Newton Bases and Event-Triggered Adaptive Control in Native Spaces

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***Abstract*— This paper extends recent methods of native space embedding for adaptive control by deriving event-driven controllers that modify the basis used for approximation of the functional uncertainty. Using the power function for the native space, trigger conditions are defined that determine when and how many new basis functions are introduced. Bases are selected using a greedy method and augmentation process at each triggering event. By using a recursive Newton basis, coordinate implementations of the adaptive law do not require the inversion of the leading Grammian matrix, which yields a substantial improvement in numerical efficiency and numerical stability. This paper derives upper bounds on the ultimate tracking performance of the closed-loop control scheme in terms of known functions of the approximation dimension after each triggering event. These upper bounds are quite general and hold for bounded uncertainty classes in a variety of reproducing kernel Hilbert spaces (RKHS). A numerical example shows the applicability of the proposed results.**

***Index Terms*— adaptive control, nonlinear estimation, greedy methods**

I. Introduction

A. Motivation

Within the framework of model reference adaptive control (MRAC), this paper presents event-triggered controllers for nonlinear plants affected by nonlinear, matched, functional uncertainties. Unlike conventional adaptive control strategies with a fixed number of regressor functions, our approach leverages a flexible event-triggered scheme introduced to enhance adaptive estimation of uncertainty. We draw upon methodologies from [1]–[9] to implement adaptive control using reproducing kernel Hilbert spaces (RKHS) for ordinary differential equations (ODEs), focusing on nonparametric uncertainty classes. An event-triggered scheme was recently employed in [10] by one of the authors of this work. However, in this paper, we use an entirely different control scheme based on an error-bounding adaptive compensator predicated on [11]. Furthermore, in contrast to [10], we employ a greedy technique for augmenting bases, maximizing the power function over a specified subset after the trigger event.

B. Problem Statement and Outline of this Work

The typical MRAC problem with matched uncertainty is concerned with systems that are governed by equations of the form

$$\dot{x}(t) = Ax(t)+B(u(t)+f(x(t))+\delta(t), \ \ x(t_0) = x_0, \ \ t \geq t_0, \tag{1}$$

where $x : [t_0, \infty) \to \mathbb{R}^n$ denotes the *plant state*, $u : [t_0, \infty) \to \mathbb{R}$ denotes the *control input*, $f : \mathbb{R}^n \to \mathbb{R}$ denotes the *matched uncertainty*, $\delta : \mathbb{R} \to \mathbb{R}^n$ is the unmodeled disturbance in the system bounded by $\delta_{\max} \geq \delta(t)$ for all $t \geq 0$, $A \in \mathbb{R}^{n\times n}$ denotes the *system matrix*, and $B \in \mathbb{R}^n$ denotes the *control matrix*. According to the MRAC methodology, we aim to design a control law for $u(\cdot)$ in (1) to steer the plant trajectory $x(\cdot)$ toward the reference trajectory $x_{\mathrm{ref}} : [t_0, \infty) \to \mathbb{R}^n$, where

$$\dot{x}_{\mathrm{ref}}(t) = A_{\mathrm{ref}}x_{\mathrm{ref}}(t) + B_{\mathrm{ref}}r(t), \ \ x_{\mathrm{ref}}(t_0) = x_{\mathrm{ref},0}, \ \ t \geq t_0, \tag{2}$$

denotes the *reference model*, $A_{\mathrm{ref}} \in \mathbb{R}^{n\times n}$ is Hurwitz and denotes the *reference state matrix*, $B_{\mathrm{ref}} \in \mathbb{R}^n$ denotes the *reference control input matrix*, and the reference *command input* $r : [t_0, \infty) \to \mathbb{R}$ is continuous and such that $|r(t)| \leq r_{\max}$ for all $t \geq t_0$ and for some $r_{\max} \geq 0$.

To assure the compatibility between the plant model (1) and the reference model (2), it is assumed that the *matching conditions* $A_{\mathrm{ref}} = A + BK_x^{\mathrm{T}}$ and $B_{\mathrm{ref}} = BK_r$ are verified for some $K_x \in \mathbb{R}^n$ and $K_r \in \mathbb{R}$. This holds for any single input systems such as those studied in this paper. To simplify the analysis in this study, we assume that A and B are known and aim to steer the plant state despite matched uncertainties through feedback linearization. However, in the general case, A and f are unknown and the control matrix is expressed in the form $B\Lambda$, where Λ is diagonal, positive-definite, and unknown. In this case, the solutions of the matching conditions K_x and K_r are unknown, and the adaptive gains $\hat{K}_x$ and $\hat{K}_r$ are introduced in the control law to account for these uncertainties; for details, see [12, Ch. 9].

