Robust Adaptive Control for Constrained Tilt-Rotor Quadcopters of Unknown Inertial Properties

Robert B. Anderson, John-Paul Burke, Julius A. Marshall, and Andrea L'Afflitto

***Abstract*— This paper analyzes the dynamics of tilt-rotor quadcopters with H-configuration and synthesizes a control system for these vehicles. Our analysis is the first to show a nonlinear effect in the vehicle's rotational dynamics due to the fact that not all propellers of a tilt-rotor are aligned to one of the vehicle's principal axes. The proposed control system relies on an original robust model reference adaptive control law to guarantee user-defined constraints on both the trajectory tracking error and the adaptive gains at all time despite parametric, matched, and unmatched uncertainties due to unknown external disturbances and dangling payloads.**

I. Introduction

In the first part of this paper, we present and analyze the equations of motion of a tilt-rotor quadcopter with H-configuration under the assumption that the vehicle's mass is known, but both its inertia matrix and the location of its center of mass are unknown. Our analysis of the rotational dynamics of a tilt-rotor quadcopter reveals that the presence of tilting and spinning propellers is captured by three terms, namely the inertial counter-torque, the gyroscopic effect, and an additional term that we named *tilt-rotor gyroscopic effect*. Both the inertial counter-torque and the gyroscopic effect characterize also the rotational dynamics of conventional quadcopters [1], whereas the tilt-rotor gyroscopic effect is not present in classical multi-rotor UAVs because their propellers are aligned to one of the vehicle's principal axes. To the authors' best knowledge, the tilt-rotor gyroscopic effect has not been highlighted in previous publications on tilt-rotors.

In the second part of this paper, we propose a robust model reference adaptive control (MRAC) law that guarantees user-defined constraints on the trajectory tracking error and the adaptive gains at all time despite parametric, matched, and unmatched uncertainties. Existing robust MRAC laws, such as the e-modification of MRAC [2], only guarantee uniform ultimate boundedness of the trajectory tracking error within bounds that can be estimated, but not imposed *a priori*. The proposed robust MRAC law, instead, guarantees user-defined constraints on the trajectory tracking error at all time.

In the third part of this paper, we design a control system for tilt-rotor quadcopters with H-configuration. Specifically, we propose a control strategy, whereby the vehicle's reference position, pitch angle, and yaw angle are user-defined and the reference roll angle is deduced accordingly. Successively,we apply our MRAC law to regulate the quadcopter's dynamics despite external disturbances and poorly modeled, unsteady payloads. A numerical simulation involving a tilt-rotor quadcopter, which transports an actuated inverted pendulum of unknown inertial properties and oscillating at unknown frequency, illustrates our theoretical results.

R. B. Anderson, J.-P. Burke, J. A. Marshall, and Andrea L'Afflitto are with the School of Aerospace and Mechanical Engineering, The University of Oklahoma, Norman, OK 73019-0390, USA (`[b.anderson, burk7119, j.marshall, a.lafflitto]@ou.edu`).

II. Notation, Definitions, and Mathematical Preliminaries

In this section, we establish notation, definitions, and review some basic results. The *interior* of the set $\mathcal{C}$ is denoted by $\mathring{\mathcal{C}}$. The ith vector of the canonical basis in $\mathbb{R}^n$ is denoted by $\mathbf{e}_{i,n}$, the *identity matrix* in $\mathbb{R}^{n\times n}$ is denoted by $\mathbf{1}_n$, the *zero* $n\times m$ *matrix* in $\mathbb{R}^{n\times m}$ is denoted by $0_{n\times m}$ or 0. We write $\|\cdot\|$ both for the *Euclidean vector norm* and the corresponding *equi-induced matrix norm*, and we define $\|B\|_{\mathrm{F},L} \triangleq \left[\mathrm{tr}\left(BLB^{\mathrm{T}}\right)\right]^{\frac{1}{2}}$ as the *weighted Frobenius norm* of $B\in\mathbb{R}^{n\times m}$, where $L\in\mathbb{R}^{m\times m}$ is symmetric and positive-definite; if $L=I$, then we write $\|B\|_{\mathrm{F}}$. The *trace* of $A\in\mathbb{R}^{n\times n}$ is denoted by $\mathrm{tr}(A)$ and the smallest eigenvalue of the symmetric matrix $Q\in\mathbb{R}^{n\times n}$ is denoted by $\lambda_{\min}(Q)$. The *Kronecker product* of $A\in\mathbb{R}^{n\times m}$ and $B\in\mathbb{R}^{p\times q}$ is denoted by $A\otimes B$. Given $x,y\in\mathbb{R}^3$, we denote the *cross-product of* x *and* y by $x^{\times}y$.

Given $\alpha\in\mathbb{R}$, for brevity we denote $\sin\alpha$, $\cos\alpha$, $\tan\alpha$, and $\sec\alpha$ by $\mathrm{s}\alpha$, $\mathrm{c}\alpha$, $\mathrm{t}\alpha$, $\mathrm{sc}\alpha$, respectively. The *Fréchet derivative* of the continuously differentiable function $V:\mathcal{D}\to\mathbb{R}$ at $x\in\mathcal{D}\subseteq\mathbb{R}^n$ is denoted by $V'(x)\triangleq\frac{\partial V(x)}{\partial x}$. The *Fréchet derivative* of the continuously differentiable function $h:\mathcal{X}\to\mathbb{R}$ at $X\in\mathcal{X}\subseteq\mathbb{R}^{n\times m}$ is given by [3]

$$\frac{\partial h(X)}{\partial X}\triangleq\begin{bmatrix}\frac{\partial h(X)}{\partial X_{1,1}} & \cdots & \frac{\partial h(X)}{\partial X_{1,m}}\\ \vdots & \ddots & \vdots\\ \frac{\partial h(X)}{\partial X_{n,1}} & \cdots & \frac{\partial h(X)}{\partial X_{n,m}}\end{bmatrix}, \tag{1}$$

where $X_{i,j}$ denotes the element of X on the ith row and jth column.

Proposition 2.1: If $A\in\mathbb{R}^{n\times m}$ and $b\in\mathbb{R}^m$, then

$$Ab = M_W(b,n)W_M(A), \tag{2}$$

where

$$M_W(b,n)\triangleq\left(b^{\mathrm{T}}\otimes\mathbf{1}_n\right)\in\mathbb{R}^{n\times nm}, \tag{3}$$

$$W_M(A)\triangleq\sum_{i=1}^{n}\left[\mathbf{e}_{i,m}\otimes\left(A\mathbf{e}_{i,m}\right)\right]\in\mathbb{R}^{nm}. \tag{4}$$

Proof: The proof of the next result is immediate and omitted for brevity. ■

III. Dynamics of a Tilt-Rotor Quadcopter

A. Problem formulation

In this section, we consider a tilt-rotor quadcopter with H-configuration, such as the one shown in Figure 1. This aircraft is composed of a frame, which we model as a rigid body, and four propellers that can be tilted independently. We also assume that the quadcopter transports some payload of known mass that is not rigidly connected to the vehicle's frame so that variations both in the position of the center of mass and the inertia matrix of the overall vehicle are unknown and not negligible.

