(t) \rightarrow 0$ as $t \rightarrow \infty$ uniformly in $t_0 \in [0, \infty)$. Lastly, (15) is verified with $\alpha > -\text{Re}(\lambda_{\min}(A_{\text{ref}})) \geq -\text{Re}(\lambda_{\max}(A_{\text{ref}}))$.

Theorem 6.2 allows to impose both the rate of convergence on the closed-loop plant’s trajectory and user-defined bounds on the closed-loop system’s trajectory tracking error and the control input, and guarantees both uniform boundedness of the adaptive gains and uniform asymptotic convergence of the trajectory tracking error to zero. To the authors’ best knowledge, there is no result within the MRAC framework that allows to meet all these specifications concurrently. The proof of Theorem 6.2 follows the same mechanisms as the proofs of Lemma 5.1 and Theorems 5.1 and 6.1 and hence, its proof is omitted for brevity. Fig. 6 provides a schematic representation of the control architecture outlined by Theorem 6.2. Remarkably, the only difference between the control scheme presented in Fig. 2 and the control scheme presented in Fig. 6 lies in the adaptive laws employed to impose the multiple, possibly

competing, user-defined requirements on the closed-loop plant’s trajectory, the control input, and the trajectory tracking error.

6.2. Illustrative numerical example

In the following, we revisit the problem of controlling the roll dynamics of an unmanned aerial vehicle discussed in Section 5.2, and illustrate the applicability of Theorem 6.2. Specifically, consider the plant

$$\begin{aligned} \begin{bmatrix} \dot{\varphi}(t) \\ \dot{\rho}(t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ \theta_1 & \theta_2 \end{bmatrix} \begin{bmatrix} \varphi(t) \\ \rho(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0.75 \end{bmatrix} \left(u(t) + \begin{bmatrix} \theta_3 \\ \theta_4 \\ \theta_5 \end{bmatrix}^T \begin{bmatrix} |\varphi(t)|\rho(t) \\ |\rho(t)|\rho(t) \\ \varphi^3(t) \end{bmatrix} \right) + \xi(t), \\ [\varphi(0), \rho(0)]^T &= [\varphi_0, \rho_0]^T, \quad t \geq 0, \end{aligned} \tag{44}$$

and note that if $\xi(t) = 0, t \geq 0$, then Eq. (44) is in the same form as Eq. (4) with $n = 2, m = 1, N = 3, \mathcal{D} = \mathbb{R}^n, x = [\varphi, \rho]^T, A = \begin{bmatrix} 0 & 1 \\ \theta_1 & \theta_2 \end{bmatrix}, B = [0, 0.75]^T, \Lambda = 1, \Theta = [\theta_3, \theta_4, \theta_5]^T, \Phi(t, x) = [|\varphi|\rho, |\rho|\rho, \varphi^3]^T, x_0 = [\varphi_0, \rho_0]^T$, and $t_0 = 0$. The unmatched uncertainty $\xi : [0, \infty) \rightarrow \mathbb{R}^2$ captures a Gaussian white noise characterized by a frequency of 100 Hz and \mathcal{L}_∞ -norm of 0.05.

Our goal is to design an MRAC law so that the closed-loop trajectory $x(\cdot)$ eventually tracks the trajectory $x_{\text{ref}}(\cdot)$ of the plant’s reference model (5) with reference command input (28). Furthermore, we wish to enforce that $(e(t), u(t)) \in \mathcal{E} \times \mathcal{U}, t \geq t_0$, where \mathcal{E} and \mathcal{U} are given by Eq. (29) and Eq. (30), respectively, with $h_e(\cdot)$ and $h_u(\cdot)$ given by Eq. (32) and Eq. (33), respectively, so that $\|e(t)\| \in [0, \sqrt{\eta_e})$ and $u(t) \in (-\sqrt{\eta_u}, \sqrt{\eta_u})$. Lastly, we want to guarantee that the trajectory tracking error’s transient dynamics is faster than the transient dynamics of the plant’s reference model. These design specifications can be met by applying Theorem 6.2 with $A_{\text{ref}} = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}, A_{\text{transient}} = \begin{bmatrix} 0 & 1 \\ k_4 & k_5 \end{bmatrix}, \theta_1 = -0.018, \theta_2 = 0.015, \theta_3 = -0.062, \theta_4 = 0.009, \theta_5 = 0.021, k_1 = 1.0002, k_2 = 1.7218, k_4 \triangleq -[\sigma \text{Re}(\lambda_{\min}(A_{\text{ref}}))]^2, k_5 \triangleq 2\sigma \text{Re}(\lambda_{\min}(A_{\text{ref}})), \sigma = 2, x_0 = 0, x_{\text{ref},0} = [0.6, 0]^T, \Gamma = 5 \cdot 10^5 I_6, \Gamma_g = 5 \cdot 10^5 I_2, Q = I_2$, and $Q_{\text{transient}} = Q$. Furthermore, let $k = 50, \eta_e = 0.85, \eta_u = 9, \beta_{\text{max}} = 1$, and $\chi = -1$.

The stochastic unmatched uncertainty $\xi(\cdot)$ has been introduced in the plant model (44) to validate the proposed theoretical results also in realistic applications, wherein external, non-deterministic disturbances are unavoidable. To increase the realism of the proposed simulations and further challenge Theorem 6.2, the state vector $x(\cdot)$ employed in the feedback control law (16) and the adaptive laws (40) and (41) is corrupted by an additive disturbance, namely a Gaussian white noise characterized by a frequency of 100 Hz and \mathcal{L}_∞ -norm of 0.02.

To further validate the results of this paper, we compare the proposed control law to the prescribed performance control law presented in [59], which guarantees *a priori* user-defined transient performance without modifying the reference model or the reference signal, despite uncertainties in the plant parameters. Applying the prescribed performance control framework, the control input is given by

$$u(t) = -\frac{k_{\text{ppc}} z_{\text{ppc}}(t)}{s_{\text{ppc}}(t)} - \frac{\hat{\Theta}_{\text{ppc}}(t)}{2\eta_{\text{ppc}}^2} \Phi^T(t, x(t)) \Phi(t, x(t)) - \frac{\hat{\epsilon}_{\text{ppc}}^2(t) s_{\text{ppc}}(t)}{\hat{\epsilon}_{\text{ppc}}(t) |s_{\text{ppc}}(t)| + \sigma_{1,\text{ppc}}}, \quad t \geq 0, \tag{45}$$