The primary focus of this paper is on designing adaptive laws to account for the matched uncertainty f, which is assumed to be an element of an infinite-dimensional RKHS as has been done in several other of our recent works [13], [14]. The primary advantage of this formulation is that we do not assume that the nonlinear matched uncertainty is parameterized using a regressor vector of fixed dimension, as it occurs in classical MRAC, or estimated online through neural networks or other data-driven methods. The functional uncertainty is just a member of an RKHS. By leveraging the convergence properties of this algorithm, we derive tracking error convergence results using an adaptive control law with some robust modifications.

This paper is organized as follows. Section I-C broadly discusses the related work and the major contribution of this

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paper. Section II presents key background material in the area of RKHSs, which is needed for the scope of this paper. We then give more detail into the event-driven control scheme in Section III as well as the iterative basis augmentation strategy in Section V. The detailed theoretical results are given in Section V, and an illustrative numerical example is given in VI followed by concluding remarks in Section VII.

C. Relevance of the Proposed Results

In recent research endeavors by the authors, MRAC strategies intended for systems influenced by nonlinear uncertainties in an RKHS have been explored [9], [13], [14]. These approaches operate under the premise that a known upper bound exists for the largest functional uncertainty magnitude, expressed in the norm of the RKHS. Nevertheless, prior studies have some limitations. To be implemented, the infinite-dimensional adaptive laws proposed in [9], [13], [14] are approximated on finite-dimensional Hilbert spaces defined by a fixed choice of basis functions. Having a pre-determined basis introduces a constraint, namely the uniform ultimate boundedness of the trajectory tracking error depends on the chosen basis.

The novel approach introduced in this paper brings forth several distinct advantages that enhance its flexibility and applicability in control design. Foremost, leveraging a greedy kernel selection strategy inspired by the works of [15], [16], the proposed framework eliminates the need for manually specifying kernel centers. Instead, the user defines a bounded set, which is estimated to contain the system's dynamics, and determines the desired level of discretization within this set. This streamlined process simplifies control design considerably. Remarkably, assuming to estimate a bounded set containing the closed-loop trajectory is a common implicit assumption in MRAC whenever a finite-dimensional regressor vector is employed to capture nonlinearities. Indeed, for problems of practical interest, the approximation provided by the regressor vector is effective only within some region of the state space.

Furthermore, the ability to update a set of basis functions via an optimality principle can lead to a significant reduction in the dimensionality of the RKHS approximation. By choosing a sparse set of kernel centers, we effectively manage the computational complexity typically associated with high-dimensional state spaces. This computational efficiency renders the proposed MRAC approach not only powerful but also well-suited for real-time applications, where timely control responses are imperative.

II. BACKGROUND

A. Native Space Embedding

A symmetric, nonnegative, bounded on the diagonal and continuous kernel is denoted by $\mathfrak{K} : X \times X \to \mathbb{R}$, where $X \triangleq \mathbb{R}^n$. This kernel function induces a scalar-valued native space $\mathcal{H}$ consisting of functions defined over the set X. In this paper, we let $\mathcal{H} \triangleq \overline{\text{span}\{\mathfrak{K}(x,\cdot) \mid x \in X\}}$, which is an RKHS of real-valued functions over X.

To produce adaptive laws in finite-dimensional spaces of dimension N, we introduce the *space of approximants*

$$\mathcal{H}_N \triangleq \text{span}\left\{\mathfrak{K}_{\xi_i} \mid \xi_i \in \Xi_N, 1 \leq i \leq N\right\}, \tag{3}$$

where $\Xi_N \subset \Omega$ is a set of centers and $\mathfrak{K}_{\xi_i}(\cdot) \triangleq \mathfrak{K}(\xi_i, \cdot)$ is a basis function centered at ξ_i. We define $\Pi_N : \mathcal{H} \to \mathcal{H}_N$ as the *$\mathcal{H}$-orthogonal projection onto $\mathcal{H}_N \subseteq \mathcal{H}$*, and we use Π_N to build approximations. The evaluation functional at a point $x \in X$ is denoted as $E_X : \mathcal{H} \to \mathbb{R}$, and it satisfies $E_x f \triangleq f(x)$ for all $f \in \mathcal{H}$. The adjoint operator, denoted as $\mathbb{E}_x^* : \mathbb{R} \to \mathcal{H}$, acts as the multiplication operator and is defined as $\mathbb{E}_x^* \alpha \triangleq \mathfrak{K}(x,\cdot)\alpha$ for all $\alpha \in \mathbb{R}$. As we discuss in the next section, the dimension N evolves in time so that $N = N(m)$ for a time-varying count m of trigger events.