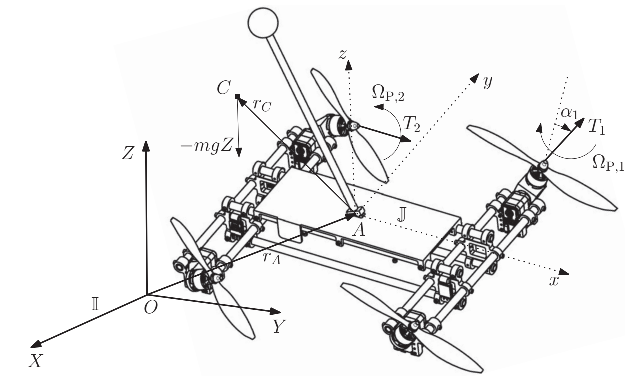

Fig. 1. Schematic representation of a tilt-rotor quadcopter with H-configuration.

To identify the position and orientation of the quadcopter's frame in space, consider the inertial reference frame $\mathbb{I} = \{O; X, Y, Z\}$. The vehicle's weight is captured by $F_{\rm g}^{\mathbb{I}} = -mgZ$, where $m > 0$ denotes the vehicle's mass and $g > 0$ denotes the gravitational acceleration. We also consider the orthonormal reference frame $\mathbb{J}(\cdot) = \{A(\cdot); x(\cdot), y(\cdot), z(\cdot)\}$ fixed with the vehicle's frame, centered at a point $A : [t_0, \infty) \to \mathbb{R}^3$ conveniently chosen on the vehicle's frame, and with axes $x, y, z : [t_0, \infty) \to \mathbb{R}^3$; we refer to $\mathbb{J}(\cdot)$ as the *body reference frame*. The reference frame $\mathbb{J}(\cdot)$ is chosen so that the propellers' arms are aligned to the $y(\cdot)$ axis; for details, see Figure 1. If a vector $a \in \mathbb{R}^3$ is expressed in the reference frame $\mathbb{I}$, then it is denoted by $a^{\mathbb{I}}$; if a vector is expressed in $\mathbb{J}(\cdot)$, then no superscript is used.

The *position* of the vehicle's reference point $A(\cdot)$ with respect to O is denoted by $r_A^{\mathbb{I}} : [t_0, \infty) \to \mathbb{R}^3$ and the *velocity* of $A(\cdot)$ with respect to the reference frame $\mathbb{I}$ is denoted by $v_A^{\mathbb{I}} : [t_0, \infty) \to \mathbb{R}^3$. Using a 3-2-1 rotation sequence, the orientation of the reference frame $\mathbb{J}(\cdot)$ with respect to the inertial reference frame $\mathbb{I}$ is captured by the *roll angle* $\phi : [t_0, \infty) \to [0, 2\pi)$, the *pitch angle* $\theta : [t_0, \infty) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and the *yaw angle* $\psi : [t_0, \infty) \to [0, 2\pi)$ [4, pp. 11]. The angular position of the ith propeller about its spin axis is denoted by $\Omega_i : [t_0, \infty) \to \mathbb{R}$, and the angular displacement of the ith propeller's spin axis, $i = 1, \ldots, 4$, about the $y(\cdot)$ axis and measured from the $z(\cdot)$ axis is denoted by $\alpha_i : [t_0, \infty) \to \mathbb{R}$; in this paper, $\alpha_i(\cdot)$ is referred to as the *ith propeller's tilt angle*.

B. Equations of motion of a tilt-rotor quadcopter

The *vector of independent generalized coordinates*

$$q = \left[r_A^{\mathbb{I},\rm T}, \phi, \theta, \psi\right]^{\rm T} \tag{5}$$

captures the position and orientation of the vehicle's frame. The *angular velocity* of the reference frame $\mathbb{J}(\cdot)$ with respect to $\mathbb{I}$ is denoted by $\omega : \mathcal{D} \times \mathbb{R}^6 \to \mathbb{R}^3$, where $\mathcal{D} \triangleq \mathbb{R}^3 \times [0, 2\pi) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times [0, 2\pi)$, and is such that [4, Th. 1.7]

$$\omega(q(t), \dot{q}(t)) = \Gamma^{-1}(q(t)) \left[\dot{\phi}(t), \dot{\theta}(t), \dot{\psi}(t)\right]^{\rm T}, \qquad t \geq t_0, \tag{6}$$

where

$$\Gamma(q) \triangleq \begin{bmatrix} 1 & {\rm s}\phi\,{\rm t}\theta & {\rm c}\phi\,{\rm t}\theta \\ 0 & {\rm c}\phi & -{\rm s}\phi \\ 0 & {\rm s}\phi\,{\rm sc}\theta & {\rm c}\phi\,{\rm sc}\theta \end{bmatrix}, \qquad q \in \mathcal{D};$$

recall that $\Gamma(q)$, $q \in \mathcal{D}$, is invertible [4, pp. 18-19].

It follows from (6) that the *kinematic equations* of a tilt-rotor quadcopter are given by

$$\dot{q}(t) = \begin{bmatrix} v_A^{\mathbb{I}}(t) \\ \Gamma(q(t))\omega(q(t), \dot{q}(t)) \end{bmatrix},$$
$$q(t_0) = \left[r_{A,0}^{\mathbb{I},\rm T}, \phi_0, \theta_0, \psi_0\right]^{\rm T}, \qquad t \geq t_0. \tag{7}$$

Proceeding as in [5], one can prove that the *dynamic equations* of a tilt-rotor quadcopter are given by

$$\mathcal{M}(t, q(t)) \begin{bmatrix} \dot{v}_A^{\mathbb{I}}(t) \\ \dot{\omega}(q(t), \dot{q}(t)) \end{bmatrix} = \begin{bmatrix} f_{\rm tran}(t, q(t), \dot{q}(t)) \\ f_{\rm rot}(t, q(t), \dot{q}(t)) \end{bmatrix} + \hat{G}(q(t))u(t),$$
$$\begin{bmatrix} v_A^{\mathbb{I}}(t_0) \\ \omega(t_0) \end{bmatrix} = \begin{bmatrix} v_{A,0}^{\mathbb{I}} \\ \omega_0 \end{bmatrix}, \qquad t \geq t_0, \tag{8}$$

where

$$u = [u_5, u_1, \ldots, u_4]^{\rm T} \tag{9}$$

denotes the *control input*, $u_5, u_1 : [t_0, \infty) \to \mathbb{R}$ denote the components of the forces produced by the propellers along the $x(\cdot)$ and $z(\cdot)$ axes, respectively, $u_2, u_3, u_4 : [t_0, \infty) \to \mathbb{R}$ denote the moment of the forces about the $x(\cdot)$, $y(\cdot)$, and $z(\cdot)$ axes, respectively,

$$\mathcal{M}(t, q) \triangleq \begin{bmatrix} m\mathbf{1}_3 & -mR(q)r_C^{\times}(t) \\ mr_C^{\times}(t)R^{\rm T}(q) & I(t) \end{bmatrix},$$
$$(t, q) \in [t_0, \infty) \times \mathcal{D}, \tag{10}$$

denotes the *generalized mass matrix*,

$$f_{\rm tran}(t, q, \dot{q}) \triangleq -mR(q)\left[\ddot{r}_C(t) + 2\omega^{\times}(q, \dot{q})\dot{r}_C(t) + \omega^{\times}(q, \dot{q})\omega^{\times}(q, \dot{q})r_C(t)\right] + [0, 0, -mg]^{\rm T}, \tag{11}$$

$$f_{\rm rot}(t, q, \dot{q}) \triangleq -\omega^{\times}(q, \dot{q})I(t)\omega(q, \dot{q}) - \dot{I}(t)\omega(q, \dot{q}) - \sum_{i=1}^{4}\left[I_{P_i}(t)\dot{\omega}_{P_i}(t) + \omega_{P_i}^{\times}(t)I_{P_i}(t)\omega_{P_i}(t)\right] - \omega^{\times}(q, \dot{q})\sum_{i=1}^{4} I_{P_i}(t)\omega_{P_i}(t) + r_C^{\times}(t)F_{\rm g}(q), \tag{12}$$

