On Calculus of Variations in Aircraft and Spacecraft Formation Flying Path Planning

Andrea L'Afflitto[*] and Cornel Sultan[†]
Department of Aerospace and Ocean Engineering
Virginia Polytechnic Institute and State University

The optimal trajectory for formations of aerospace vehicles in terms of *energy consumption* and *fuel consumption* is addressed by mean of Calculus of Variations. Accounting for environmental models of increasing complexity, mission scenarios examined require each vehicle to leave from unspecified positions, pass through assigned waypoints, avoid static obstacles, and intercept allotted surfaces. Tasks are modularly optimized to be effortlessly recombined by varying a finite amount of coefficients. Compactness of the formation as well as collision avoidance is always guaranteed.

Nomenclature

a	=	acceleration of the vehicle
a_c	=	acceleration of the vehicle due to control forces
f	=	generic integrand of the cost function
f_x	=	derivative of f with respect to x
f_r	=	derivative of f with respect to x'
f_{r2}	=	derivative of f with respect to x''
g	=	gravitational acceleration
J	=	optimization/cost/performance index
k_{1p}, k_{2p}…	=	integration constants
m	=	number of waypoints
N	=	number of vehicles of the formation
n	=	dimension of the state vector
r	=	position vector
t	=	time
t_k, t_{k+1}	=	initial and final time
v	=	velocity vector
x	=	state vector
y, z	=	generic smooth functions
λ_0, λ…	=	multipliers
μ	=	gravitational parameter

I. Introduction

CALCULUS of Variations (CV) is the branch of mathematics concerned with finding extremals of functionals. Applying CV to the trajectory optimization problem for single spacecraft and aircraft moving between two given fixed endpoints in realistic environmental conditions has already been discussed[1]. The present paper extends these results to formations of aircraft and spacecraft, constantly ensuring both the compactness of the formation and the collision avoidance between its members. The mission scenario analyzed requires each member of a formation of N aerospace vehicles to leave from an unspecified position at an unspecified velocity and reach an assigned waypoint at given velocity (Task T1). The vehicles are then imposed to pass through m waypoints (Task T2), to

[*] Ph.D. Student, Department of Aerospace and Ocean Engineering, Virginia Tech, 215 Randolph Hall, Blacksburg, VA 24061, andrea.lafflitto@vt.edu, AIAA Space Logistics TC Associate Member, AIAA Senior Member.
[†] Assistant Professor, Department of Aerospace and Ocean Engineering, Virginia Tech, 215 Randolph Hall, Blacksburg, VA 24061, csultan@vt.edu, AIAA Senior Member.

avoid a static obstacle (Task T3), and finally to intercept a given surface (Task T4). The mission is modularly optimized in order to expeditiously recombine the tasks by varying a finite number of constant parameters. While modeling the vehicles as point masses moving in realistic environmental conditions, trajectories found guarantee that the *consumed energy* and the *fuel consumption* are always optimized.

With the advent of the computer era several approaches to the optimal path planning problem, which involve selecting arbitrary parameterizations for the trajectories and searching for purely numerical solutions, have been attempted[4]. The main disadvantage of these techniques is that the selected parameterizations often have no relevance to the performance index of interest. Calculus of Variations, instead, is used to construct trajectory parameterizations that guarantee the optimization of a performance index, eliminating the heuristics of selecting arbitrary parameters.

This paper illustrates the rigorous application of CV to trajectory optimization problems in the framework of a systematic study by using an analytical approach[1-3].

II. Calculus of Variations in Engineering Applications

Meeting design requirements while optimizing assigned cost indices has always assumed a central role in engineering and extensively needed CV, also called in early days *isoperimetric method*, to be tackled. A survey about the history of CV from the mathematical point of view is provided by Sargent[4]. From the engineering perspective one of the first known problems in CV dates back to 800 b.C.: *Queen Dido's problem*[4]. In northern Africa queen Dido received as much land as it could be enclosed within the hide of a bull and she cut the hide into thin strips and laid them out along a circular arc using the Mediterranean coast as supplementary boundary. The first relevant problem of CV in the modern era was formulated in 1638 by G. Galilei who wanted to find the trajectory that minimizes the time of descent of a point mass moving between two given points placed at different height under the effect of the gravitational acceleration (*brachistochrone problem*[6]). Another problem of great interest for aerospace and naval engineering was formulated and solved by I. Newton in his *Principia* (1687): find the shape of the surface of a solid of revolution body moving with constant velocity in the direction of the axis of revolution in a perfect incompressible fluid such that the total pressure drag is minimized, supposing that the frictional force at any point on the surface is proportional to the square of the normal component of the velocity[4]. The theory of linear and non-linear deformation of structural elements can be completely addressed by mean of CV and nowadays it is also applied to solve environmental impact problems like finding the optimal rate to harvest fish or wood maximizing the profits and minimizing the impact of the human action on the environment[4,7].