III. THE EVENT-DRIVEN ADAPTIVE CONTROL LAW

The control strategy in this paper uses a trigger event that signals that we must update the basis functions for improved approximation properties. As time progresses, we obtain a sequence of trigger times $\{t_m\}_{m=1}^{L}$, where L may be either finite or infinite. Let $\mathcal{I}_m \triangleq [t_m, t_{m+1}) \subset [t_0, \infty)$ denote the m-th time interval, and let $\mathcal{H}_{N(m)}$ denote a subspace of dimension $N(m)$ employed during the time interval $\mathcal{I}_m$, where the $N(m)$ centers lie in $\Xi_{N(m)} \triangleq \{\xi_1, \ldots, \xi_{N(m)}\}$.

We set the control input in (1) as

$$u(t) = u_{N(m)}(t) \quad \text{for all } (t,m) \in \mathcal{I}_m \times \{1,\ldots,M\}, \tag{4}$$

where $\tilde{x}(t) \triangleq x(t) - x_{\text{ref}}(t)$ denotes the *trajectory tracking error*,

$$\begin{aligned} u_{N(m)}(t) \triangleq{}& K_x^{\mathrm{T}} x(t) + (1-\mu(x(t)) K_r^{\mathrm{T}} r(t) \\ &- \hat{f}_{N(m)}(t, x(t)) + \mu(x(t)) u_{\text{SL}}(t) \\ &+ v_{N(m)}(x(t), \tilde{x}(t)), \end{aligned} \tag{5}$$

$\hat{f}_{N(m)}(t,\cdot) \in \mathcal{H}_{N(m)} \subset \mathcal{H}$ denotes the adaptive gain associated with the functional uncertainty f at time t, and is such that the adaptive law

$$\begin{aligned} \frac{\partial \hat{f}_{N(m)}(t,\cdot)}{\partial t} &= -\gamma_f \Pi_{N(m)} \mathfrak{K}(x(t),\cdot) B^T P \tilde{x}(t), \\ & \hat{f}_{N(m)}(t_0,\cdot) = f_0, \end{aligned} \tag{6}$$

is verified for all $t \geq t_0$, $\gamma_f > 0$ denotes the *adaptive rate*,

$$v_{N(m)}(\cdot, \tilde{x}(t)) \triangleq \begin{cases} \text{sign}(B^{\mathrm{T}} P \tilde{x}(t)) \bar{e}_{N(m)}(\cdot) & \text{if } |B^{\mathrm{T}} P \tilde{x}(t)| \geq \varepsilon, \\ \frac{1}{\varepsilon} B^{\mathrm{T}} P \tilde{x}(t) \bar{e}_{N(m)}(\cdot) & \text{otherwise}, \end{cases} \tag{7}$$

denotes the *compensator*, $\varepsilon > 0$ is user-defined, $u_{\text{SL}}(t) \triangleq -K_r r_{\max} \text{sgn}(B^{\mathrm{T}} P \tilde{x}(t))$ denotes the *state-limiting term*, $\bar{e}_{N(m)} \triangleq \mathcal{P}_{N(m)}(x) \|f\|_{\mathcal{H}}$, and

$$\mathcal{P}_{N(m)}(x) \triangleq \sqrt{\mathfrak{K}(x,x) - \mathfrak{K}_{\Xi_{N(m)}^{\mathrm{T}}}(x) \mathbb{K}_{N(m)}^{-1} \mathfrak{K}_{\Xi_{N(m)}}(x)} \tag{8}$$

denotes the *power function* of the subspace $\mathcal{H}_{N(m)} \subseteq \mathcal{H}$ [16]–[18], $\mathfrak{K}_{\Xi_{N(m)}}(x) \triangleq [\mathfrak{K}_{\xi_1}(x), \ldots, \mathfrak{K}_{\xi_{N(m)}}(x)]^{\mathrm{T}} \in \mathbb{R}^{N(m)}$ denotes the *vector of basis functions*, and $\mathbb{K}_{N(m)} \triangleq [\mathfrak{K}(\xi_i, \xi_j)] \in \mathbb{R}^{N(m) \times N(m)}$ denotes the *Grammian matrix*.

The *blending function* $\mu(\cdot)$ in (5) is presented in [12, Ch. 12] and omitted for brevity. Defining the functional uncertainty error $\tilde{f}_{N(m)}(t,\cdot) \triangleq f(\cdot) - \hat{f}_{N(m)}(t,\cdot) \in \mathcal{H}$, $t \in \mathcal{I}_{N(m)}$, and using a gradient adaptive law, we obtain the error equations

$$\begin{bmatrix} \dot{\tilde{x}}(t) \\ \frac{\partial \tilde{f}_{N(m)}(t,\cdot)}{\partial t} \end{bmatrix} = \begin{bmatrix} A & BE_{x(t)} \\ -\gamma_f \Pi_{N(m)} \mathfrak{K}(x(t),\cdot) B^{\mathrm{T}} P & 0 \end{bmatrix} \cdot \begin{bmatrix} \tilde{x}(t) \\ \tilde{f}_{N(m)}(t,\cdot) \end{bmatrix} + B \begin{bmatrix} K_x^{\mathrm{T}} x(t) + (1-\mu(x(t))) K_r^{\mathrm{T}} r(t) \\ 0 \end{bmatrix} + B \begin{bmatrix} \mu u_{\mathrm{SL}}(t) + v_{N(m)}(x(t), \tilde{x}(t)) + \delta(t) \\ 0 \end{bmatrix} \tag{9}$$