$r_C : [t_0, \infty) \to \mathbb{R}^3$ denotes the position of the vehicle's center of mass with respect to the reference point $A(\cdot)$,

$$R(q) \triangleq \begin{bmatrix} {\rm c}\psi & -{\rm s}\psi & 0 \\ {\rm s}\psi & {\rm c}\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {\rm c}\theta & 0 & {\rm s}\theta \\ 0 & 1 & 0 \\ -{\rm s}\theta & 0 & {\rm c}\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & {\rm c}\phi & -{\rm s}\phi \\ 0 & {\rm s}\phi & {\rm c}\phi \end{bmatrix}, \tag{13}$$

and

$$\hat{G}(q) = \begin{bmatrix} R(q)\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} & 0_{3\times 3} \\ 0_{3\times 2} & \mathbf{1}_3 \end{bmatrix}. \tag{14}$$

The *tiltrotor's equations of motion* are given by (7) and (8).

It follows from (8) and (12) that the effect of the propellers' motion is captured by the *inertial* *counter-torque* $\sum_{i=1}^{4} I_{P_i}(\cdot)\dot{\omega}_{P_i}(\cdot)$, the *gyroscopic effect* $\omega^{\times}(\cdot)\sum_{i=1}^{4} I_{P_i}(\cdot)\omega_{P_i}(\cdot)$, and $\sum_{i=1}^{4} \omega_{P_i}^{\times}(\cdot)I_{P_i}(\cdot)\omega_{P_i}(\cdot)$. For conventional quadcopters, $\sum_{i=1}^{4} \omega_{P_i}^{\times}(t)I_{P_i}(t)\omega_{P_i}(t) = 0$, $t \geq t_0$, since $\alpha_i(t) = 0$, $i = 1,\ldots,4$, $I_{P_i}(\cdot)$ is a diagonal matrix, and the propellers' thrust force is exerted along the $z(\cdot)$ axis of the reference frame $\mathbb{J}(\cdot)$ [5]. For tilt-rotor quadcopters, $\omega_{P_i}^{\times}(t)I_{P_i}(t)\omega_{P_i}(t) \not\equiv 0$, $t \geq t_0$, $i = 1,\ldots,4$, since $\alpha_i(t) \not\equiv 0$. To the authors' best knowledge, existing results on tilt-rotor quadcopters do not account for any effect due to the presence of spinning and tilting propellers. Indeed, the term $\sum_{i=1}^{4} \omega_{P_i}^{\times}(\cdot)I_{P_i}(\cdot)\omega_{P_i}(\cdot)$ is not identified by a specific name and henceforth, it will be referred to as the *tilt-rotor gyroscopic effect*.

IV. Control Design for a Tilt-Rotor Quadcopter

A. Robust Model Reference Adaptive Control for Constrained Dynamical Systems

Consider the unknown dynamical system

$$\dot{x}(t) = Ax(t) + B\left[u(t) + \Theta^{\mathrm{T}}\Phi(x(t))\right] + \xi(t), \quad x(t_0) = x_0, \quad t \geq t_0, \quad (15)$$

where $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $t \geq t_0$, denotes the *system's trajectory*, $u(t) \in U \subseteq \mathbb{R}^m$ denotes the *control input*, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $\Theta \in \mathbb{R}^{N\times m}$, the *regressor vector* $\Phi : \mathcal{D} \to \mathbb{R}^N$ is Lipschitz continuous, and $\xi : [t_0,\infty) \to \mathbb{R}^n$. The matrices A and Θ are unknown and capture parametric and matched uncertainties, and the continuous function $\xi(\cdot)$ is unknown, captures unmatched uncertainties, and is such that $\|\xi(t)\| \leq \xi_{\max}$, $t \geq t_0$, where $\xi_{\max} \geq 0$ is known.

Consider also the *reference dynamical model*

$$\dot{x}_{\mathrm{ref}}(t) = A_{\mathrm{ref}}x_{\mathrm{ref}}(t) + B_{\mathrm{ref}}r(t), \quad x_{\mathrm{ref}}(t_0) = x_{\mathrm{ref},0}, \quad t \geq t_0, \quad (16)$$

where $x_{\mathrm{ref}}(t) \in \mathbb{R}^n$, $t \geq t_0$, $r(t) \in \mathbb{R}^m$ is bounded and denotes the *command input*, $A_{\mathrm{ref}} \in \mathbb{R}^{n\times n}$ is Hurwitz, $B_{\mathrm{ref}} \in \mathbb{R}^{n\times m}$, and the pair $(A_{\mathrm{ref}}, B_{\mathrm{ref}})$ is controllable. We assume that there exist $(K_x, K_r) \in \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times m}$ such that

$$A_{\mathrm{ref}} = A + BK_x^{\mathrm{T}}, \quad (17)$$

$$B_{\mathrm{ref}} = BK_r^{\mathrm{T}}. \quad (18)$$

The *matching conditions* (17) and (18) guarantee that if both A and Θ were known and $u(t) = K\pi(t)$, $t \geq t_0$, where $K \triangleq \left[K_x^{\mathrm{T}}, K_r^{\mathrm{T}}, -\Theta^{\mathrm{T}}\right]$, then $\lim_{t\to\infty} e(t) = 0$ [6, Ch. 8].

Consider the feedback control law

$$\phi(\pi,\hat{K}) = \hat{K}\pi, \quad (\pi,\hat{K}) \in \mathbb{R}^{n+m+N} \times \mathbb{R}^{m\times(n+m+N)}, \quad (19)$$

where $\hat{K} : [t_0,\infty) \to \mathbb{R}^{m\times(n+m+N)}$. A design specification for the *adaptive gain matrix* $\hat{K}(\cdot)$ would be that the *adaptive gain's error* $\widetilde{\Delta K}(t) \triangleq \hat{K}(t) - K$, $t \geq t_0$, verifies some constraints assigned *a priori*. The matrix K is unknown, and $\widetilde{\Delta K}(\cdot)$ cannot be computed. However, in problems of practical interest it is possible to find $K_{\mathrm{e}} \in \mathbb{R}^{m\times(n+m+N)}$ that provides an estimate of K, that is, such that $\|K_{\mathrm{e}} - K\|_{\mathrm{F}} \leq \varepsilon$, for some $\varepsilon \geq 0$. In this paper, we provide $\hat{K}(\cdot)$ so that both the trajectory tracking error $e(\cdot)$ and the *estimated adaptive gain's error* $\Delta K(t) \triangleq \hat{K}(t) - K_{\mathrm{e}}$, $t \geq t_0$, verify user-defined constraints at all time. In particular, we consider the compact, connected *constraint set*

$$\mathcal{C} \triangleq \big\{(e,\Delta K) \in \mathbb{R}^n \times \mathbb{R}^{m\times(n+m+N)} : h(e^{\mathrm{T}}Me, \Delta K\Gamma^{-1}\Delta K^{\mathrm{T}}) \geq 0\big\}, \quad (20)$$

where $M \in \mathbb{R}^{n\times n}$ is symmetric and positive-definite, $\Gamma \in \mathbb{R}^{(n+m+N)\times(n+m+N)}$ is symmetric and positive-definite, and $h : \mathbb{R}\times\mathbb{R}^{m\times m} \to \mathbb{R}$ is continuously differentiable and such that $h(0,0) > 0$.