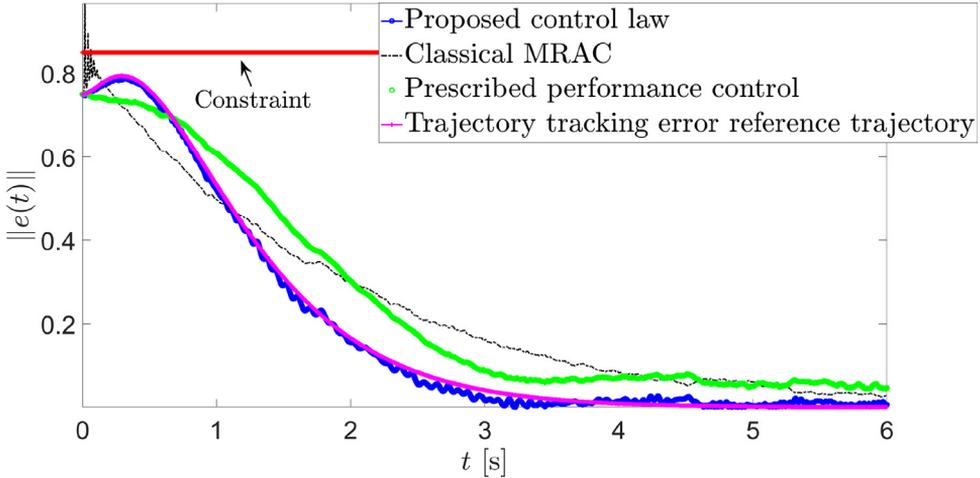


Fig. 7. Norms of the trajectory tracking errors obtained by applying Theorems 6.2, 3.1, and the prescribed performance control method. Applying either the proposed adaptive control framework or the prescribed performance control framework, the norm of the trajectory tracking error does not violate the user-defined constraint. Applying the classical MRAC framework, the constraint on the trajectory tracking error is violated. Applying Theorem 6.2, the trajectory tracking error converges to zero faster than applying prescribed performance control, which in turn is faster than the trajectory tracking error achieved by applying Theorem 3.1. Remarkably, applying Theorem 6.2, the plot of the norm of the trajectory tracking error $e(\cdot)$ substantially overlaps with the plot of the norm of the trajectory tracking error reference trajectory $e_{ref,transient}(\cdot)$.

where $k_{ppc}, \eta_{ppc}, \sigma_{1,ppc} > 0$ are user-defined,

$$z_{ppc}(t) \triangleq \Lambda_{ppc} \ln \left(1 + \frac{e_{ppc}(t)}{1 - e_{ppc}(t)} \right) + s_{ppc}(t) [[0, 1](x(t) - x_{ref}(t)) - e_{ppc}(t)\dot{\rho}(t)], \tag{46}$$

$$s_{ppc}(t) \triangleq \frac{1}{2\rho(t)} \left[\frac{1}{e_{ppc}(t) + 1} - \frac{1}{e_{ppc}(t) - 1} \right], \tag{47}$$

$\Lambda_{ppc} > 0$ is user-defined, $e_{ppc}(t) \triangleq \rho^{-1}(t)[1, 0][x(t) - x_{ref}(t)]$, $\rho(t) \triangleq (\rho_0 - \rho_\infty)e^{-\lambda_{ppc}(t-t_0)} + \rho_\infty$ captures user-defined constraints on the trajectory tracking error, $\rho_0, \rho_\infty > 0$, $\lambda_{ppc} > 0$ captures the user-defined decay rate,

$$\dot{\hat{\Theta}}_{ppc}(t) = \Gamma_{1,ppc}s_{ppc}(t) \left(\frac{z_{ppc}^2(t)}{2\eta_{ppc}^2} \Phi^T(t, x(t))\Phi(t, x(t)) - \sigma_{2,ppc}\hat{\Theta}_{ppc}(t) \right), \quad \hat{\Theta}_{ppc}(0) = \hat{\Theta}_{ppc,0}, \tag{48}$$

$$\dot{\hat{\epsilon}}_{ppc}(t) = \Gamma_{2,ppc}s_{ppc}(t) [|z_{ppc}(t)| - \sigma_{3,ppc}\hat{\epsilon}_{ppc}(t)], \quad \hat{\epsilon}_{ppc}(0) = \hat{\epsilon}_{ppc,0}, \tag{49}$$

and $\Gamma_{1,ppc}, \Gamma_{2,ppc}, \sigma_{2,ppc}, \sigma_{3,ppc} > 0$ are user-defined. For additional details, see [13,14,59]. To ensure that the system converges to the reference model before the transient has decayed, the decay rate of the constraint function is set as $\lambda_{ppc} = -\text{Re}(\lambda_{\min}(A_{ref}))$, and the additional user-defined parameters are $k_{ppc} = 1, \eta_{ppc} = 0.01, \sigma_{ppc} = 0.1, \Lambda_{ppc} = 1, \Gamma_{ppc1} = \Gamma_{ppc2} = 1000, \rho_0 = 1$, and $\rho_\infty = 0.03$.

Fig. 7 shows the norm of the trajectory tracking error applying the Theorem 3.1, that is, Eq. (11) with adaptive law (12), the MRAC law for prescribed performance proposed in

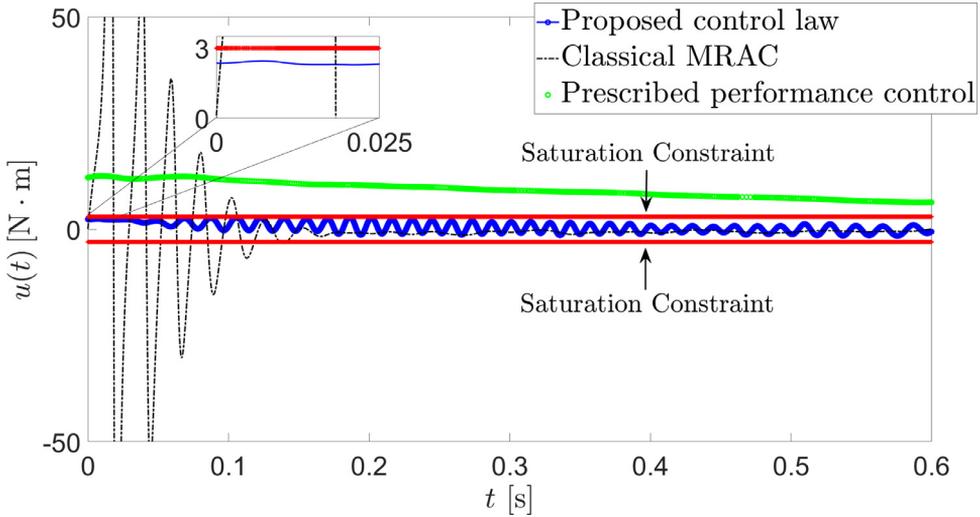


Fig. 8. Control inputs obtained applying Theorems 6.2, 3.1, and the prescribed performance control method. Applying the proposed adaptive control framework, the control input verifies the saturation constraints. These constraints are violated applying both the classical MRAC law and the prescribed performance control technique.