III. Mathematical Background

A. The Simplest Problem of Calculus of Variations

Define $x(\cdot):[t_k,t_{k+1}]\subset\mathbb{R}\to\mathbb{R}^n$ as the *state vector* and let $\tilde{x}(\cdot)=\left[x^T(\cdot)\;\; x^{'T}(\cdot)\right]^T$, with $k\in\mathbb{N}$ and $x^{'}(\cdot)$ the derivative of $x(\cdot)$ with respect to the independent variable. Among the piecewise smooth (PWS) functions[‡] $\tilde{x}(\cdot):[t_k,t_{k+1}]\subset\mathbb{R}\to\mathbb{R}^{2n}$, define the set $\Omega=\left\{\tilde{x}(\cdot)\in PWS(t_k,t_{k+1})\,|\,\tilde{x}(t_k)=\tilde{x}_k,\tilde{x}(t_{k+1})=\tilde{x}_{k+1}\right\}$ with $\tilde{x}_k$ and $\tilde{x}_{k+1}$ given. Assume[§] $f(\cdot,\cdot,\cdot):D_f\subseteq\mathbb{R}\times\mathbb{R}^{2n}\times\mathbb{R}^{2n}\to\mathbb{R}$ and let[8] $f(z,w_1,w_2)$ be continuous in w_i, $i\in\{1,2\}$, and

[‡]A function $z(\cdot):[t_k,t_{k+1}]\to\mathbb{R}^n$ is called piecewise smooth if for each of its components $z_i(\cdot)$ there exist a piecewise continuous (PWC) function $g_i(\cdot)$ and a constant c_i such that for all $t\in[t_k,t_{k+1}]$, $z_i(t)=c_i+\int_{t_k}^{t} g_i(s)\,ds$. A function $g_i(\cdot):[t_k,t_{k+1}]\to\mathbb{R}$ is piecewise continuous on $[t_k,t_{k+1}]$ if $g_i(\cdot)$ is bounded on $[t_k,t_{k+1}]$, the right hand limit $g_i(t^+)$ exists and is finite on $[t_k,t_{k+1})$, the left hand limit $g_i(t^-)$ exists and is finite on $(t_k,t_{k+1}]$ and there is a finite partition of $[t_k,t_{k+1}]$, $t_k=\hat{t}_1<...<\hat{t}_m=t_{k+1}$ such that $g_i(\cdot)$ is continuous on each subinterval.

[§] Given a function $q(\cdot,...,\cdot)$ of *n* variables, it should be reported as $q\left(\left[w_1,...,w_n\right]^T\right)$. For the sake of compactness it will be expressed as $q(w_1,...,w_n)$.

measurable** in *z*. Define the cost index $J(\cdot)$,

$$J(x(\cdot)) = \int_{t_k}^{t_{k+1}} f\left(t, \tilde{x}(t), \tilde{x}'(t)\right) dt. \quad (1)$$

The functional $J(\cdot)$ is known as *variational integral* and its integrand as *variational integrand* or *Lagrangian.*

The scope of the Simplest Problem of Calculus of Variations[9,10] (SPCV) is to find $x_*(\cdot)$ such that $\tilde{x}_*(\cdot) \in \Omega$ and

$$J(x_*(\cdot)) \leq J(x(\cdot)) \quad (2)$$

for all $\tilde{x}(\cdot) \in \Omega$. The function $x_*(\cdot)$ is called *minimizer* of $J(x(\cdot))$ on Ω.

A more general formulation of the problem considers as cost index

$$J(x(\cdot)) = g\left(t_k, x(t_k), t_{k+1}, x(t_{k+1})\right) + \int_{t_k}^{t_{k+1}} f\left(t, \tilde{x}(t), \tilde{x}'(t)\right) dt.$$

The *terminal cost function* $g(\cdot,\cdot,\cdot,\cdot)$ is aimed at penalizing certain terminal states in spite of others[9], which is quite useful in the presence of conservative vector fields. Considering that this paper is aimed at finding trajectories for aerospace vehicles in realistic environmental conditions characterized by non-conservative forces, such as the aerodynamic ones, suppressing the terminal cost function is not inconvenient.