Lastly, define

$$h_e(e^{\mathrm{T}}Me, \Delta K\Gamma^{-1}\Delta K^{\mathrm{T}}) \triangleq \left.\frac{\partial h(\beta,X)}{\partial \beta}\right|_{\substack{\beta=e^{\mathrm{T}}Me\\ X=\Delta K\Gamma^{-1}\Delta K^{\mathrm{T}}}}, \quad (e,\Delta K) \in \mathring{\mathcal{C}}, \quad (21)$$

$$h_X(e^{\mathrm{T}}Me, \Delta K\Gamma^{-1}\Delta K^{\mathrm{T}}) \triangleq \left.\frac{\partial h(\beta,X)}{\partial X}\right|_{\substack{\beta=e^{\mathrm{T}}Me\\ X=\Delta K\Gamma^{-1}\Delta K^{\mathrm{T}}}}, \quad (22)$$

where $\frac{\partial h(\beta,X)}{\partial X}$ is given by (1), and assume that

$$h_e(e^{\mathrm{T}}Me, \Delta K\Gamma^{-1}\Delta K^{\mathrm{T}}) \leq 0, \quad (23)$$

$$h_X(e^{\mathrm{T}}Me, \Delta K\Gamma^{-1}\Delta K^{\mathrm{T}}) \leq 0, \quad (24)$$

that is, $h_X(\cdot,\cdot)$ is symmetric and nonpositive-definite, and

$$(0_n, 0_{m\times(n+m+N)}) = \underset{(e,\Delta K)\in\mathbb{R}^n\times\mathbb{R}^{m\times(n+m+N)}}{\arg\max} h(e^{\mathrm{T}}Me, \Delta K\Gamma^{-1}\Delta K^{\mathrm{T}}). \quad (25)$$

Conditions (23)–(25) are verified, for example, by

$$h(e^{\mathrm{T}}Me, \Delta K\Gamma^{-1}\Delta K^{\mathrm{T}}) = h_{\max} - e^{\mathrm{T}}Me - \|\Delta K\|^2_{\mathrm{F},\Gamma^{-1}}, \quad (e,\Delta K) \in \mathbb{R}^m \times \mathbb{R}^{m\times(n+m+N)}, \quad (26)$$

where $h_{\max} > 0$.

The next theorem provides an adaptive law $\hat{K}(\cdot)$ for the feedback control law (19) so that if $u(t) = \phi(\pi(t),\hat{K}(t))$, $t \geq t_0$, then both the trajectory tracking error $e(\cdot)$ and the estimated adaptive gain's error $\Delta K(\cdot)$ verify the user-defined constraints captured by (20) at all time, despite parametric, matched, and unmatched uncertainties. For the statement of this result, note that it follows from (15), (19), and (16) that

$$\dot{e}(t) = A_{\mathrm{ref}}e(t) + B\widetilde{\Delta K}(t)\pi(t) + \xi(t), \quad e(t_0) = x_0 - x_{\mathrm{ref},0}, \quad t \geq t_0. \quad (27)$$

In addition, define the positive-definite function

$$V(e,\Delta K) \triangleq \frac{e^{\mathrm{T}}Pe + \mathrm{tr}\left(\Delta K\Gamma^{-1}\Delta K^{\mathrm{T}}\right)}{h(e^{\mathrm{T}}Me, \Delta K\Gamma^{-1}\Delta K^{\mathrm{T}})}, \quad (e,\Delta K) \in \mathring{\mathcal{C}}, \quad (28)$$

where $P \in \mathbb{R}^{n\times n}$ denotes the symmetric, positive-definite solution of the algebraic Lyapunov equation

$$0 = A_{\mathrm{ref}}^{\mathrm{T}}P + PA_{\mathrm{ref}} + Q_1, \quad (29)$$

and $Q_1 \in \mathbb{R}^{n\times n}$ is symmetric and positive-definite, and let

$$\mathcal{S}_\pi \triangleq \big\{(e,\Delta K) \in \mathbb{R}^n \times \mathbb{R}^{m\times(n+m+N)} : S_\pi(e,\Delta K) > 0\big\}, \quad (30)$$

where

$$S_\pi(e,\Delta K) \triangleq -\alpha\|e\|^2 - \sigma\|e^{\mathrm{T}}PB\|^p\|\Delta K\|^2_{\mathrm{F}} + 2\left(\varepsilon\|\pi\| + \xi_{\max}\right)\|R(e,\Delta K)\|_{\mathrm{F}}, \quad (31)$$

$p \in \mathbb{N}$, $\sigma > 0$, $\alpha \triangleq \lambda_{\min}(Q_1)$, and

$$R(e, \Delta K) \triangleq e^{\mathrm{T}}\big[P - V(e, \Delta K) \cdot h_e(e^{\mathrm{T}}Me, \Delta K\Gamma^{-1}\Delta K^{\mathrm{T}})M\big]B. \quad (32)$$

Lastly, note that the trajectory $x_{\rm ref}(t)$, $t \geq t_0$, of (16) is bounded, since $r(t)$ is bounded and $A_{\rm ref}$ is Hurwitz, and if $e(t)$ is bounded, then $x(t)$ is bounded and there exists a compact set $\Pi \subset \mathbb{R}^{n+m+N}$ such that $\pi(t) = \big[x^{\mathrm{T}}(t), r^{\mathrm{T}}(t), -\Phi^{\mathrm{T}}(x(t))\big]^{\mathrm{T}} \in \Pi$ for all $t \geq t_0$. Therefore, since $S_\pi(\cdot,\cdot)$ is continuous in π, it follows from Weierstrass theorem that

$$\pi^*(e, \Delta K) \triangleq \mathrm{argmax}_{\pi\in\Pi} S_\pi(e, \Delta K), \quad (e, \Delta K) \in \mathbb{R}^n \times \mathbb{R}^{m\times(n+m+N)}, \quad (33)$$

exists and is finite.