Theorem 6.2, that is, Eq. (26) with adaptive laws (40) and (41), and prescribed performance control, that is, Eq. (45) with adaptive laws (48) and (49). Employing either Theorem 6.2 or the prescribed performance control technique, the trajectory tracking error does not violate the constraint captured by Eqs. (29) and (32), whereas applying Theorem 3.1, the constraint on the trajectory tracking error is violated. Applying Theorem 6.2, the trajectory tracking error converges to zero faster than applying either Theorem 3.1 or using the prescribed performance control method, and the prescribed performance control guarantees a higher convergence rate than the classical MRAC law. It is worthwhile to note that the plot of the norm of the trajectory tracking error $e(\cdot)$ substantially overlaps with the plot of the norm of the trajectory tracking error reference trajectory $e_{\text{ref,transient}}(\cdot)$.

Fig. 8 shows the control input applying the proposed framework, the classical MRAC law, and the prescribed performance control law. It is apparent that, applying Theorem 6.2, the control input verifies the saturation constraints captured by Eqs. (30) and (33) at all time. These constraints, however, are exceeded by both the classical MRAC law and the prescribed performance control law. Applying (26) with adaptive laws (40) and (41) and setting $\lambda_{\text{ppc}} = -\lambda_{\text{min}}(A_{\text{transient}})$, even faster convergence can be achieved. However, simulation results show that, in this case, the trajectory tracking error violates the user-defined constraints due to large excursions of the error velocity, and even larger control inputs are realized. The differentiator employed in this example to approximate the time derivative of $h_u(\phi(t)) = \eta_u^2 - \phi^2(t)$, $t \in [0, \infty)$, is based on the classical Parks-McClellan optimal finite impulse response filter [51, Ch. 7].

7. Conclusion

In this paper, we presented a unified MRAC framework, which guarantees that both the trajectory tracking error and the adaptive gains are uniformly bounded, the trajectory tracking

error asymptotically converges to zero, the plant’s actuators do not saturate, the plant’s trajectory does not exceed its null-controllable region, and the plant’s trajectory tracking error reaches the reference trajectory at the user-defined rate of convergence. To the authors’ best knowledge, this result is the first of this kind.

Future work directions involve extending the proposed direct MRAC laws for prescribed performance to estimate the unknown plant parameters. These mixed direct-indirect adaptive controls will be produced extending the concurrent learning adaptive control and the composite learning adaptive control frameworks.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgment

This work was partly supported by the National Science Foundation, DARPA, and the Office of Naval Research under Grants no. 1700640, D18AP00069, and N00014-19-1-2422, respectively.

Appendix A

Proof of Proposition 2.1: The solution of Eq. (2) is given by $x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^t e^{A(t-\tau)}Bu(\tau)d\tau$, $t \geq t_0$ [60, p. 33]. Thus, applying the triangle inequality and the Cauchy–Schwarz inequality, it holds that

$$\|x(t)\| \leq \|e^{A(t-t_0)}\| \|x_0\| + \|B\| \sup_{t \in [t_0, \infty)} \|u(t)\| \int_{t_0}^t \|e^{A(t-\tau)}\| d\tau, \quad t \geq t_0. \tag{50}$$

Now, it follows from [61] that there exist $\xi : [t_0, \infty) \rightarrow \mathbb{R}$ and $\gamma_1 \geq 1$ such that $\|e^{A(t-t_0)}\| \leq \gamma_1 e^{\xi(t)(t-t_0)}$, $t \geq t_0$, where $\lim_{t \rightarrow \infty} (\xi(t) - \text{Re}(\lambda_{\max}(A))) = 0^+$. Hence, setting $u(t) = 0$, it follows from Eq. (1) that $\alpha_{\max}(x(\cdot)) = -\text{Re}(\lambda_{\max}(A))$. Furthermore, since $\|I_n\| = 1$, it follows from Fact 11.18.8 of [62] that $\|e^{A(t-\tau)}\| \leq \gamma_1 e^{\gamma_2 \text{Re}(\lambda_{\max}(A))(t-\tau)}$ for all $t \in [t_0, \infty)$ and for all $\tau \in [t_0, t]$, where $\gamma_2 \in (0, 1)$. Thus, Eq. (3) follows from Eq. (50) and the fact that

$$\int_{t_0}^t e^{\gamma_2 \text{Re}(\lambda_{\max}(A))(t-\tau)} d\tau \leq \int_{t_0}^{\infty} e^{\gamma_2 \text{Re}(\lambda_{\max}(A))(t-\tau)} d\tau = -(\gamma_2 \text{Re}(\lambda_{\max}(A)))^{-1}, \quad t \geq t_0.$$

Proof of Theorem 4.1: Consider the Lyapunov function candidate

$$V(t, e, \hat{K}, \hat{K}_g) \triangleq e^T P e + \text{tr}(\Delta K^T \Gamma^{-1} \Delta K \Lambda) + \text{tr}(\hat{K}_g^T \Gamma_g^{-1} \hat{K}_g \Lambda), \tag{51}$$

$$(t, e, \hat{K}, \hat{K}_g) \in [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^{(n+m+N) \times m} \times \mathbb{R}^{p \times m},$$

where $P \in \mathbb{R}^{n \times n}$ denotes the symmetric, positive-definite solution of Eq. (10). It follows from Eqs. (10) and (19) that

$$\dot{V}(t, e, \hat{K}, \hat{K}_g) = -e^T Q e + 2\text{tr}(\Delta K^T \Gamma^{-1} \dot{\hat{K}}(t) \Lambda) + 2\text{tr}(\Delta K^T \pi(t, x(t), r(t)) e^T P B \Lambda)$$

$$\begin{aligned}
 &+ 2\text{tr}\left(\hat{K}_g^T g(t, x(t), e, r(t))e^T P B \Lambda\right) \\
 &+ 2\text{tr}\left(\hat{K}_g^T \Gamma_g^{-1} \dot{\hat{K}}_g(t) \Lambda\right),
 \end{aligned} \tag{52}$$

for all $(t, e, \hat{K}, \hat{K}_g) \in [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^{(n+m+N) \times m} \times \mathbb{R}^{p \times m}$, and, applying the adaptive laws (12) and (17), it holds that

$$\dot{V}(t, e, \hat{K}, \hat{K}_g) = -e^T Q e, \tag{53}$$

which is non-positive definite in $(t, e, \hat{K}, \hat{K}_g)$. Therefore, it follows from Corollary 4.1 of [43] that the nonlinear time-varying dynamical system given by Eqs. (19), (12), and (17) is uniformly Lyapunov stable and hence, $e(\cdot)$, $\hat{K}(\cdot)$, and $\hat{K}_g(\cdot)$ are uniformly bounded.