It is worth to briefly recall the following notions: a norm on $PWS(t_k, t_{k+1})$ is introduced as[8]

$$\|\tilde{x}(t)\| = \sup_{t_k \leq t \leq t_{k+1}} \left\{ \|\tilde{x}(t)\|_2 \right\} \quad (3)$$

where $\|\cdot\|_2$ denotes the Euclidean norm. Given $(\tilde{y}(\cdot), \tilde{z}(\cdot)) \in \Omega^2$, we define two metrics on $PWS(t_k, t_{k+1})$

$$d_0(\tilde{y}(\cdot), \tilde{z}(\cdot)) = \|\tilde{y}(\cdot) - \tilde{z}(\cdot)\| \quad (4)$$

$$d_1(\tilde{y}(\cdot), \tilde{z}(\cdot)) = d_0(\tilde{y}(\cdot), \tilde{z}(\cdot)) + \|\tilde{y}'(\cdot) - \tilde{z}'(\cdot)\|. \quad (5)$$

Some authors[12] prefer a more conservative norm defined as

$$\|\tilde{x}(t)\| = \max_{i \in \{1,..,n\}} \left(\sup_{t_k \leq t \leq t_{k+1}} \left(|\tilde{x}_i(t)| \right) \right) \quad (6)$$

** Given a set $A \subset \mathbb{R}^n$, a function $g(\cdot): A \to \mathbb{R}$ is measurable if the set $\{y \in A : g(y) < \lambda\}$ is measurable $\forall \lambda \in \mathbb{R}$. Given the intervals $[a_i, b_i] \subset \mathbb{R}$, $i \in \{1,...,n\}$, an interval in $\mathbb{R}^n$ is defined as $I = [a_1, b_1] \times ... \times [a_n, b_n]$. A measure of *I* is defined as $\mu(I) = \prod_{i=1}^{n} (b_i - a_i)$. Given a finite or a countable family of intervals I_k, a pluri-interval is defined as $P = \bigcup_k I_k$. We define the measure of *P* as $\mu(P) = \sum_k I_k$. Given a set $\Theta \subset \mathbb{R}^n$, the measure of Θ is defined as the infimum of the measure of the pluri-intervals containing Θ: $\mu^*(\Theta) = \inf_{P \supset \Theta} \mu(P)$. A set $\Theta \subset \mathbb{R}^n$ is measurable if for any interval $I \subset \mathbb{R}^n$, $\mu(I) = \mu^*(I \cap \Theta) + \mu^*(I \setminus \Theta)$.

where $\tilde{x}_i(\cdot)$, $i \in \{1,...,2n\}$, is the *i*-th component of $\tilde{x}(\cdot)$.

Disserting about the opportunity of employing (3) instead of (6) is beyond the scopes of the present work.

Given $\delta > 0$, the $U_0(\tilde{x}(\cdot),\delta)$-neighborhood of $\tilde{x}(\cdot)$ is the open ball

$$U_0(\tilde{x}(\cdot),\delta) = \{\hat{x}(\cdot) \in PWS(t_k,t_{k+1}) \mid d_0(\tilde{x}(\cdot),\hat{x}(\cdot)) < \delta\} \tag{7}$$

while the $U_1(\tilde{x}(\cdot),\delta)$-neighborhood of $x(\cdot)$ is the open ball

$$U_1(\tilde{x}(\cdot),\delta) = \{\hat{x}(\cdot) \in PWS(t_k,t_{k+1}) \mid d_1(\tilde{x}(\cdot),\hat{x}(\cdot)) < \delta\}. \tag{8}$$

If (2) yields for all $\tilde{x}(\cdot) \in \Omega$, then $\tilde{x}_*(\cdot)$ provides a *global minimum.* If (2) yields for all $\tilde{x}(\cdot) \in \Omega \cap U_0(\tilde{x}_*(\cdot),\delta)$, then $x_*(\cdot)$ provides a *strong local minimum* for $J(x(\cdot))$ on Ω while if it yields for all $\tilde{x}(\cdot) \in \Omega \cap U_1(\tilde{x}_*(\cdot),\delta)$, then $x_*(\cdot)$ provides a *weak local minimum* for $J(x(\cdot))$ on Ω.

It is remarkable that, because of the problems herein addressed, the space of PWS functions has been assumed as the class of admissible functions where the minimizer for (1) might exist. In general the class of admissible functions should be found while solving the optimization problem.

By the Euler-Poisson Necessary Condition[9,11] (EPNC), if $x_*(\cdot)$ is a local minimum for $J(x(\cdot))$ on Ω, then

E1. Between corners[††] the functions $f_x(\cdot,\cdot,\cdot)$, $f_r(\cdot,\cdot,\cdot)$, and $f_{r2}(\cdot,\cdot,\cdot)$ are differentiable, and

$$f_x(t,\tilde{x}_*(t),\tilde{x}_*'(t)) - \frac{d}{dt} f_r(t,\tilde{x}_*(t),\tilde{x}_*'(t)) + \frac{d^2}{dt^2} f_{r2}(t,\tilde{x}_*(t),\tilde{x}_*'(t)) = 0 \tag{9}$$

E2. $\tilde{x}_*(t_k) = \tilde{x}_k$ and $\tilde{x}_*(t_{k+1}) = \tilde{x}_{k+1}$.