Theorem 4.1: Consider the trajectory tracking error dynamics (27), the constraint set (20), and the set $\mathcal{S}_\pi$ given by (30). Let $x_0 \in \mathbb{R}^n$, $x_{{\rm ref},0} \in \mathbb{R}^n$, and $\hat{K}_0 \in \mathbb{R}^{m\times(n+m+N)}$ be such that $(x_0 - x_{{\rm ref},0}, \hat{K}_0 - K_{\rm e}) \in \mathring{\mathcal{C}} \setminus \{(0,0)\}$, and let

$$\begin{aligned}\dot{\hat{K}}^{\mathrm{T}}(t) = -\Gamma\big[&\pi(t)e^{\mathrm{T}}(t)(P - V(e(t), \Delta K(t)) \\ &\cdot h_e(e^{\mathrm{T}}(t)Me(t), \Delta K(t)\Gamma^{-1}\Delta K^{\mathrm{T}}(t))M)B \\ &+ \sigma\|e^{\mathrm{T}}(t)PB\|^p \Delta K^{\mathrm{T}}(t)\big] \\ &\cdot \big[\mathbf{1}_m - V(e(t), \Delta K(t)) \\ &\cdot h_X(e^{\mathrm{T}}(t)Me(t), \Delta K\Gamma^{-1}\Delta K^{\mathrm{T}}(t))\big]^{-1}, \\ \hat{K}(t_0) = \hat{K}_0, &\qquad t \geq t_0, \end{aligned} \quad (34)$$

where $P \in \mathbb{R}^{n\times n}$ denotes the symmetric, positive-definite solution of (29), $V(\cdot,\cdot)$ is given by (28), $h_e(\cdot,\cdot)$ is given by (21), and $h_X(\cdot,\cdot)$ is given by (22). If $\mathcal{S}_{\pi^*} \subset \mathring{\mathcal{C}}$, where $\pi^*(\cdot,\cdot)$ is given by (33), then (15) with control law (19) and adaptive law (34) is such that $(e(t), \Delta K(t)) \in \mathring{\mathcal{C}}$, $t \geq t_0$.

Theorem 4.1, whose proof is omitted for brevity, provides sufficient conditions for the control law (19) and the adaptive law (34) to steer the trajectory of the dynamical system (15) and guarantee user-defined levels of performance, which are captured by the constraint set (20), despite uncertainties in the dynamical model.

B. Control strategy for tilt-rotor quadcopters

Tilt-rotor quadcopters with H-configuration allow to control directly both the forces along the $x(\cdot)$ and $z(\cdot)$ axes and the moment of the forces about the $x(\cdot)$, $y(\cdot)$, and $z(\cdot)$ axes by regulating the propellers' tilt angles and angular velocities. Therefore, the following control strategy is proposed.

The twice continuously differentiable *reference trajectory* $r_{\rm ref}^{\mathbb{I}} : [t_0, \infty) \to \mathbb{R}^3$, the twice continuously differentiable *reference pitch angle* $\theta_{\rm ref} : [t_0, \infty) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and the twice continuously differentiable *reference yaw angle* $\psi_{\rm ref} : [t_0, \infty) \to [0, 2\pi)$ are considered as user-defined. The *translational equivalent control input* $v_{\rm tran} : [t_0, \infty) \to \mathbb{R}^3$ and the *reference roll angle* $\phi_{\rm ref}(\cdot)$ are defined as

$$v_{\rm tran}(t) \triangleq R(q_{\rm ref}(t))\,[u_5(t), 0, u_1(t)]^{\mathrm{T}}, \qquad t \geq t_0, \quad (35)$$

$$\phi_{\rm ref}(t) \triangleq -\tan^{-1}\frac{\tilde{v}_{{\rm tran},2}(t)}{\tilde{v}_{{\rm tran},3}(t)}, \quad (36)$$

where $R(\cdot)$ is given by (13),

$$q_{\rm ref} \triangleq \left[r_{\rm ref}^{\mathbb{I},\mathrm{T}}, \phi_{\rm ref}, \theta_{\rm ref}, \psi_{\rm ref}\right]^{\mathrm{T}} \quad (37)$$

denotes the *vector of reference generalized coordinates*, $\tan^{-1}(\cdot)$ denotes the *signed inverse tangent function*, and

$$\begin{bmatrix}\tilde{v}_{{\rm tran},1}(t)\\ \tilde{v}_{{\rm tran},2}(t)\\ \tilde{v}_{{\rm tran},3}(t)\end{bmatrix} = \begin{bmatrix}{\rm c}\theta_{\rm ref}(t) & 0 & -{\rm s}\theta_{\rm ref}(t)\\ 0 & 1 & 0\\ {\rm s}\theta_{\rm ref}(t) & 0 & {\rm c}\theta_{\rm ref}(t)\end{bmatrix} \cdot \begin{bmatrix}{\rm c}\psi_{\rm ref}(t) & {\rm s}\psi_{\rm ref}(t) & 0\\ -{\rm s}\psi_{\rm ref}(t) & {\rm c}\psi_{\rm ref}(t) & 0\\ 0 & 0 & 1\end{bmatrix} v_{\rm tran}(t). \quad (38)$$

In this case, it follows from (35) that the equations of motion (7) and (8) are equivalent to

$$\begin{aligned}\mathcal{M}(t, q(t)) &\begin{bmatrix}\mathbf{1}_3 & 0\\ 0 & \Gamma^{-1}(q(t))\end{bmatrix}\ddot{q}(t) \\ &= \begin{bmatrix}f_{\rm tran}(t, q(t), \dot{q}(t))\\ f_{\rm rot}(t, q(t), \dot{q}(t))\end{bmatrix} + \Big[\hat{G}(q(t)) - \hat{G}(q_{\rm ref}(t))\Big]u(t) \\ &\quad + \mathcal{M}(t, q(t))\begin{bmatrix}0_{3\times1}\\ \Gamma^{-1}(q(t))\dot{\Gamma}(q(t))\omega(q(t), \dot{q}(t))\end{bmatrix} + v(t), \\ &\big[q^{\mathrm{T}}(t_0), \dot{q}^{\mathrm{T}}(t_0)\big]^{\mathrm{T}} = \big[q_0^{\mathrm{T}}, q_{{\rm d},0}^{\mathrm{T}}\big]^{\mathrm{T}}, \qquad t \geq t_0,\end{aligned} \quad (39)$$

where the *equivalent control input*

$$v(t) \triangleq [v_{\rm tran}^{\mathrm{T}}(t), u_2(t), u_3(t), u_4(t)]^{\mathrm{T}}, \quad (40)$$

is designed so that $q(\cdot)$ tracks $q_{\rm ref}(\cdot)$ within user-defined bounds. Finally, $u(\cdot)$ is deduced from (35) and (40).

C. Control laws for tilt-rotor quadcopters

Assuming that both the location of the center of mass and the inertia matrix of the quadcopter are unknown, we define the twice continuously differentiable functions $\overline{r}_C : [t_0, \infty) \to \mathbb{R}^3$ and $\Delta r_C : [t_0, \infty) \to \mathbb{R}^3$ so that

$$r_C(t) = \overline{r}_C(t) + \Delta r_C(t), \qquad t \geq t_0. \quad (41)$$

The function $\overline{r}_C(\cdot)$ is considered as known and is deduced using, for example, analytical models of the vehicle's configuration, and $\Delta r_C(\cdot)$ is unknown. Similarly, we define the continuously differentiable and symmetric matrix functions $\overline{I}_{\rm quad}$, I_{P_i}, $\Delta I : [t_0, \infty) \to \mathbb{R}^{3\times3}$ with $\overline{I}(\cdot)$ and $I_{P_i}(\cdot)$ positive-definite so that

$$I(t) = \overline{I}_{\rm quad}(t) + \sum_{i=1}^{4} I_{P_i}(t) + \Delta I(t), \qquad t \geq t_0, \quad (42)$$

where $\overline{I}_{\rm quad}(\cdot)$ is considered as known and is deduced using, for example, analytical models of the vehicle's configuration, the ith propeller's inertia matrix $I_{P_i}(\cdot)$, $i = 1, \ldots, 4$, is known, since each propeller's mass and geometry is known, each propeller's location is known, and the tilt angles $\alpha_i(\cdot)$, $i = 1, \ldots, 4$, are known, and $\Delta I(\cdot)$ is unknown.