Next, we apply Barbalat’s lemma [43, Lemma 4.1] to prove uniform convergence of the closed-loop trajectory tracking error $e(\cdot)$ to zero. To this goal, we must prove that $\dot{V}(t, e(t), \hat{K}(t), \hat{K}_g(t))$, $t \geq t_0$, is uniformly continuous, and a sufficient condition for the uniform continuity of $\dot{V}(t, e(t), \hat{K}(t), \hat{K}_g(t))$ is that $\ddot{V}(t, e(t), \hat{K}(t), \hat{K}_g(t))$ is bounded for all $t \geq t_0$ [43, p. 506]. To prove boundedness of the second time derivative of Eq. (51) along the trajectories of Eqs. (12), (17), and (19), we note that since A_{ref} is Hurwitz and $r(t)$, $t \geq t_0$, is bounded, it follows from Eq. (5) that both $x_{\text{ref}}(t)$ and $\dot{x}_{\text{ref}}(t)$ are bounded [43, p. 245]. Hence, $x(t) = e(t) + x_{\text{ref}}(t)$, $t \geq t_0$, is bounded. Furthermore, since $g(\cdot, x, e, r)$ is bounded by assumption for all $t \in [t_0, \infty)$, it follows from Eq. (16) that $u(\cdot)$ given by Eq. (18) is bounded and hence, it follows from Eq. (4) that $\dot{x}(\cdot)$ is bounded. Therefore, $\dot{e}(t) = \dot{x}(t) - \dot{x}_{\text{ref}}(t)$, $t \geq t_0$, is bounded and $\ddot{V}(t, e(t), \hat{K}(t), \hat{K}_g(t)) = -2e^T(t) Q \dot{e}(t)$ is bounded along the trajectories of Eqs. (12), (17), and (19). Consequently, $\dot{V}(t, e(t), \hat{K}(t), \hat{K}_g(t))$, $t \geq t_0$, is uniformly continuous. Hence, it follows from Barbalat’s lemma that $\dot{V}(t, e(t), \hat{K}(t), \hat{K}_g(t)) \rightarrow 0$ as $t \rightarrow \infty$ and hence, it follows from (53) that $e(t) \rightarrow 0$ as $t \rightarrow \infty$, which concludes the proof.

Proof of Lemma 5.1: Consider the Lyapunov function candidate

$$\begin{aligned}
 V(t, \varepsilon, \hat{K}, \hat{K}_g) &= \varepsilon^T P_{\text{transient}} \varepsilon + \text{tr}\left(\widetilde{\Delta K}^T \widetilde{\Gamma}^{-1} \widetilde{\Delta K} \Lambda\right), \\
 (t, \varepsilon, \hat{K}, \hat{K}_g) &\in [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^{(n+m+N) \times m} \times \mathbb{R}^{n \times m},
 \end{aligned} \tag{54}$$

where $\widetilde{\Delta K}(t) \triangleq [\Delta K(t), \Delta K_g(t)]$ and $\widetilde{\Gamma} \triangleq \text{blockdiag}(\Gamma, \Gamma_g)$. Analyzing the time derivative of Eq. (54) along the trajectories of Eqs. (21), (23), and (24) and by reasoning as in the proof of Theorem 4.1, it is possible to prove that $\varepsilon(\cdot)$, $\hat{K}(\cdot)$, and $\hat{K}_g(\cdot)$ are uniformly bounded, and $\varepsilon(t) \rightarrow 0$ as $t \rightarrow \infty$ uniformly in $t_0 \in [0, \infty)$. Moreover, it follows from the triangle inequality [43, Def. 2.6] that

$$\|\varepsilon(t)\| \geq \left| \|e(t)\| - \|e_{\text{ref,transient}}(t)\| \right|, \quad t \geq t_0, \tag{55}$$

and, since $e_{\text{ref,transient}}(\cdot)$ is uniformly bounded and $e_{\text{ref,transient}}(t) \rightarrow 0$ as $t \rightarrow \infty$, $e(\cdot)$ is uniformly bounded and $e(t) \rightarrow 0$ as $t \rightarrow \infty$ uniformly in $t_0 \in [0, \infty)$.

Proof of Lemma 5.2: It follows from Lemma 5.1 that $\varepsilon(\cdot)$, $e(\cdot)$, $\Delta K(\cdot)$, and $\Delta K_g(\cdot)$ are uniformly bounded. Furthermore, both $r(\cdot)$ and $\Phi(\cdot, x)$ are bounded in t over $[t_0, \infty)$ by assumption, and since $x(t) = e(t) + x_{\text{ref}}(t)$, $t \geq t_0$, $x(\cdot)$ is uniformly bounded. Therefore, $\Delta K^T(t) \pi(t, x(t), r(t)) + \Delta K_g^T(t) e(t)$, $t \geq t_0$, can be considered as a bounded input for Eq. (21), and since $e(t) = \varepsilon(t) + e_{\text{ref,transient}}(t)$, it follows from Eqs. (20), (21), the fact that $\log(a + b) = \log a + \log(1 + ba^{-1})$, $a, b > 0$, and Proposition 2.1 that there exist

$\xi : [t_0, \infty) \rightarrow \mathbb{R}$, $\gamma_1 \geq 1$, and $\gamma_2 \in (0, 1)$ such that $\lim_{t \rightarrow \infty} \xi(t) = \text{Re}(\lambda_{\max}(A_{\text{transient}}))$ and

$$\begin{aligned} & \log(\|e(t)\|) \\ & \leq \log(\gamma_1 \|x_0 - x_{\text{ref},0}\|) + \xi(t)(t - t_0) \\ & + \log\left(1 - \frac{\gamma_1 \|B\Lambda\|}{\gamma_2 \text{Re}(\lambda_{\max}(A_{\text{transient}}))} \sup_{\tau \in [t_0, \infty)} \|\Delta K^T(\tau)\pi(\tau, x(\tau), r(\tau))\right. \\ & \left. + \Delta K_g^T(\tau)(\varepsilon(\tau) + e_{\text{ref},\text{transient}}(\tau))\| \cdot [e^{\xi(t)(t-t_0)} \gamma_1 \|x_0 - x_{\text{ref},0}\|]^{-1}\right), \quad t \geq t_0. \end{aligned} \tag{56}$$

Now, it follows from Lemma 5.1 that

$e(t) \rightarrow 0$ as $t \rightarrow \infty$ uniformly in $t_0 \in [0, \infty)$. Therefore, since $\lim_{t \rightarrow \infty} \frac{\log(1 - ke^{\lambda(t-t_0)})}{t_0 - t} = 0$ for all $k > 0$ and $\lambda < 0$, and $\lim_{t \rightarrow \infty} \xi(t) = \text{Re}(\lambda_{\max}(A_{\text{transient}}))$, it follows from Eq. (56) and Definition 2.1 that $\alpha_{\max}(e(\cdot)) \geq -\text{Re}(\lambda_{\max}(A_{\text{transient}})) > -\text{Re}(\lambda_{\min}(A_{\text{ref}})) \geq -\text{Re}(\lambda_{\max}(A_{\text{ref}})) > 0$, and the result is proven.

Proof of Theorem 6.1: The proof of this result is divided in two parts. Firstly, we assume that $(e(t), \phi(t)) \in \mathring{\mathcal{E}} \times \mathring{\mathcal{U}}$, $t \geq t_0$, and proceed as in the proof of Theorem 4.1 to prove both the boundedness of $e(\cdot)$, $\hat{K}(\cdot)$, and $\hat{K}_g(\cdot)$, and the asymptotic convergence of $e(\cdot)$ to zero. Then, we use a contradiction argument to prove that if $(e(t_0), \phi(t_0)) \in \mathring{\mathcal{E}} \times \mathring{\mathcal{U}}$, then $(e(t), \phi(t)) \in \mathring{\mathcal{E}} \times \mathring{\mathcal{U}}$, $t \geq t_0$.