This theorem holds both for weak and for strong local minima. The subscripts *x*, *r*, and *r2* identify the partial derivatives with respect to $x(\cdot)$, $x'(\cdot)$, and $x''(\cdot)$ respectively. Any $x(\cdot)$ that verifies eq. (9) is an *extremal*. Note that E2 follows from the fact that we are solving the SPCV, i.e. $\tilde{x}_*(t_k)$ and $\tilde{x}_*(t_{k+1})$ are imposed a priori, and it must be verified also for the Legendre-Hadamard and the Jacobi conditions exposed hereafter.

The EPNC is invariant of the reference system: let $T(\cdot,\cdot): \mathbb{R}\times\mathbb{R}^{2n} \to \mathbb{R}\times\mathbb{R}^{2n}$ be smooth non-singular[‡‡] such that $T(t,\tilde{x}(t)) = [\tau(t,\tilde{x}(t)) \;\; \tilde{\chi}^T(t,\tilde{x}(t))]^T$. By applying $T(\cdot,\cdot)$, the cost index (1) becomes[12]

[††] A function $x(\cdot)$ defined on the interval $[t_k,t_{k+1}]$ has a corner in $\hat{t} \in (t_k,t_{k+1})$ if $\left.\frac{dx^+(t)}{dt}\right|_{t=\hat{t}} \neq \left.\frac{dx^-(t)}{dt}\right|_{t=\hat{t}}$.

[‡‡] Let $r(\cdot,\cdot): \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}\times\mathbb{R}^n$ be such that $r(\cdot,\cdot) = [r_1(\cdot,\cdot) \;\; r_2^T(\cdot,\cdot)]^T$, where $r_1(\cdot,\cdot): \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$, and $r_2(\cdot,\cdot): \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$. If $\rho_2(\cdot): \mathbb{R} \to \mathbb{R}^n$, then $r(\rho_1,\rho_2(\rho_1))$ is *smooth* if $\frac{\partial r_i(\rho_1,\rho_2(\rho_1))}{\partial \rho_j}$, $(i,j) \in \{1,2\}^2$, are continuous. The function $r(\cdot,\cdot)$ is *non-singular* if $\det\left(\frac{\partial r(\rho_1,\rho_2(\rho_1))}{\partial([\rho_1,\rho_2^T]^T)}\right) \neq 0$.

$$J\left(\chi(\cdot,\cdot)\right)=\int_{\tau_k}^{\tau_{k+1}}\left(f\left(\tau\left(t,\tilde{x}(t)\right),\tilde{\chi}\left(t,\tilde{x}(t)\right),\tilde{\chi}^{'}\left(t,\tilde{x}(t)\right)\right)\frac{\partial t\left(\tau,\tilde{\chi}(\tau)\right)}{\partial\tau}\right)d\tau \tag{10}$$

where $\tau_j=\tau\left(t_j,\tilde{x}\left(t_j\right)\right)$, $j\in\{k,k+1\}$. Given $\tilde{\chi}_j$, $x_*(\cdot)$ is an extremal for (1) on Ω if and only if $\chi_*(\cdot,\cdot)$ is an extremal for (10) on[12] $\Psi=\left\{\tilde{\chi}(\cdot,\cdot)\in PWS(\tau_k,\tau_{k+1})\mid\tilde{\chi}\left(t_k,\tilde{x}(t_k)\right)=\tilde{\chi}_k,\tilde{\chi}\left(t_{k+1},\tilde{x}(t_{k+1})\right)=\tilde{\chi}_{k+1}\right\}$.

The Legendre Hadamard Condition[11] (LHC) states that if $x_*(\cdot)$ is a weak local minimum for $J\left(x(\cdot)\right)$ on **Ω**, then

L1. $f_{r2r2}\left(t,\tilde{x}_*(t),\tilde{x}_*^{'}(t)\right)\succeq 0$ (11)

where the curly inequality sign indicates that the matrix in (11) is positive semidefinite.

If the curly inequality sign is strictly positive, then the *strengthen* LHC is verified. If $\frac{\partial^2}{\partial\left(y_3\right)^2}f\left(w,w_1,w_2\right)\succ 0$, for any *w, y*$_1$, *y*$_2$, *y*$_3$ such that $w_1=\left[y_1^T\ \ y_2^T\right]^T\in\mathbb{R}^{2n}$, $w_2=\left[y_2^T\ \ y_3^T\right]^T\in\mathbb{R}^{2n}$, and for any $w\in\mathbb{R}$, then $f(\cdot,\cdot,\cdot)$ is called *regular*. Furthermore if $x_*(\cdot)$ is an extremal and $f_{r2r2}\left(t,\tilde{x}_*(t),\tilde{x}_*^{'}(t)\right)\succ 0$ for all *t* where $\tilde{x}_*^{'}(\cdot)$ is defined, then $\tilde{x}_*(\cdot)$ is called *regular*. If $f(\cdot,\cdot,\cdot)$ is regular, all extremals do not have corners. If $\frac{\partial^2}{\partial\left(y_3\right)^2}f\left(w,w_1,w_2\right)\neq 0$, then $f(\cdot,\cdot,\cdot)$ is called *non-singular*. In addition, if $x_*(\cdot)$ is an extremal such that $\tilde{x}_*(\cdot)\in PWS\left(t_k,t_{k+1}\right)$ and $f_{r2r2}\left(t,\tilde{x}_*(t),\tilde{x}_*^{'}(t)\right)\neq 0$ at all points where $\tilde{x}_*^{'}(\cdot)$ is defined, then $\tilde{x}_*(\cdot)$ is *non-singular*.