To feedback linearize (39), define $e(t) \triangleq \big[q^{\mathrm{T}}(t) - q_{\rm ref}^{\mathrm{T}}(t), \dot{q}^{\mathrm{T}}(t) - \dot{q}_{\rm ref}^{\mathrm{T}}(t)\big]^{\mathrm{T}}$, $t \geq t_0$, where $q(\cdot)$ is given by (5) and $q_{\rm ref}(\cdot)$ is given by (37), define

$$\overline{\mathcal{M}}(t, q) \triangleq \begin{bmatrix}m\mathbf{1}_3 & -mR(q)\overline{r}_C^{\times}(t)\\ m\overline{r}_C^{\times}(t)R^{\mathrm{T}}(q) & \overline{I}(t)\end{bmatrix}, \quad (t, q) \in [t_0, \infty) \times \mathcal{D}, \quad (43)$$

$$\Delta\mathcal{M}(t, q) \triangleq \begin{bmatrix}0_{3\times3} & -mR(q)\Delta r_C^{\times}(t)\\ m\Delta r_C^{\times}(t)R^{\mathrm{T}}(q) & \Delta I(t)\end{bmatrix}, \quad (44)$$

so that $\mathcal{M}(t,q) = \overline{\mathcal{M}}(t,q) + \Delta\mathcal{M}(t,q)$ and $\overline{\mathcal{M}}(\cdot,\cdot)$ is invertible, define

$$\overline{f}_{\rm tran}(t,q,\dot q) \triangleq -mR(q)\big[\ddot{\overline{r}}_C(t) + 2\omega^\times(q,\dot q)\dot{\overline{r}}_C(t) + \omega^\times(q,\dot q)\omega^\times(q,\dot q)\overline{r}_C(t)\big] + [0,0,-mg]^{\rm T}, \quad (45)$$

$$\Delta f_{\rm tran}(t,q,\dot q) \triangleq -mR(q)\big[\Delta\ddot r_C(t) + 2\omega^\times(q,\dot q)\Delta\dot r_C(t) + \omega^\times(q,\dot q)\omega^\times(q,\dot q)\Delta r_C(t)\big], \quad (46)$$

$$\overline{f}_{\rm rot}(t,q,\dot q) \triangleq -\omega^\times(q,\dot q)\overline{I}(t)\omega(q,\dot q) - \dot{\overline{I}}(t)\omega(q,\dot q) - \omega^\times(q,\dot q)\sum_{i=1}^{4} I_{P_i}(t)\omega_{P_i}(t) + \overline{r}_C^\times(t)F_{\rm g}(q) - \sum_{i=1}^{4}\left[I_{P_i}(t)\dot\omega_{P_i}(t) + \omega_{P_i}^\times(t)I_{P_i}(t)\omega_{P_i}(t)\right], \quad (47)$$

$$\Delta f_{\rm rot}(t,q,\dot q) \triangleq -\omega^\times(q,\dot q)\Delta I(t)\omega(q,\dot q) - \Delta\dot I(t)\omega(q,\dot q) + \Delta r_C^\times F_{\rm g}(q), \quad (48)$$

so that

$$f_{\rm tran}(t,q,\dot q) = \overline{f}_{\rm tran}(t,q,\dot q) + \Delta f_{\rm tran}(t,q,\dot q), \quad (49)$$

$$f_{\rm rot}(t,q,\dot q) = \overline{f}_{\rm rot}(t,q,\dot q) + \Delta f_{\rm rot}(t,q,\dot q), \quad (50)$$

and define

$$\beta(t,q,q_{\rm ref},v_2) \triangleq \overline{\mathcal{M}}(t,q)\begin{bmatrix}\mathbf{1}_3 & 0_{3\times3}\\ 0_{3\times3} & \Gamma^{-1}(q)\end{bmatrix}\left(\ddot q_{\rm ref} - \begin{bmatrix}0_{3\times1}\\ \dot\Gamma(q)\omega(q,\dot q)\end{bmatrix} - K_{\rm P}\left[q(t) - q_{\rm ref}(t)\right] - K_{\rm D}\left[\dot q - \dot q_{\rm ref}\right] + v_2\right) - \left[\overline{f}^{\rm T}_{\rm tran}(t,q,\dot q), \overline{f}^{\rm T}_{\rm rot}(t,q,\dot q)\right]^{\rm T}, \quad (51)$$

where $v_2 \in \mathbb{R}^6$ and the user-defined matrices of control gains $K_{\rm P}$ and $K_{\rm D} \in \mathbb{R}^{6\times6}$ are symmetric and positive-definite.

If $v(t) = \beta(t,q(t),q_{\rm ref}(t),v_2(t))$, $t \geq t_0$, then (39) is feedback-linearized and the trajectory tracking error dynamics is given by

$$\dot e(t) = \begin{bmatrix}0_{6\times6} & \mathbf{1}_6\\ -K_{\rm P} & -K_{\rm D}\end{bmatrix}e(t) + \begin{bmatrix}0_{6\times6}\\ \mathbf{1}_6\end{bmatrix}\left[v_2(t) + \Theta^{\rm T}\Phi(q(t),\dot q(t))\right] + \hat\xi(t),$$
$$e(t_0) = \left[q^{\rm T}(t_0), \dot q^{\rm T}(t_0)\right]^{\rm T} - \left[q^{\rm T}_{\rm ref}(t_0), \dot q^{\rm T}_{\rm ref}(t_0)\right]^{\rm T}, \ t \geq t_0, \quad (52)$$

where $v_2(t) \in \mathbb{R}^6$ denotes the *virtual control input*, $\hat\xi(t) = \left[0_{1\times6}, \hat\xi^{\rm T}_{\rm dyn}(t)\right]^{\rm T} \in \mathbb{R}^{12}$,

$$\hat\xi_{\rm dyn}(t) = \begin{bmatrix}\mathbf{1}_3 & 0_{3\times3}\\ 0_{3\times3} & \Gamma(q(t))\end{bmatrix}\overline{\mathcal{M}}^{-1}(t,q)\left(\begin{bmatrix}\Delta f_{\rm tran}(t,q(t),\dot q(t))\\ \Delta f_{\rm rot}(t,q(t),\dot q(t))\end{bmatrix} + \left[\hat G(q(t)) - \hat G(q_{\rm ref}(t))\right]u(t) - \Delta\mathcal{M}(t,q)\begin{bmatrix}\mathbf{1}_3 & 0_{3\times3}\\ 0_{3\times3} & \Gamma^{-1}(q(t))\end{bmatrix}\ddot q(t)\right), \quad (53)$$