for $t \in \mathcal{I}_m$, where P denotes the symmetric, positive-definite solution of the *Lyapunov equation*

$$PA_{\mathrm{ref}} + A_{\mathrm{ref}}^{\mathrm{T}} P = -Q \tag{10}$$

and $Q \in \mathbb{R}^{n \times n}$ is user-defined, symmetric, and positive definite. We also study robust modifications of (9).

The governing equations (9) depend on a trigger event that defines both the intervals $\mathcal{I}_m$ and the subspace $\mathcal{H}_{N(m)}$. Let $t > t_0$ and assume that the last trigger event before t occurred at $t_m > t_0$ for some $m \in \mathbb{N}$. The next trigger event depends on four design parameters, namely a *starting time* $T_m > 0$, a *termination time* $\Delta_m > 0$, a compact *subset of interest* $\Omega_m \subset X$, and a *threshold* $c_m > 0$. We say that the trajectory $t \mapsto x(t)$ of the closed-loop system satisfies the *next event trigger at* $t \geq t_m + T_M$ *over* Ω_m *for a threshold*

$$c_m \leq (1-\sigma) \frac{\lambda_{\min}(Q)}{2} \|PB\| \|f\|_{\mathcal{H}}, \tag{11}$$

with $\sigma \in (0,1)$, if

$$\sup_{\xi \in \Omega_m} \mathcal{P}_{N(m)}(\xi) \geq c_m \|\tilde{x}(t)\|. \tag{12}$$

The termination time Δ_m defines when we turn off function adaptation altogether at $t_m + T_m + \Delta_m$ This means that after the event t_m the possible occurrence of the next trigger event is monitored over the interval $[t_m + T_M, t_m + T_M + \Delta_M)$ when Δ_m is finite. It is possible to choose $\Delta_m = \infty$ so that we never stop checking for the next trigger event.

Note that a direct implementation of (8) in the greedy update process implies an $O(N^3(m))$ computational cost owing to the need to factor and solve equations using $\mathbb{K}_{N(m)}$. In implementations, an alternative, computationally more efficient form of the power function in terms of the Newton basis is used [15] as discussed in Section IV.

When the event trigger first occurs at time $t > t_m + T_m$, we set $t_{m+1} = t$ and augment the subset of centers $\Xi_{N(m)}$ to form a new set of centers $\Xi_{N(m+1)}$. We use the greedy algorithm where we recursively add ℓ basis functions according to

$$\xi_{m+\ell} \triangleq \underset{\xi \in \Omega_{N(m+1)}}{\operatorname{argmax}} \; \mathcal{P}_{N(m)+\ell-1}(\xi) \text{ for each } \ell = 1, 2, \ldots, L. \tag{13}$$

This process terminates when the trigger condition no longer holds, that is, when

$$\sup_{\xi \in \Omega_{N(m+1)}} \mathcal{P}_{N(m+1)} \leq c_{m+1} \|\tilde{x}(t)\|,$$

where $N(m+1) \triangleq N(m) + L$.

IV. Implementation and Recursive Newton Bases

Suppose we choose the "standard basis" of translates of the kernel section over the interval $\mathcal{I}_m \triangleq [t_m, t_{m+1})$ for the subspace

$$\mathcal{H}_{N(m)} = \operatorname{span}\left\{ \mathfrak{K}_{\xi_i} \mid \xi_i \in \Xi_{N(m)}, 1 \leq i \leq N(m) \right\}, \tag{14}$$

so that the adaptive gain can be parameterized as $\hat{f}_{N(m)}(t,\cdot) = \sum_{i=1}^{N(m)} \mathfrak{K}_{\xi_i}(\cdot) \hat{\theta}_i(t)$, where $\Xi_{N(m)} \triangleq \left\{ \xi_1, \ldots, \xi_{N(m)} \right\}$. Then, the coordinate implementation of the adaptive law (6) becomes

$$\dot{\hat{\Theta}}(t) = \gamma_f \mathbb{K}_{N(m)}^{-1} \mathfrak{K}_{\Xi_{N(m)}}(x(t)) B^{\mathrm{T}} P \tilde{x}(t), \quad \hat{\Theta}(t_0) = \hat{\Theta}_0, \tag{15}$$

where $\hat{\Theta}(t) \triangleq [\theta_1(t), \ldots, \theta_N(t)]^{\mathrm{T}}$. Again, as in the greedy update calculation, the direct implementation of this learning law implies an $O(N(m)^3)$ computational cost. Also, we often replace (15) with its robust modifications, which are studied in the recent papers [1]–[9], [11], [19].