Let $\eta(\cdot):\mathbb{R}\to\mathbb{R}^n$, $\tilde{\eta}^T(\cdot)=\left[\eta^T(\cdot)\ \ \eta^{'T}(\cdot)\right]$, $\mathrm{N}^T\left(\eta(t)\right)=\left[\eta^T(t)\,\eta^{'T}(t)\,\eta^{''T}(t)\right]$, $X_*^T(t)=\left[x_*^T(t)\,x_*^{'T}(t)\,x_*^{''T}(t)\right]$, and $\Phi\left(f\left(t,\tilde{x}_*(t),\tilde{x}_*^{'}(t)\right)\right)=\frac{\partial^2}{\partial X_*^2}f\left(t,\tilde{x}_*(t),\tilde{x}_*^{'}(t)\right)$. Given the *accessory* cost index[9]

$$\frac{1}{2}\int_{t_k}^{t_{k+1}}\mathrm{N}^T\left(\eta(t)\right)\Phi\left(f\left(t,\tilde{x}_*(t),\tilde{x}_*^{'}(t)\right)\right)\mathrm{N}\left(\eta(t)\right)dt, \tag{12}$$

the *secondary problem* requires to find a *secondary extremal* $\eta_*(\cdot)$ not identically zero such that $\tilde{\eta}_*(\cdot)\in\Omega_\eta:=\left\{\tilde{\eta}(\cdot)\in PWS\left(t_k,t_{k+1}\right)\mid\tilde{\eta}\left(t_k\right)=0,\tilde{\eta}\left(t_c\right)=0\right\}$. A scalar t_c, if any, is a *focal point* conjugate to t_k if it is the first root greater than t_k such that $\tilde{\eta}_*\left(t_c\right)=0$ and $\tilde{\eta}_*(t)\neq 0$ for all $t\in\left(t_k,t_c\right)$. The Jacobi Necessary Condition[9,11] (JNC) states that if $x_*(\cdot)$ provides a weak local minimum for $J\left(x(\cdot)\right)$ on Ω and if $\tilde{x}_*(\cdot)$ is smooth and regular, then there cannot exist t_c conjugate to t_k such that $t_c<t_{k+1}$. The Strengthen JNC is verified if $t_c\leq t_{k+1}$.

The JNC therefore imposes a condition on the interval $\left[t_k,t_c\right]$ where a candidate weak local minimum for (1) may exist. The Erdman corner condition, the Weierstrass Necessary Condition[9], and the Bolza Necessary Condition[11], which is rarely used, are not presented as they are not needed for the scopes of the problems herein addressed. According to Ewing[11], although there possibly remain other undiscovered necessary conditions, searching for them seems unlikely to be a profitable endeavor.

Three fundamental sufficient conditions of CV can be formulated[9-12]

S1. If $x_*(\cdot)\in\Omega$ is smooth and satisfies condition EPNC, the strengthen LHC, and the strengthen JNC, then $x_*(\cdot)$ provides a *weak* local minimum for $J\left(x(\cdot)\right)$ on **Ω**.

S2. If $f(\cdot,\cdot,\cdot)$ is regular, the ENC and the strengthen JNC yield, then $x_*(\cdot)$ provides a *strong* local minimum for $J(x(\cdot))$ on Ω.

S3. Define $\Omega_x = \{(\tilde{y}_1,\tilde{y}_2) \mid (t,\tilde{y}_1,\tilde{y}_2) \in D_f\}$ for all $t \in [t_k, t_{k+1}]$. If Ω_x is convex, if $f(t,\tilde{y}_1,\tilde{y}_2)$ is convex in the variables $(\tilde{y}_1,\tilde{y}_2)$ such that $(t,\tilde{y}_1,\tilde{y}_2) \in [t_k,t_{k+1}] \times \Omega_x$, and if $x_*(\cdot)$ satisfies the sufficient conditions S1 and S2, then $x_*(\cdot)$ is a global minimum for $J(x(\cdot))$ on Ω.

There not exist more general sufficient conditions for global minima that are as operative as S3. This point still represents an open field of research in mathematics[13].