$\Theta^{\rm T} = \begin{bmatrix}\Theta_{\rm tran} & 0_{3\times30}\\ 0_{3\times9} & \Theta_{\rm rot}\end{bmatrix}$, $\Phi(q,\dot q) = \left[\Phi^{\rm T}_{\rm tran}(q,\dot q), \Phi^{\rm T}_{\rm rot}(q,\dot q)\right]^{\rm T}$, $(q,\dot q) \in \mathcal{D}\times\mathbb{R}^6$,

$$\Theta_{\rm tran} = M_W(\overline{\overline{r}}_C, 3), \quad (54)$$

$$\Theta_{\rm rot} = \left[M_W\left(W_M(\overline{\overline{I}}), 3\right), \overline{\overline{r}}_C^\times\right], \quad (55)$$

$$\Phi_{\rm tran}(q,\dot q) = -mW_M(R(q)\omega^\times(q,\dot q)\omega^\times(q,\dot q)), \quad (56)$$

$$\Phi_{\rm rot}(q,\dot q) = \left[-W_M^{\rm T}\left(\omega^\times(q,\dot q)M_W(\omega(q,\dot q))\right), F^{\rm T}_{\rm g}(q)\right]^{\rm T}, \quad (57)$$

$\overline{\overline{r}}_C \in \mathbb{R}^3$ and $\overline{\overline{I}} \in \mathbb{R}^{3\times3}$ are unknown, $M_W(\cdot,\cdot)$ is given by (3), and $W_M(\cdot)$ is given by (4) so that $\Theta_{\rm tran}\Phi_{\rm tran}(q,\dot q) = -mR(q)\omega^\times(q,\dot q)\omega^\times(q,\dot q)\overline{\overline{r}}_C$ and $\Theta_{\rm rot}\Phi_{\rm rot}(q,\dot q) = -\omega^\times(q,\dot q)\overline{\overline{I}}\omega(q,\dot q) + \overline{\overline{r}}_C^\times F_{\rm g}(q)$.

The next theorem applies the adaptive law (34) to regulate the equations of motion (39) of a tilt-rotor quadcopter, whose inertial properties are partly unknown, and bound both the trajectory tracking error $e(\cdot)$ and the estimated adaptive gains' error $\Delta K(\cdot)$ within the user-defined constraint set (20). For the statement of the next result, note that if

$$v_2(t) = \Delta K(t)\pi(t), \qquad t \geq t_0, \quad (58)$$

where $\pi(t) = \left[q^{\rm T}(t), \dot q^{\rm T}(t), 0_{1\times6}, -\Phi^{\rm T}(q(t),\dot q(t))\right]^{\rm T}$, $\Delta K(t) = \hat K(t) - K_{\rm e}$, $\hat K : [t_0,\infty) \to \mathbb{R}^{6\times57}$, $K_{\rm e} = -\left[K_{\rm P}, K_{\rm D}, 0_{6\times6}, \Theta^{\rm T}_{\rm e}\right]$, and $\Theta_{\rm e} \in \mathbb{R}^{39\times6}$ denotes an estimate of Θ, that is, $\|\Theta_{\rm e} - \Theta\| \leq \varepsilon$ with $\varepsilon \geq 0$ arbitrarily small, then (52) is in the same form as (27) with $n = 12$, $m = 6$, $N = 39$, $\widetilde{\Delta K}(t) = \Delta K(t) + K_{\rm e} - K$, $K = -\left[K_{\rm P}, K_{\rm D}, 0_{6\times6}, \Theta^{\rm T}\right]$,

$$\xi(t) = \hat\xi(t) + \begin{bmatrix}0_{6\times1}\\ (\Theta - \Theta_{\rm e})^{\rm T}\Phi(q(t),\dot q(t))\end{bmatrix}, \quad (59)$$

$x_0 = \left[q_0^{\rm T}, q_{\rm d,0}^{\rm T}\right]^{\rm T}$, $A_{\rm ref} = \begin{bmatrix}0_{6\times6} & \mathbf{1}_6\\ -K_{\rm P} & -K_{\rm D}\end{bmatrix}$, and $B = \begin{bmatrix}0_{6\times6}\\ \mathbf{1}_6\end{bmatrix}$. Furthermore, the matching conditions (17) and (18) are verified with $A = \begin{bmatrix}0_{6\times6} & \mathbf{1}_6\\ 0_{6\times6} & 0_{6\times6}\end{bmatrix}$, $B_{\rm ref} = 0_{12\times6}$, $K_x^{\rm T} = -[K_{\rm P}, K_{\rm D}]$, and $K_r = 0_{6\times6}$.

Theorem 4.2: Consider the equations of motion of a tilt-rotor quadcopter given by (39) with $v(t) = \beta(t,q(t),q_{\rm ref}(t),v_2(t))$, $t \geq t_0$, where $\beta(\cdot,\cdot,\cdot,\cdot)$ is given by (51) and $v_2(\cdot)$ is given by (58). Furthermore, consider the the trajectory tracking error dynamics given by (52) and the constraint set given by (20). If $\hat K(\cdot)$ verifies (34) for some $\sigma > 0$ and $p \in \mathbb{N}$ and the conditions of Theorem 4.1 are verified, then $(e(t), \Delta K(t)) \in \mathring{\mathcal{C}}$, $t \geq t_0$.

Proof: The result is a direct consequence of Theorem 4.1 applied to the trajectory tracking error dynamics (52) with $v_2(\cdot)$ given by (58). ■

D. Realization of the desired control inputs

The forces and moments needed for $q(\cdot)$ to track $q_{\rm ref}(\cdot)$ must be realized by generating the appropriate thrust forces $T_i(\cdot)$, $i = 1,\ldots,4$ and tilting the propellers' axes by $\alpha_i(\cdot)$. To this goal, we model the ith propeller's thrust force as

$$T_i(t) = k\dot\Omega_i^2(t), \qquad i = 1,\ldots,4, \qquad t \geq t_0, \quad (60)$$

where $k > 0$ [1]. We also model the ith propeller's moment of the aerodynamic drag as

$$D_i(t) = k_{\rm T}T_i(t), \qquad i = 1,\ldots,4, \qquad t \geq t_0, \quad (61)$$

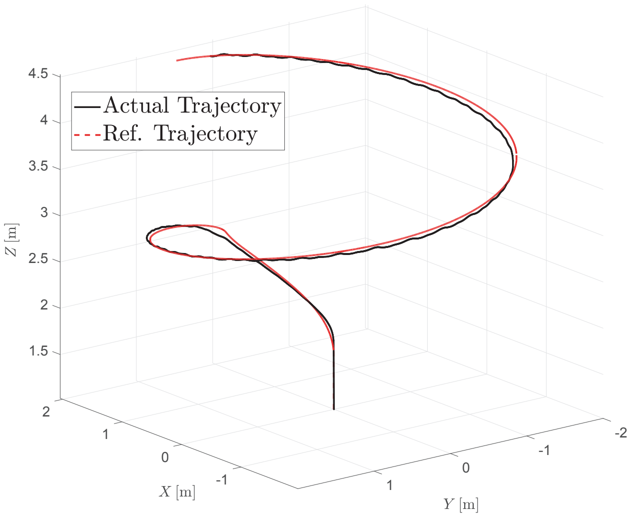

Fig. 2. Trajectory of a tilt-rotor quadcopter following a cylindrical spiral reference trajectory