In addition to the computational cost associated with the appearance of $\mathbb{K}_{N(m)}$ in the algorithm, the numerical conditioning of $\mathbb{K}_{N(m)}$ is likewise important. As noted in [15], [17], [20], [21], the Grammian matrix $\mathbb{K}_{N(m)}$ can become ill-conditioned due to the smoothness of the kernel $\mathfrak{K}$, decreases in the *separation distance*

$$q_{\Xi_{N(m)}} \triangleq \min\left\{ \|\xi_i - \xi_j\| \mid \xi_i \neq \xi_j, \xi_i, \xi_j \in \Xi_{N(m)} \right\}$$

between the samples, or poor choices for any hyper-parameters in the definition of the kernel $\mathfrak{K}(\cdot,\cdot)$. In the previous work [15], [20], [21], the condition number of the Grammian matrix is kept reasonable by assuring a lower bound on the minimal separation of basis centers *a priori*. As noted in the introduction and by the authors of [10], however, it can be substantially more challenging to maintain good conditioning of the Grammian during basis augmentation, especially as the fill distance decreases; the *fill distance* in a set Ω is given by

$$h_{\Xi_{N(m)},\Omega} \triangleq \sup_{\xi \in \Omega} \min_{\xi_i \in \Xi_{N(m)}} d_\Omega(\xi, \xi_i), \tag{16}$$

and it measures the size of the space not occupied by centers in the set $\Omega \supset \Xi_{N(m)}$, which is where the trajectory is expected to lie. In this section, we outline steps for a greedy method that creates sequences of nested spaces where $h_{\Xi_N,\Omega} \to 0$ as $N(m) \to \infty$, while assuring that $q_{\Xi_{N(m)}} > C h_{\Xi_{N(m)},\Omega}$ for some constant $C > 0$.

The approach in this paper does not create implementations using the standard basis $\{\mathfrak{K}_{\xi_1}, ..., \mathfrak{K}_{\xi_{N(m)}}\}^{\mathrm{T}}$, but rather uses the Newton basis $\{\mathfrak{b}_1, ..., \mathfrak{b}_{N(m)}\}^{\mathrm{T}}$ for $\mathcal{H}_{N(m)}$. A detailed analysis of this basis can be found in [15]. The Newton basis is constructed using a greedy method for updating the set of bases at each triggering event. Suppose we select a collection of $M \geq N(m)$ candidate center locations $\Omega_M \triangleq \{x_i \mid 1 \leq j \leq M\} \subset \Omega$. The recursion to define the centers and Newton basis chooses $\xi_1 \in \arg\max_{x_j \in \Omega_M} \mathfrak{K}(\xi_j, \xi_j)$ and then recursively selects

$$\xi_N = \underset{x_j \in \Omega_M}{\arg\max}\, \mathcal{P}_{N-1}^2(x_j), \tag{17}$$

$$\mathfrak{b}_N(\cdot) = \frac{1}{\|\mathfrak{u}_N(\cdot)\|_{\mathcal{H}}} \left(\mathfrak{K}(\xi_N, \cdot) - \sum_{i=1}^{N} \mathfrak{b}_i(\cdot)\mathfrak{b}_i(\xi_N) \right), \tag{18}$$

where $\|\mathfrak{u}_N(\cdot)\|_{\mathcal{H}} = \max_{x_j \in \Omega_M} \mathcal{P}_{N-1}(x_j)$ and $\{w_j\}_{j=1}^{M} = \left\{ \sum_{i=1}^{N} \mathfrak{b}_i^2(x_j) \right\}_{j=1}^{M}$. Note here that each function $\mathfrak{b}_N(\cdot)$ depends only on the previously chosen basis functions. Also, the resultant set of Newton bases is $H(\Omega)$-orthogonal, as a result of the orthogonalization step in (18). Consequently, we can adaptively augment the basis through this recursive structure. Applying Netwon bases, the adaptive law (15) can be replaced by

$$\dot{\hat{\Theta}}(t) = \gamma_f \mathfrak{b}_{\Xi_{N(m)}}(x(t)) B^{\mathrm{T}} P \tilde{x}(t), \quad \hat{\Theta}(t_0) = \hat{\Theta}_0, \tag{19}$$

which does not require the inversion of the lead Grammian matrix. A pseudocode for selecting the kernel centers after the triggering event is given in Algorithm 1