The order used to present these necessary and sufficient conditions of CV is the one used by Bliss[10], who used to name the EPNC as (I), the LHC as (III), and the JNC as (IV). Most of the literature in CV still adopts Bliss' nomenclature but it might be confusing as Bolza in his *Lectures on Calculus of Variations* and several scholars after him identify the JNC as (II) and the LHC as (III). The approach herein adopted is known as *indirect method*: minimizers for (1) are found by applying the EPNC, the LHC, and the JNC. *Direct methods*, instead, are aimed at establishing directly the existence of minimizers by means of set-theoretic arguments[8].

In the following some modifications to the theory of the SPCV are discussed: endpoints are not assigned a priori but can be unspecified (Free Endpoint Problem), or can be generically imposed to lay upon a surface (Point to Surface Problem), or the optimal trajectories can also be imposed to satisfy equality and inequality constraints.

B. The Free Endpoint Problem

For $(i,j) \in \{k,k+1\}^2$ and $i \neq j$, define $\Omega_F = \{\tilde{x}(\cdot) \in PWS(t_k,t_{k+1}) \mid \tilde{x}(t_i) = \tilde{x}_i\}$ where $\tilde{x}_i$ is given. The scope of the Free Endpoint Problem[9] (FEP) is to find $x_*(\cdot)$ such that $\tilde{x}_*(\cdot) \in \Omega_F$ and (2) holds for all $\tilde{x}(\cdot) \in \Omega_F$. It is worth to stress that for the FEP, in spite of the SPCV, the boundary condition $\tilde{x}(t_j) = \tilde{x}_j$ is not assigned.

The norm defined in (3), the metrics (4) and (5), the neighborhoods introduced in (7) and (8), and finally the definitions of global, strong and weak local minima still hold on Ω_F and not on Ω.

The EPNC, LHC, and JNC still hold but instead of E2 the conditions to impose are[9]

$$\tilde{x}_*(t_i) = \tilde{x}_i, \;\; f_{r2}\left(t_j,\tilde{x}_*(t_j),\tilde{x}_*'(t_j)\right) = 0, \;\; f_r\left(t_j,\tilde{x}_*(t_j),\tilde{x}_*'(t_j)\right) = \frac{d}{dt} f_{r2}\left(t,\tilde{x}_*(t),\tilde{x}_*'(t)\right)\Big|_{t=t_j}. \tag{13}$$

C. The Point to Surface Problem

Let $PWS[t_k,+\infty) := \{x(\cdot) : [t_k,+\infty) \subset \mathbb{R} \to \mathbb{R}^n : x(\cdot) \in PWS(t_k,T), \forall T > t_k\}$ and assume $S(\cdot) : z \in \mathbb{R}^n \to \mathbb{R}$ is a smooth surface such that $S(z) = 0$. The scope of the Point to Surface Problem[9] (PSP) is to find a function $x_*(\cdot) \in \Omega_S := \{\tilde{x}(\cdot) \in PWS[t_k,+\infty) \mid \tilde{x}(t_k) = \tilde{x}_k, S(x(t_{k+1}^*)) = 0\}$ such that (2) is verified for all $x(\cdot) \in \Omega_S$ and for $t_{k+1}^* > t_k$ unknown. Two cases are possible: when $x_*'(t_{k+1}^*)$ is assigned and when $x_*'(t_{k+1}^*)$ is arbitrary. The latter case is herein addressed. Definitions (3) - (8) and those of global, strong, and weak local minima hold on Ω_S. The EPNC, LHC, and JNC are the same as for the SPCV but E2 becomes:

$$\begin{gathered}\tilde{x}_*(t_k) = \tilde{x}_k, \; S\left(x_*(t_{k+1}^*)\right) = 0, \;\; f\left(t_{k+1}^*,\tilde{x}_*(t_{k+1}^*),\tilde{x}_*'(t_{k+1}^*)\right) = \Upsilon^T x_*'(t_{k+1}^*) + f_{r2}^T\left(t_{k+1}^*,\tilde{x}_*(t_{k+1}^*),\tilde{x}_*'(t_{k+1}^*)\right) x_*''(t_{k+1}^*) \\ \Upsilon = \lambda\left(\frac{\partial^2 S(z)}{\partial z^2}\Big|_{z=x_*(t_{k+1}^*)} x_*'(t_{k+1}^*)\right) + \mu \frac{\partial S(z)}{\partial z}\Big|_{z=x_*(t_{k+1}^*)}, \;\; f_{r2}\left(t_{k+1}^*,\tilde{x}_*(t_{k+1}^*),\tilde{x}_*'(t_{k+1}^*)\right) = \lambda \frac{\partial S(z)}{\partial z}\Big|_{z=x_*(t_{k+1}^*)}\end{gathered} \tag{14}$$

where $\Upsilon := f_r\left(t_{k+1}^*,\tilde{x}_*(t_{k+1}^*),\tilde{x}_*'(t_{k+1}^*)\right) - \frac{d}{dt} f_{r2}\left(t_{k+1}^*,\tilde{x}_*(t_{k+1}^*),\tilde{x}_*'(t_{k+1}^*)\right)$, λ and μ constants such that $|\lambda| + |\mu| \neq 0$. The optimal final time t_{k+1}^* must be deduced from (14).