Let Q be symmetric, positive-definite, and such that $Q \geq \zeta I_n$, where $\zeta > 0$. Define also

$$Q_1(e) \triangleq V_e(e)h'_e(e^T M e)(A_{\text{ref}}^T M + M A_{\text{ref}}), \quad e \in \mathring{\mathcal{E}}, \tag{57}$$

$$\tilde{Q}(e) \triangleq Q + Q_1(e), \tag{58}$$

where $V_e(\cdot)$ is given by Eq. (35). It follows from Eq. (31) that $A_{\text{ref}}^T M + M A_{\text{ref}}$ is symmetric and nonpositive-definite and hence, since $h'_e(e^T M e) \leq 0$, $e \in \mathring{\mathcal{E}}$, $Q_1(\cdot)$ is symmetric and nonnegative-definite. Therefore, $\tilde{Q}(\cdot)$ is symmetric, $\tilde{Q}(e) \geq \zeta I_n$, $e \in \mathring{\mathcal{E}}$, and it follows from Eqs. (57), (58), and (10) that

$$-\dot{\tilde{Q}}(e) = A_{\text{ref}}^T [P - V_e(e)h'_e(e^T M e)M] + [P - V_e(e)h'_e(e^T M e)M]A_{\text{ref}}, \quad e \in \mathring{\mathcal{E}}. \tag{59}$$

Next, assume that $(e(t), \phi(t)) \in \mathring{\mathcal{E}} \times \mathring{\mathcal{U}}$, $t \geq t_0$, and consider the barrier Lyapunov function candidate

$$V(t, e, \hat{K}, \hat{K}_g) = V_e(e) + \text{tr}(\Delta K^T \Gamma^{-1} \Delta K) + V_g(t, \hat{K}_g), \tag{60}$$

for all $(t, e, \hat{K}, \hat{K}_g) \in [t_0, \infty) \times \mathring{\mathcal{E}} \times \mathbb{R}^{(n+m+N) \times m} \times \mathbb{R}^{p \times m}$. Since $z_1^T z_2 = \text{tr}(z_2 z_1^T)$ for all $z_1, z_2 \in \mathbb{R}^n$, it follows from Eqs. (10), (19), (59), (39), (37), and (38) that

$$\begin{aligned} \dot{V}(t, e, \hat{K}, \hat{K}_g) & = -h_e^{-1}(e^T Q e)e^T \tilde{Q}(e)e \\ & + 2h_e^{-1}(e^T M e)\text{tr}\left(\Delta K^T \pi(t, x(t), r(t))e^T [P - V_e(e)h'_e(e^T M e)M]B\right) \\ & + 2\text{tr}\left(\Delta K^T \Gamma^{-1} \dot{\hat{K}}(t)\right) \\ & + 2h_e^{-1}(e^T M e)\text{tr}\left(\hat{K}_g^T(t)g(t, x(t), e, r(t))e^T [P - V_e(e)h'_e(e^T M e)M]B\right) \end{aligned}$$

$$\begin{aligned}
 & -h_u^{-2}(\phi(t))[\gamma^T(t)h_{du}(t) + \beta(t)]\text{tr}^{\frac{1}{2}}\left(\hat{K}_g^T\Gamma_g^{-1}\hat{K}_g\right) \\
 & +h_u^{-1}(\phi(t))\text{tr}^{-\frac{1}{2}}\left(\hat{K}_g^T\Gamma_g^{-1}\hat{K}_g\right)\text{tr}\left(\hat{K}_g^T\Gamma_g^{-1}\dot{\hat{K}}_g(t)\right) \\
 & = -h_e^{-1}(e^T Q e)e^T \tilde{Q}(e)e + \frac{\chi - \beta(t)}{h_u^2(\phi(t))}\text{tr}^{\frac{1}{2}}\left(\hat{K}_g^T\Gamma_g^{-1}\hat{K}_g\right), \\
 & (t, e, \hat{K}, \dot{\hat{K}}_g) \in [t_0, \infty) \times \mathcal{E} \times \mathbb{R}^{(n+m+N) \times m} \times \mathbb{R}^{p \times m}.
 \end{aligned} \tag{61}$$

Therefore, $\dot{V}(\cdot, \cdot, \cdot, \cdot)$ is non-positive definite and, by proceeding as in the proof of [Theorem 4.1](#), we verify that if $(e(t), \phi(t)) \in \mathring{\mathcal{E}} \times \mathring{\mathcal{U}}, t \geq t_0$, then $e(\cdot), \hat{K}(\cdot)$, and $\dot{\hat{K}}_g(\cdot)$ are uniformly bounded, and $e(t) \rightarrow 0$ as $t \rightarrow \infty$ uniformly in $t_0 \in [0, \infty)$.

Next, we prove that if $e(t_0) \in \mathring{\mathcal{E}}$ and $u(t) \in \mathring{\mathcal{U}}, t \geq t_0$, with $u(\cdot)$ given by [Eq. \(18\)](#), then $e(t) \in \mathring{\mathcal{E}}$. To this goal, assume *ad absurdum* that there exists a finite-time $T_1 > t_0$ such that $h_e(e^T(T_1)Me(T_1)) = 0$, that is, such that $e(T_1) \in \partial\mathcal{E}$. Since $0 \in \mathring{\mathcal{E}}$ and P is symmetric and positive-definite, it holds that $\lim_{t \rightarrow T_1^-} e^T(t)Pe(t) = e^T(T_1)Pe(T_1) > 0$. Therefore, it follows from [Eq. \(35\)](#) that $V_e(e(t)) \rightarrow \infty$ as $t \rightarrow T_1^-$ and, since $V(\cdot, \cdot, \cdot, \cdot)$ is the sum of positive-definite terms, $V(t, e(t), \hat{K}(t), \dot{\hat{K}}_g(t)) \rightarrow \infty$ as $t \rightarrow T_1^-$. However, it follows from [Eq. \(61\)](#) that $V(t, e(t), \hat{K}_x(t), \dot{\hat{K}}_g(t)), t \geq t_0$, is uniformly bounded along the trajectories of [Eqs. \(19\), \(37\), and \(38\)](#), which is a contradiction. Thus, $e(t) \in \mathring{\mathcal{E}}, t \geq t_0$.