Algorithm 1 Proposed greedy method for a basis selection

input: Centers $\Xi_{N(m-1)}$, ϵ, Ω_M
initialization: $\Xi_{N(m)} = \Xi_{N(m-1)}$
$z_j \triangleq \mathfrak{K}(x_j, x_j) \quad \forall x_j \in \Omega_M$
$\mathcal{P}_n(x_j) \triangleq z_j$
while $\max_{x_j \in \Omega_M} \mathcal{P}_n(x_j) > \epsilon$ **do**
  if $\Xi_{N(m-1)} = \varnothing$ **then**
    $\Xi_{N(m)} = \Xi_1 \triangleq \arg\max\{\mathfrak{K}(x, x) \mid x \in \Omega_M\}$
    for $x_j \in \Omega_M$ **do**
      $\mathfrak{b}_1(x_j) = \mathfrak{K}(\xi_1, x_j)/\sqrt{\mathfrak{K}(\xi_1, \xi_1)}$
      $w_j = \mathfrak{b}_1^2(x_j)$
    end for
  else
    $\xi = \max_{x_j \in \Omega_M} \mathcal{P}_{N(m)}(x_j)(x_j)$
    $\Xi_{N(m)} = \Xi_{N(m)} \cup \{\xi\}$
    for $x_j \in \Omega_M$ **do**
      $\mathfrak{u}_{N(m)}(x_j) = \mathfrak{K}(\xi_{N(m)}, x_j) - w_j$
      $\|\mathfrak{u}_{N(m)}\|_{\mathcal{H}} = z_{N(m)} - w_{N(m)}$
      $\mathfrak{b}_{N(m)}(x_j) = \mathfrak{u}_{N(m)}(x_j)/\|u_{N(m)}\|_{\mathcal{H}}$
      $w_j = w_j + \mathfrak{b}_{N(m)}^2(x_j)$
    end for
  end if
end while
return $\Xi_{N(m)}$

V. Stability and Convergence Results

Theorem 1 below is the key result of this paper as it proves the effectiveness of Algorithm 1, the adaptive law (19) and the control law (5). This result also guarantees that regions of approximation are positive invariant. It extends the results in [12, Ch. 15] or [22, Ch. 4, 6, 7]. While the authors in [12, Ch, 15] use a variable structure controller for all of the uncertainties in the problem, here, we use the variable structure mode controller (7) only to account for the departure of the state from the region of approximation. We also include a bounded control similar to that in Reference [22, Ch. 4, 6, 7], but modified as in [8] to make use of the power function that is available in the native space.

For the statement of this result, recall that, since $r(\cdot)$ is bounded and A_{ref} is Hurwitz, $x_{\mathrm{ref}}(\cdot)$ is bounded, that is, $\|x_{\mathrm{ref}}(t)\| \leq x_{r,\max}$ for all $t \geq t_0$. Furthermore, let P denote the solution of (10),

$$\bar{e}_1 \geq 2\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} \frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)} \cdot \left(\delta_{\max} + 2 \sup_{\xi \in S_\epsilon} \mathcal{P}_N(\xi) \|B\| \|f\|_{\mathcal{H}} \right), \tag{20}$$

$$S_\epsilon \triangleq \left\{ x \in \mathbb{R}^n \mid |B^{\mathrm{T}} P \tilde{x}| < \epsilon \right\} \tag{21}$$

for some $\epsilon > 0$,

$$b \triangleq \bar{e}_1 + \max_{t \geq T} \|x_{\mathrm{ref}}(t)\|_{\mathbb{R}^n} \tag{22}$$

for some $T \geq t_0$, and $\mathcal{B}_b \triangleq \{x \in \mathbb{R}^n \mid \|x\|_{\mathbb{R}^n} \leq b\}$ denoting the *approximation region*. Finally, note that, with the control law (5), the trajectory tracking error dynamics are given by

$$\begin{aligned} \dot{\tilde{x}}(t) =& A_{\mathrm{ref}} \tilde{x}(t) \\ &+ B\Big(E_{x(t)}(f - \hat{f}(t, \cdot)) + v_N(x(t), \tilde{x}(t)) \\ &\qquad + \mu(x(t))(u_{SL} - K_r^{\mathrm{T}} r(t)) + \delta(t) \Big), \\ \tilde{x}(t_0) =& x_0 - x_{\mathrm{ref},0}, \quad t \geq t_0. \end{aligned} \tag{23}$$

Theorem 1: Consider the trajectory tracking error dynamics (23) and the adaptive law (6). Suppose that the termination time $\Delta_m = \infty$. If there is a last finite trigger time $t_m < \infty$, then there is a time $T > t_0$ such that

$$\|\tilde{x}(t)\| \leq \bar{e}_1 \quad \text{for all } t \geq T. \tag{24}$$

Proof: The result is proven in the full-length paper [23]. ■

VI. Numerical Example

To demonstrate the effectiveness of the proposed control system, we implement the event-triggered controller and the greedy basis augmentation to the Van der Pol oscillator with external forcing. The system is characterized by

$$\dot{x}_1(t) = x_2(t), \quad x_1(t_0) = x_{10}, \quad t \geq t_0, \tag{25}$$

$$\dot{x}_2(t) = -x_1(t) + \eta x_2(t) - \eta x_1^2(t) x_2(t) + u(t), \quad x_2(t_0) = x_{20}, \tag{26}$$

where $\eta = 3$. A trajectory of (25) and (26) with $u(t) \equiv 0$, $t \in [0, \infty)$, is shown in Figure 1.