It can be proven that the FEP is a special case of the PSP[9]. Furthermore both the FEP and the PSP belong to the broad category of *problems with variable endpoints*.

D. Pointwise Inequality Constraints

The optimal trajectory for the SPCV, the FEP, or the PSP can be constrained by $\theta(\cdot):[t_k,t_{k+1}]\to\mathbb{R}^n$, such that[4,14] $\tilde{\theta}(\cdot):=\left[\theta^T(\cdot)\ \theta^{'T}(\cdot)\right]^T\in PWS(t_k,t_{k+1})$ and $x_*(t)\le\theta(t)$ for $t\in[t_k,t_{k+1}]$. The inequality is meant component-wise. If $x_*(\cdot)$ is an extremal for the constrained SPCV, FEP, or PSP, and if an extremal for the corresponding unconstrained problem violates the inequality constraints in $(\tilde{t}_k,\tilde{t}_{k+1})\subseteq[t_k,t_{k+1}]$, then $x_*(\cdot)$ consists of segments where $x_*(t)=\theta(t)$ and segments where $x_*(\cdot)$ is an extremal for (1). At the *switching points* that join these segments, $\tilde{x}_*(\cdot)$ is continuous and they are optimally located at t_s such that

$$\begin{aligned}&f\left(t_s,\tilde{x}_*(t_s),\tilde{x}_*^{'}(t_s)\right)-f\left(t_s,\tilde{\theta}(t_s),\tilde{\theta}^{'}(t_s)\right)=\\&=\left[x_*^{'}(t_s)-\theta^{'}(t_s)\right]^T f_{r2}\left(t_s,\tilde{x}_*(t_s),\tilde{x}_*^{'}(t_s)\right)-\left[x_*(t_s)-\theta(t_s)\right]^T f_r\left(t_s,\tilde{x}_*(t_s),\tilde{x}_*^{'}(t_s)\right).\end{aligned}\tag{15}$$

E. Pointwise Equality Constraints – The Theorem of Multipliers

Given *m* functions $\varphi_i(\cdot,\cdot):[t_k,t_{k+1}]\times\mathbb{R}^n\to\mathbb{R}$, with $i\in\{1,2,...,m\}\subset\mathbb{N}$ and $m<n$, the SPCV, the FEP, and the PSP can be altered by imposing that $x_*(\cdot)$ satisfies *m* equality constraints[10,15] $\varphi_i(t,x_*(t))=0$, $\forall t\in[t_k,t_{k+1}]$.

The following theorem, known as *Theorem of Multipliers*, holds: if $x_*(\cdot)$ is a local or global minimizer for the SPCV, FEP, or PSP and it satisfies the *m* constraints $\varphi_i(t,x_*(t))=0$ for all $t\in[t_k,t_{k+1}]$, then there exist a constant λ_0 and a function $\lambda(\cdot):[t_k,t_{k+1}]\to\mathbb{R}^m$, such that $|\lambda_0|+\|\lambda(t)\|_2\ne 0$ for all $t\in[t_k,t_{k+1}]$, and $x_*(\cdot)$ is an extremal of

$$I(x(\cdot))=\int_{t_k}^{t_{k+1}}\left(\lambda_0 f\left(t,\tilde{x}(t),\tilde{x}^{'}(t)\right)+\lambda^T(t)\varphi\left(t,x(t)\right)\right)dt\tag{16}$$

where $\varphi(\cdot,\cdot)=\left[\varphi_1(\cdot,\cdot)...\varphi_m(\cdot,\cdot)\right]^T$.

In addition if for all $(s,y)\in\{(s,y):\varphi(s,y)=0\}$, $\dfrac{\partial\varphi(s,y)}{\partial y}$ is full ranked, then $\lambda_0\ne 0$ and (16) becomes

$$I(x(\cdot))=\int_{t_1}^{t_2}\left(f\left(t,\tilde{x}(t),\tilde{x}^{'}(t)\right)+\lambda^T(t)\varphi\left(t,x(t)\right)\right)dt.\tag{17}$$

If $\lambda_0\ne 0$ then the problem is said to be *normal* otherwise it is *abnormal*. In case of abnormal problems, $x_*(\cdot)$ does not depend on the cost function (1) but on the constraints only. Concerning normal problems, the theorem of multipliers recasts the original problem of finding *m* constrained components of $x_*(\cdot)$ and the remaining unconstrained ones into an *unconstrained* problem with $n+m$ unknowns, namely the *n* components of $x_*(\cdot)$ and the *m* components of $\lambda(\cdot)$ called *multipliers*[7, 10,11]. The properties of the multipliers[4,10] are beyond the scopes of the present work but it is worth to stress how they measure the sensitivity[15] of the cost function to variations of $\varphi(\cdot)$.