where $k_{\rm T} > 0$ [1]. Next, assume that adjacent propellers spin in opposite directions and that the propellers are centered at $r_{\rm prop,1} = [L_x, L_y, L_z]^{\rm T}$, $r_{\rm prop,2} = [-L_x, L_y, L_z]^{\rm T}$, $r_{\rm prop,3} = [-L_x, -L_y, L_z]^{\rm T}$, and $r_{\rm prop,4} = [L_x, -L_y, L_z]^{\rm T}$ with respect to the reference point $A(\cdot)$, where $L_x, L_y, L_z \geq 0$ and $L_x L_y > 0$. In this case, it holds that

$$u(t) = MT(t), \qquad t \geq t_0, \tag{62}$$

where $T(t) \triangleq [T_{1,{\rm c}}(t), T_{1,{\rm s}}(t), T_{2,{\rm c}}(t), T_{2,{\rm s}}(t), T_{3,{\rm c}}(t), T_{3,{\rm s}}(t), T_{4,{\rm c}}(t), T_{4,{\rm s}}(t)]^{\rm T}$ denotes the *vector of thrust forces*, $T_{i,{\rm c}}(t) \triangleq T_i(t)\,{\rm c}\alpha_i(t)$, $i = 1, \ldots, 4$, $T_{i,{\rm s}}(t) \triangleq T_i(t)\,{\rm s}\alpha_i(t)$, and

$$M \triangleq \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ L_y & k_{\rm T} & L_y & -k_{\rm T} & -L_y & k_{\rm T} & -L_y & -k_{\rm T} \\ -L_x & L_z & L_x & L_z & L_x & L_z & -L_x & L_z \\ k_{\rm T} & -L_y & -k_{\rm T} & -L_y & k_{\rm T} & L_y & -k_{\rm T} & L_y \end{bmatrix}. \tag{63}$$

Since $L_x L_y > 0$ and $k_{\rm T} > 0$, the matrix M given by (63) is full-rank and the Moore-Penrose inverse of M is given by $M^+ = M^{\rm T}\left[MM^{\rm T}\right]^{-1}$. Thus, given $u : [t_0, \infty) \to \mathbb{R}^5$, we compute the vector of thrust forces as

$$T^*(t) = M^+ u(t), \qquad t \geq t_0, \tag{64}$$

and we compute the propellers' tilt angles as

$$\alpha_i(t) = \tan^{-1}\frac{T^*_{i,{\rm s}}(t)}{T^*_{i,{\rm c}}(t)}, \qquad i = 1, \ldots, 4, \tag{65}$$

where $\tan^{-1}(\cdot)$ denotes the signed inverse tangent function.

V. Illustrative Numerical Example

To illustrate the theoretical results of this paper, we present a numerical simulation that involves the quadcopter shown in Figure 1. This vehicle comprises a frame, four propellers, and an actuated inverted pendulum, which we consider as a payload oscillating at a frequency of $0.5\,{\rm Hz}$ and amplitude of 10 degrees. The length and mass of the pendulum are $0.308\,{\rm m}$ and $0.05\,{\rm kg}$, respectively. The mass of the quadcopter, excluding its payload, is $1.667\,{\rm kg}$, and its principal moments of inertia along the $x(\cdot)$, $y(\cdot)$, and $z(\cdot)$ axes are 0.0317, 0.0643, and $0.0319\,{\rm kg\,m^2}$, respectively.

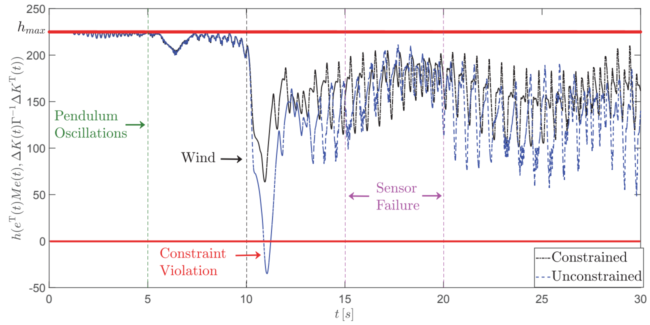

Fig. 3. Constraint function

Figure 2 shows the result of a numerical simulation, whereby the vehicle's reference trajectory is given by $r^{\mathbb{I}}_{\rm ref}(t) = \left[2\cos(0.35(t-10)), 2\sin(0.35(t-10)), 2 + (t-4)/30\right]^{\rm T}$, $t \geq 0$, the reference pitch angle is $\theta_{\rm ref}(t) = 0$, and the reference yaw angle is $\psi_{\rm ref}(t) = 0$. To verify the robustness of the robust MRAC law (34), the vehicle was perturbed by simulating the effect of wind blowing along the X axis of the reference frame $\mathbb{I}$ at a velocity of $10\,{\rm m/s}$ for all $t \geq 10\,{\rm s}$, simulating a failure of the inertial measurement unit, whereby the measured yaw, pitch, and roll angles are equal to zero in the time interval $[15, 20]{\rm s}$, underestimating the vehicle's mass and inertia matrix of 10%.

This simulation has been performed capturing constraints on the trajectory tracking error and the estimated adaptive gains error by means of (20) and (26) with $h_{\max} = 225$ and $M = \mathbf{1}_{12}$. The constraint function $h(\cdot,\cdot)$ is plotted in Figure 3. The root mean square of the path following error is $12 \cdot 10^{-3}\,{\rm m}$. The same simulation has been performed assuming that $h(e^{\rm T}Me, \Delta K\Gamma^{-1}\Delta K^{\rm T}) = 1$, $(e, \Delta K) \in \mathbb{R}^n \times \mathbb{R}^{m \times (n+m+N)}$, and $p = 1$. In this case, as shown by Figure 3, the constraint given by (20) and (26) is violated. A video of this numerical simulation is available at [7].

Acknowledgment

This work was supported in part by the National Science Foundation, the US Army Research Lab, and DARPA under Grants no. 1700640, 40304747, and D18AP00069.

References

[1] A. L'Afflitto, R. B. Anderson, and K. Mohammadi, "An introduction to nonlinear robust control for unmanned quadrotor aircraft," *IEEE Control Systems Magazine*, vol. 38, no. 3, pp. 102–121, 2018.

[2] K. Narendra and A. Annaswamy, "A new adaptive law for robust adaptation without persistent excitation," *IEEE Transactions on Automatic Control*, vol. 32, no. 2, pp. 134–145, 1987.

[3] J. R. Magnus, "On the concept of matrix derivative," *Journal of Multivariate Analysis*, vol. 101, no. 9, pp. 2200 – 2206, 2010.

[4] A. L'Afflitto, *A Mathematical Perspective on Flight Dynamics and Control*. London, UK: Springer, 2017, doi:10.1007/978-3-319-47467-0.

[5] A. L'Afflitto and K. Mohammadi, "Equations of motion of rotary-wing UAS with time-varying inertial properties," *AIAA Journal of Guidance, Control, and Dynamics*, vol. 41, no. 2, pp. 559–564, 2018.

[6] E. Lavretsky and K. Wise, *Robust and Adaptive Control: With Aerospace Applications*. London, UK: Springer, 2012.

[7] R. A. Anderson, J.-P. Burke, and J. A. Marshall. (2018) Robust adaptive control for constrained tilt-rotor quadcopters of unknown inertial properties. [Online]. Available: https://youtu.be/50k3lxd_y1w