Next, we prove that if $\phi(t_0) \in \mathring{\mathcal{U}}$ and $e(t) \in \mathring{\mathcal{E}}, t \geq t_0$, then $\phi(t) \in \mathring{\mathcal{U}}$. To this goal, assume *ad absurdum* that there exists a finite-time $T_2 > t_0$ such that $h_u(\phi(T_2)) = 0$, that is, such that $\phi(T_2) \in \partial\mathcal{U}$. Now, two alternative cases must be considered, namely $\text{tr}\left(\hat{K}_g^T(T_2)\Gamma_g^{-1}\hat{K}_g(T_2)\right) \neq 0$ and $\text{tr}\left(\hat{K}_g^T(T_2)\Gamma_g^{-1}\hat{K}_g(T_2)\right) = 0$. If $\lim_{t \rightarrow T_2^-} h_u(\phi(t)) = 0$ and $\lim_{t \rightarrow T_2^-} \text{tr}\left(\hat{K}_g^T(t)\Gamma_g^{-1}\hat{K}_g(t)\right) \neq 0$, then it follows from [Eq. \(36\)](#) that $V_g(t, \hat{K}_g(t)) \rightarrow \infty$ as $t \rightarrow T_2^-$, which implies that $V(t, e(t), \hat{K}(t), \dot{\hat{K}}_g(t)) \rightarrow \infty$ as $t \rightarrow T_2^-$. However, this conclusion contradicts the uniform boundedness of $V(t, e(t), \hat{K}(t), \dot{\hat{K}}_g(t)), t \geq t_0$, and hence $\phi(t) \in \mathring{\mathcal{U}}$. Alternatively, if $h_u(\phi(T_2)) = 0$ and $\text{tr}\left(\hat{K}_g^T(T_2)\Gamma_g^{-1}\hat{K}_g(T_2)\right) = 0$, then it follows from l’Hôpital’s rule that

$$\lim_{t \rightarrow T_2^-} V_g(t, \hat{K}_g(t)) = \lim_{t \rightarrow T_2^-} \frac{\frac{d}{dt}\text{tr}^{\frac{1}{2}}\left(\hat{K}_g^T(t)\Gamma_g^{-1}\hat{K}_g(t)\right)}{\dot{h}_u(\phi(t))}, \tag{62}$$

and it follows from [Eq. \(38\)](#) that

$$\lim_{t \rightarrow T_2^-} \frac{d}{dt}\text{tr}^{\frac{1}{2}}\left(\hat{K}_g^T(t)\Gamma_g^{-1}\hat{K}_g(t)\right) = \lim_{t \rightarrow T_2^-} [(\gamma^T(t)h_{du}(t) + \chi)V_g(t, \hat{K}_g(t))], \tag{63}$$

$$\lim_{t \rightarrow T_2^-} \dot{h}_u(\phi(t)) = \omega(T_2) + \lim_{t \rightarrow T_2^-} \left[\frac{\gamma^T(t)h_{du}(t) + \chi}{h_u(\phi(t))} \cdot \frac{\partial h_u(\phi(t))}{\partial u} \hat{K}_g^T(t)g(t, x(t), e(t), r(t)) \right], \tag{64}$$

where $\omega(t) \triangleq \frac{\partial h_u(\phi(t))}{\partial u} \left(\dot{\hat{K}}^T(t)\pi(t, x(t), r(t)) + \hat{K}^T(t)\dot{\pi}(t, x(t), r(t)) \right), t \geq t_0$. Now, two instances must be considered, namely $\omega(T_2) \neq \gamma^T(T_2)h_{du}(T_2) + \chi$ and $\omega(T_2) = \gamma^T(T_2)h_{du}(T_2) + \chi$. Assume that $\omega(T_2) \neq \gamma^T(T_2)h_{du}(T_2) + \chi$. In this case, since $\gamma^T(t)h_{du}(t)$ is continuous over $t \in [t_0, T_2]$, it follows from the Weierstrass’ theorem

[43, Th. 2.11] that $\gamma^T(t)h_{du}(t)$ is bounded over $[t_0, T_2]$. Thus, it follows from Eqs. (62)–(64) that $\lim_{t \rightarrow T_2^-} V_g^{-1}(t, \hat{K}_g(t)) = \left[1 - \frac{\omega(T_2)}{\gamma^T(T_2)h_{du}(T_2) + \chi} \right]^{-1} \lim_{t \rightarrow T_2^-} \frac{\partial h_u(\phi(t))}{\partial u} \text{sign}(\hat{K}_g^T(t))g(t)$, $x(t), e(t), r(t) = 0$, where the *signum function* of \hat{K}_g is defined so that if $\hat{K}_g \neq 0$, then $\text{sign}(\hat{K}_g) = \hat{K}_g \|\hat{K}_g\|_{F, \Gamma_g^{-1}}^{-1}$, and if $\hat{K}_g = 0$, then $\text{sign}(\hat{K}_g) = 0$, and $\|\hat{K}_g\|_{F, \Gamma_g^{-1}} \triangleq \text{tr}^{\frac{1}{2}}(\hat{K}_g^T \Gamma_g^{-1} \hat{K}_g)$ denotes the *weighted Frobenius norm* of \hat{K}_g . Therefore, if $\omega(T_2) \neq \gamma^T(T_2)h_{du}(T_2) + \chi$, then $\lim_{t \rightarrow T_2^-} V_g(t, \hat{K}_g(t)) = \infty$, which contradicts the uniform boundedness of $V(t, e(t), \hat{K}_x(t), \hat{K}_g(t))$, $t \geq t_0$, along the trajectories of Eqs. (19), (37), and (38). Alternatively, if $\omega(T_2) = \gamma^T(T_2)h_{du}(T_2) + \chi$, then it follows from Eqs. (62) and (63) that $\lim_{t \rightarrow T_2^-} \dot{h}_u(\phi(t)) = \lim_{t \rightarrow T_2^-} \gamma^T(t)h_{du}(t) + \chi$. However, in this case, it follows from Eq. (39) that $\lim_{t \rightarrow T_2^-} \beta(t) = \chi$, which contradicts the assumption that χ is such that $\chi < -\beta_{\max}$. Therefore, $\phi(t) \in \mathring{U}$, $t \geq t_0$.

Using similar arguments, the proof is completed by showing that if $(e(t_0), \phi(t_0)) \in \mathring{E} \times \mathring{U}$ and Assumptions 6.1 and 6.2 hold, then $(e(t), \phi(t)) \in \mathring{E} \times \mathring{U}$, $t \geq t_0$.