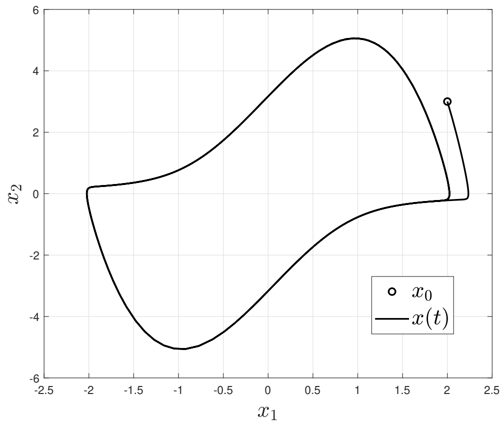


Fig. 1: Phase portrait of the Van der Pol oscillator (25) and (26) with $u(t) \equiv 0$, $t \in [0, \infty)$ and $x(0) = \{2, 3\}^{\mathrm{T}}$

For this problem, we seek to control the system represented by (25) and (26) and track the reference trajectory $x_{\mathrm{ref}}(t) = [x_{r,1}(t), x_{r,2}(t)]^{\mathrm{T}}$ such that

$$\begin{bmatrix} \dot{x}_{r,1}(t) \\ \dot{x}_{r,2}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}}_{A_{\mathrm{ref}}} \begin{bmatrix} x_{r,1}(t) \\ x_{r,2}(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B_{\mathrm{ref}}} r(t). \tag{27}$$

By selecting

$$r(t) = x_2(t), \quad t \geq t_0, \tag{28}$$

the reference trajectory follows a circular path with radius $R = \|x_{\mathrm{ref}}(t_0)\|_{\mathbb{R}^n}$. For this problem, we set $t_0 = 0$ and $x_{\mathrm{ref}}(0) = [2, 2]^{\mathrm{T}}$ so that the reference trajectory is a circle of radius $R = 2\sqrt{2}$.

We can express (25) and (26) concisely in state space as

$$\dot{x}(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} (u(t) + f(x(t))), \tag{29}$$

where $x = [x_1, x_2]^{\mathrm{T}}$ and

$$f(x) = (1 + \eta)x_2 - \eta x_1^2 x_2, \quad x \in \mathbb{R}^2. \tag{30}$$

For this tutorial example, the matrices A and B are known, $A = A_{\mathrm{ref}}$ and is Hurwitz, and $B = B_{\mathrm{ref}}$. For a general MRAC problem, we do not know A and B, and thus the ideal K_x and K_r to satisfy the matching conditions. In this more general case, we can introduce adaptive gains as proxies of K_x and K_r in (5) and apply classical adaptive laws to determine these laws; for details, see [12, Ch. 9]. The scope of this example is to solely highlight the control performance of the online adaptive law associated with the matched uncertainty. For this example, the desired control law can be achieved solely through feedback linearization by

$$u_{N(m)}(t) = r(t) - \hat{f}_{N(m)}(t, \cdot) + u_{\mathrm{SL}} + v_{N(m)}(x(t), \tilde{x}(t)), \quad t \geq t_0, \tag{31}$$

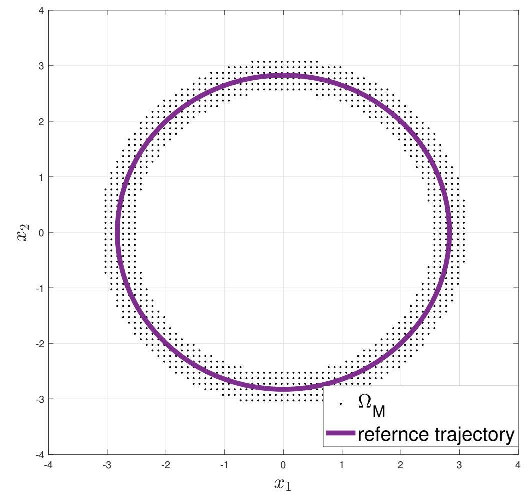


Fig. 2: the set Ω as well as the reference trajectory. The controller will perform best when the set Ω, which defines the candidate set of possible centers, covers the trajectory.

where $\hat{f}_{N(m)}(t, \cdot)$ represents the $N(m)$-dimensional online function estimate of f.