References

- [1] E. Lavretsky, K. Wise, *Robust and Adaptive Control: With Aerospace Applications*, Springer, London, UK, 2012.
- [2] B. Peterson, K. Narendra, Bounded error adaptive control, *IEEE Trans. Autom. Control* 27 (6) (1982) 1161–1168, doi:10.1109/TAC.1982.1103112.
- [3] P. Ioannou, P. Kokotovic, *Adaptive Systems with Reduced Models*, Springer, New York, NY, 1983.
- [4] K. Narendra, A. Annaswamy, A new adaptive law for robust adaptation without persistent excitation, *IEEE Trans. Autom. Control* 32 (2) (1987) 134–145, doi:10.1109/TAC.1987.1104543.
- [5] G. Navarro-Guerrero, Y. Tang, Fractional order model reference adaptive control for anesthesia, *Int. J. Adapt. Control Signal Process.* 31 (9) (2017) 1350–1360, doi:10.1002/acs.2769.
- [6] C. Ma, J. Lam, F.L. Lewis, Trajectory regulating model reference adaptive controller for robotic systems, *IEEE Trans. Control Syst. Technol.* 27 (6) (2019) 2749–2756, doi:10.1109/TCST.2018.2858203.
- [7] S.A. Davari, J. Rodriguez, Predictive direct voltage control of induction motor with mechanical model consideration for sensorless applications, *IEEE J. Emerg. Sel. Top. Power Electron.* 6 (4) (2018) 1990–2000, doi:10.1109/JESTPE.2018.2815652.
- [8] E. Arabi, B.C. Gruenwald, T. Yucelen, N.T. Nguyen, A set-theoretic model reference adaptive control architecture for disturbance rejection and uncertainty suppression with strict performance guarantees, *Int. J. Control* 91 (5) (2018) 1195–1208, doi:10.1080/00207179.2017.1312019.
- [9] A. L'Afflitto, Barrier Lyapunov functions and constrained model reference adaptive control, *IEEE Control Syst. Lett.* 2 (3) (2018) 441–446, doi:10.1109/LCSYS.2018.2842148.
- [10] J. Na, G. Herrmann, K. Zhang, Improving transient performance of adaptive control via a modified reference model and novel adaptation, *Int. J. Robust Nonlinear Control* 27 (8) (2017) 1351–1372, doi:10.1002/rnc.3636.
- [11] B. Mirkin, P.-O. Gutman, Tube model reference adaptive control, *Automatica* 49 (4) (2013) 1012–1018, doi:10.1016/j.automatica.2013.01.022.
- [12] M.C. Turner, J. Sofrony, E. Prempain, Anti-windup for model-reference adaptive control schemes with rate-limits, *Syst. Control Lett.* 137 (2020) 104630, doi:10.1016/j.sysconle.2020.104630.
- [13] C.P. Bechlioulis, G.A. Rovithakis, Robust adaptive control of feedback linearizable MIMO nonlinear systems with prescribed performance, *IEEE Trans. Autom. Control* 53 (9) (2008) 2090–2099, doi:10.1109/TAC.2008.929402.
- [14] C.P. Bechlioulis, G.A. Rovithakis, Adaptive control with guaranteed transient and steady state tracking error bounds for strict feedback systems, *Automatica* 45 (2) (2009) 532–538, doi:10.1016/j.automatica.2008.08.012.
- [15] M.A. Duarte, K.S. Narendra, Combined direct and indirect approach to adaptive control, *IEEE Trans. Autom. Control* 34 (10) (1989) 1071–1075, doi:10.1109/9.35278.
- [16] K.S. Narendra, M.A. Duarte, Application of robust adaptive control using combined direct and indirect methods, *Int. J. Adapt. Control Signal Process.* 3 (2) (1989) 131–142, doi:10.1002/acs.4480030206.

- [17] G. Kreisselmeier, Adaptive observers with exponential rate of convergence, *IEEE Trans. Autom. Control* 22 (1) (1977) 2–8, doi:[10.1109/TAC.1977.1101401](https://doi.org/10.1109/TAC.1977.1101401).
- [18] N.T. Nguyen, *Model-Reference Adaptive Control*, Springer, London, UK, 2018.
- [19] P. Ioannou, J. Sun, *Robust Adaptive Control*, Dover, Englewood Cliffs, NJ, 2012.
- [20] G. Chowdhary, E.N. Johnson, Concurrent learning for convergence in adaptive control without persistency of excitation, in: *Proceedings of the IEEE Conference on Decision and Control*, 2010, pp. 3674–3679, doi:[10.1109/CDC.2010.5717148](https://doi.org/10.1109/CDC.2010.5717148).
- [21] G. Chowdhary, T. Yucelen, M. Mühlegg, E.N. Johnson, Concurrent learning adaptive control of linear systems with exponentially convergent bounds, *Int. J. Adapt. Control Signal Process.* 27 (4) (2013) 280–301, doi:[10.1002/acs.2297](https://doi.org/10.1002/acs.2297).
- [22] G. Chowdhary, M. Mühlegg, E.N. Johnson, Exponential parameter and tracking error convergence guarantees for adaptive controllers without persistency of excitation, *Int. J. Control* 87 (8) (2014) 1583–1603, doi:[10.1080/00207179.2014.880128](https://doi.org/10.1080/00207179.2014.880128).
- [23] G.V. Chowdhary, E.N. Johnson, Theory and flight-test validation of a concurrent-learning adaptive controller, *J. Guid. Control Dyn.* 34 (2) (2011) 592–607, doi:[10.2514/1.46866](https://doi.org/10.2514/1.46866).
- [24] J.F. Quindlen, G. Chowdhary, J.P. How, Hybrid model reference adaptive control for unmatched uncertainties, in: *Proceedings of the American Control Conference*, 2015, pp. 1125–1130, doi:[10.1109/ACC.2015.7170884](https://doi.org/10.1109/ACC.2015.7170884).
- [25] Y. Pan, H. Yu, Composite learning from adaptive dynamic surface control, *IEEE Trans. Autom. Control* 61 (9) (2015) 2603–2609.
- [26] Y. Pan, J. Zhang, H. Yu, Model reference composite learning control without persistency of excitation, *IET Control Theory Appl.* 10 (16) (2016) 1963–1971.
- [27] T. Yucelen, G. De La Torre, E.N. Johnson, Improving transient performance of adaptive control architectures using frequency-limited system error dynamics, *Int. J. Control* 87 (11) (2014) 2383–2397, doi:[10.1080/00207179.2014.922702](https://doi.org/10.1080/00207179.2014.922702).
- [28] T.E. Gibson, A.M. Annaswamy, E. Lavretsky, On adaptive control with closed-loop reference models: transients, oscillations, and peaking, *IEEE Access* 1 (2013) 703–717, doi:[10.1109/ACCESS.2013.2284005](https://doi.org/10.1109/ACCESS.2013.2284005).
- [29] J. Yang, Y. Liu, J. Na, G. Gao, Springer, Singapore, 2018, pp. 331–343.
- [30] N.T. Nguyen, Optimal control modification for robust adaptive control with large adaptive gain, *Syst. Control Lett.* 61 (4) (2012) 485–494, doi:[10.1016/j.sysconle.2012.01.009](https://doi.org/10.1016/j.sysconle.2012.01.009).
- [31] A. Datta, P.A. Ioannou, Performance analysis and improvement in model reference adaptive control, *IEEE Trans. Autom. Control* 39 (12) (1994) 2370–2387, doi:[10.1109/9.362856](https://doi.org/10.1109/9.362856).
- [32] J. Yang, J. Na, G. Gao, Robust adaptive control with a modified controller for transient response improvement, in: *Proceedings of the International Conference on Modelling, Identification and Control*, 2017, pp. 929–934, doi:[10.1109/ICMIC.2017.8321589](https://doi.org/10.1109/ICMIC.2017.8321589).
- [33] Y. Yang, X. Chen, C. Li, Transient performance improvement in model reference adaptive control using H_∞ optimal method, *J. Frankl. Inst.* 352 (1) (2015) 16–32, doi:[10.1016/j.jfranklin.2014.09.014](https://doi.org/10.1016/j.jfranklin.2014.09.014).
- [34] H. Wang, J. Sun, Modified model reference adaptive control with saturated inputs, in: *Proceedings of the IEEE Conference on Decision and Control*, volume 4, 1992, pp. 3255–3256, doi:[10.1109/CDC.1992.371227](https://doi.org/10.1109/CDC.1992.371227).
- [35] F.Z. Chaoui, F. Giri, L. Dugard, J.M. Dion, M. M'saad, Adaptive tracking with saturating input and controller integral action, *IEEE Trans. Autom. Control* 43 (11) (1998) 1638–1643, doi:[10.1109/9.728887](https://doi.org/10.1109/9.728887).
- [36] C.C. Palerm, B.W. Bequette, Direct model reference adaptive control and saturation constraints, *IFAC Proc. Vol.* 35 (1) (2002) 409–414, doi:[10.3182/20020721-6-ES-1901.01054](https://doi.org/10.3182/20020721-6-ES-1901.01054). IFAC World Congress
- [37] M. Kanamori, M. Tomizuka, Model reference adaptive control of linear systems with input saturation, in: *Proceedings of the IEEE International Conference on Control Applications*, volume 2, 2004, pp. 1318–1323, doi:[10.1109/CCA.2004.1387556](https://doi.org/10.1109/CCA.2004.1387556).
- [38] E. Lavretsky, N. Hovakimyan, Stable adaptation in the presence of input constraints, *Syst. Control Lett.* 56 (11) (2007) 722–729, doi:[10.1016/j.sysconle.2007.05.002](https://doi.org/10.1016/j.sysconle.2007.05.002).
- [39] E.N. Johnson, A.J. Calise, Neural network adaptive control of systems with input saturation, in: *Proceedings of the American Control Conference*, volume 5, 2001, pp. 3527–3532, doi:[10.1109/ACC.2001.946179](https://doi.org/10.1109/ACC.2001.946179).
- [40] S. Kannan, E.N. Johnson, Adaptive control with a nested saturation reference model, in: *Proceedings of the AIAA Guidance, Navigation, and Control Conference*, 2003, pp. 1–11, doi:[10.2514/6.2003-5324](https://doi.org/10.2514/6.2003-5324).
- [41] G. Kreisselmeier, K. Narendra, Stable model reference adaptive control in the presence of bounded disturbances, *IEEE Trans. Autom. Control* 27 (6) (1982) 1169–1175, doi:[10.1109/TAC.1982.1103093](https://doi.org/10.1109/TAC.1982.1103093).
- [42] J.B. Pomet, L. Praly, Adaptive nonlinear regulation: estimation from the Lyapunov equation, *IEEE Trans. Autom. Control* 37 (6) (1992) 729–740, doi:[10.1109/9.256328](https://doi.org/10.1109/9.256328).