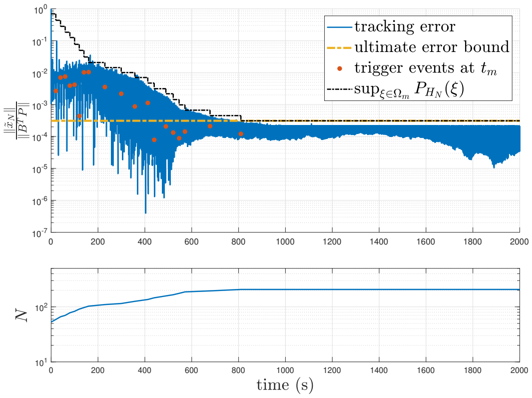
The domain of the state space depends on the initial conditions $x(t_0)$, and, ideally, the ω-limit set depends on the desired reference trajectory $x_{\mathrm{ref}}(t)$, which in this case, follows a circular trajectory of radius R. The set of centers $\Xi_{N(m)}$ comes from the set Ω_m which in this case is a constant and given by

$$\Omega = \{x \in \mathbb{R}^n \mid R_i < \|x\|_{\mathbb{R}^n} < R_o\}, \tag{32}$$

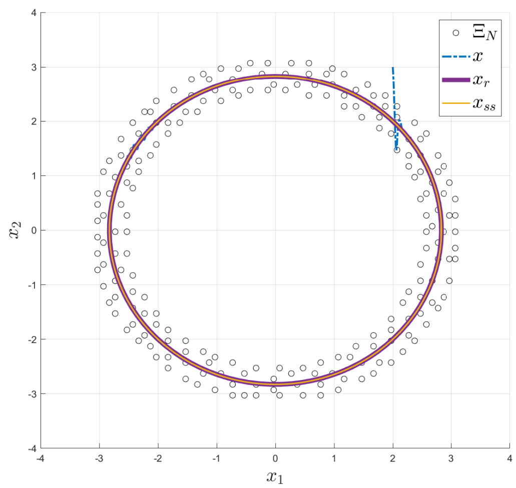
where $R_i = 2\sqrt{2} - 0.3$ and $R_o = 2\sqrt{2} + 0.3$; Figure 2 shows the set Ω as well as the reference trajectory. The set Ω was chosen to cover sufficiently well the majority of the transient dynamics as well as the ω-limit set.

The proposed greedy method is applied to select an initial set of centers $\Xi_{N(0)}$ from $\Omega_{N(0)} = \Omega$ until $\sup_{x_j \in \Omega_{N(0)}} \mathcal{P}_{N(0)}(x_j)\|f\|_{\mathcal{H}} \leq c_0\|\tilde{x}(t_0)\|_{\mathbb{R}^n}$, where $c_0 = 0.7$. After each triggering event, a set of greedy centers from Ω is selected at sequential maximums of the power function as given in Section V. The simulation is run with the adaptive law (19) with $\gamma_f = 10^5$. As discussed in [12, Ch. 15], the Lyapunov analysis shows that certain robust modifications must be implemented in (19) to ensure that the Lyapunov function remains positive along the closed-loop trajectories. In this example, we used smooth switching bounded control, which adds a term $v_{N(m)}(\cdot)$ to the control input (7) with $\varepsilon \approx O(\bar{e}_{N(m)})$. In this example, Q in (10) was chosen to be the identity matrix. With this bounded control term, the error dynamics are shown to be exponentially stable outside the set $\{x \in \mathbb{R}^n : \|\tilde{x}\|_{\mathbb{R}^n} < \varepsilon \approx O(\bar{e}_{N(m)})\}$.

The greedy basis augmentation is triggered if the tracking error $\|\tilde{x}(t)\|_{\mathbb{R}^n} \leq c_m \sup_{x_j \in \Omega_0} \mathcal{P}_{N(0)}(x_j)\|f\|_{\mathcal{H}}$ for a period of 20 s to prevent triggering basis augmentation during the transient phase of the control implementation. Figure 3a



(a)



(b)

Fig. 3: (a) The normalized tracking error and dimension N of the approximation space over time with a learning rate $\gamma_f = 10^5$. The error is ultimately bounded by $\sup_{x_j \in \Omega_m} P(x_j)$. (b) A comparison of the state trajectory versus the reference model over the set of centers used after the final triggering event. The actual state trajectory quickly converges to the reference trajectory as a result of the large learning rate and stays in the region over the centers $\Omega_m = \Omega\ \forall m \in \mathbb{N}^+$.

shows the tracking error over time. The simulation is run for 2000 s to demonstrate that the tracking error remains bounded below the ultimate error bound as $t \to \infty$.

VII. Conclusion

This paper introduces an innovative MRAC framework that harnesses RKHS basis functions with adaptively chosen kernel centers based on a greedy method criterion. The adaptive nature of our approach makes it particularly well-suited for systems with matched uncertainties, and our experimental results confirm its efficacy. We believe that this work opens up new avenues for research in adaptive control, offering promising prospects for real-world applications where robustness and adaptability are essential.

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