- [43] W.M. Haddad, V. Chellaboina, *Nonlinear Dynamical Systems and Control: A Lyapunov-Based Approach*, Princeton Univ. Press, Princeton, NJ, 2008.
- [44] S. Boyd, L. Ghaoui, E. Feron, V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SIAM, Philadelphia, PA, 1994.
- [45] M. Spong, S. Hutchinson, M. Vidyasagar, *Robot Modeling and Control*, Wiley, New York, NY, 2005.
- [46] A. Loria, E. Panteley, M. Maghenem, Strict Lyapunov functions for model reference adaptive control: application to Lagrangian systems, *IEEE Trans. Autom. Control* 64 (7) (2019) 3040–3045, doi:10.1109/TAC.2018.2874723.
- [47] H.K. Khalil, *Nonlinear Systems*, Prentice Hall, Princeton, NJ, 2002.
- [48] G. Joshi, G. Chowdhary, Hybrid direct-indirect adaptive control of nonlinear system with unmatched uncertainty, in: *Proceedings of the International Conference on Control, Decision and Information Technologies*, 2019, pp. 127–132, doi:10.1109/CoDIT.2019.8820301.
- [49] A. L'Afflitto, *A Mathematical Perspective on Flight Dynamics and Control*, Springer, London, UK, 2017.
- [50] C. Alexander, M. Sadiku, *Fundamentals of Electric Circuits*, McGraw Hill, New York, NY, 2008.
- [51] A.V. Oppenheim, W. Schaffer Ronald, J.R. Buck, *Discrete-Time Signal Processing*, Prentice Hall, Upper Saddle River, NJ, 1999.
- [52] A. Gelb, *Applied Optimal Estimation*, MIT press, Cambridge, MA, 1974.
- [53] T. Homer, M. Mahmood, P. Mhaskar, A trajectory-based method for constructing null controllable regions, *Int. J. Robust Nonlinear Control* 30 (2) (2020) 776–786, doi:10.1002/rnc.4805.
- [54] L. Zaccarian, A.R. Teel, *Modern Anti-windup Synthesis: Control Augmentation for Actuator Saturation*, Princeton University Press, Princeton, NJ, 2011.
- [55] E. Sontag, H.J. Sussmann, Nonsmooth control-Lyapunov functions, in: *Proceedings of the IEEE Conference on Decision and Control*, volume 3, 1995, pp. 2799–2805, doi:10.1109/CDC.1995.478542.
- [56] T. Homer, P. Mhaskar, Using null controllable regions to stabilize nonlinear systems, in: *Proceedings of the American Control Conference*, 2018, pp. 3141–3146, doi:10.23919/ACC.2018.8431531.
- [57] R. Vrabel, Local null controllability of the control-affine nonlinear systems with time-varying disturbances, *Eur. J. Control* 40 (2018) 80–86, doi:10.1016/j.ejcon.2017.12.004.
- [58] E.B. Lee, L. Markus, *Foundations of Optimal Control Theory*, SIAM, Philadelphia, PA, 1967.
- [59] J. Na, Q. Chen, X. Ren, Y. Guo, Adaptive prescribed performance motion control of servo mechanisms with friction compensation, *IEEE Trans. Ind. Electron.* 61 (1) (2013) 486–494.
- [60] P.J. Antsaklis, A.N. Michel, *A Linear Systems Primer*, Birkhäuser, Boston, MA, 2007.
- [61] B. Kågström, Bounds and perturbation bounds for the matrix exponential, *IEEE Trans. Biomed. Eng.* 17 (1) (2015) 39–57, doi:10.1007/BF01932398.
- [62] D.S. Bernstein, *Matrix Mathematics: Theory, Facts, and Formulas*, Second Edition, Princeton University Press, Princeton, NJ, 2009.