Once $x_*(\cdot)$, λ_0, and $\lambda(\cdot)$ have been determined, the LHC, the JNC, and the related sufficient conditions for local and global minima (S1-S3) can be applied to verify if any of the extremals of (16), or (17), is a weak, strong or global minimum for the constrained problem.

IV. Applications – Cost Indices Optimization

In this section some applications of the previous theory are presented. Specifically the problem of finding the optimal trajectory in terms of *consumed energy* and *fuel consumption* for a formation of N aircraft or spacecraft schematized as geometrical point masses is tackled. The mission scenario studied foresees four consecutive tasks:

T1. Each element of the formation leaves from an unspecified position with unspecified velocity to reach an assigned position at prescribed velocity;
T2. Each vehicle passes through m assigned waypoints with designated velocity at a given time;
T3. A static obstacle is avoided by the formation;
T4. Vehicles reach an assigned surface at unknown velocity to conclude the mission.

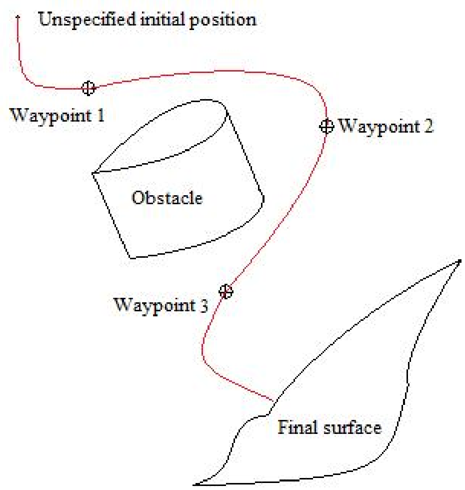

Figure 1. Sketch of tasks T1 – T4 for a generic vehicle of the formation

Several environmental conditions are considered: *constant gravity* (the vehicle moves in a constant gravitational vector field), *constant gravity with aerodynamic forces* (the constant gravity model is improved by accounting also for aerodynamic forces acting on the vehicle), *radial gravity* (the vehicle moves in the gravitational field generated by a central massive point mass under Keplerian assumptions[§§,16]) and *radial gravity with aerodynamic forces* (the radial gravity environment is improved accounting for perturbing aerodynamic forces).

Collision avoidance among the elements of the formation is enforced by imposing that each vehicle maintains a given distance from the other ones, which is always valuable for several applications like remote sensing[17].

A. Consumed Energy Optimization – Single Vehicle

For a point mass model, if $r_p(\cdot)$ is the position vector of the *p*-th vehicle of the formation in an inertial reference frame, $v_p(\cdot)$, $a_p(\cdot)$, and $F_p(\cdot)$ the velocity, acceleration, and resultant force respectively, the equations of motion are given by $F_p(t) = m_p a_p(t) = m_p \dfrac{dv_p(t)}{dt} = m_p \dfrac{d^2 r_p(t)}{dt^2}$ where m_p is the mass of the vehicle assumed constant.

A common performance index is the consumed energy that can be expressed as[18]

$$J_p = \int_{t_k}^{t_{k+1}} a_p^T(t) a_p(t) dt. \tag{18}$$

We want to find the trajectory of the *p*-th vehicle that minimizes the index (18) without accounting for the collision avoidance constraint. In par. IV.B (18) is minimized considering also the collision avoidance constraint.

---

§§ A central massive body is assumed to be reduced to a point mass and to generate a gravitational field. The mass of the vehicle moving in this field is assumed to be negligible with respect to the mass of the central body.

Let $x_p(\cdot) \equiv r_p(\cdot)$, then $f\left(t, \tilde{x}_p(t), \tilde{x}_p^{'}(t)\right) = a_p^T(t) a_p(t) = x_p^{"T}(t) x_p^{"}(t)$. From the Euler Poisson Necessary Condition, by applying eq. (9), it follows that candidate minimizers for (18) are solutions of

$$\frac{\partial}{\partial x_p}\left(x_p^{"T}(t) x_p^{"}(t)\right) - \frac{d}{dt}\frac{\partial}{\partial x_p^{'}}\left(x_p^{"T}(t) x_p^{"}(t)\right) + \frac{d^2}{dt^2}\frac{\partial}{\partial x_p^{"}}\left(x_p^{"T}(t) x_p^{"}(t)\right) = 0$$

which leads to

$$a_{p*}(t) = k_{1p}(t - t_k) + k_{2p}. \tag{19}$$

Consequently candidate optimal trajectories for the cost index (18) on a time interval $[t_k, t_{k+1}]$ are given by

$$r_{p*}(t) = x_{p*}(t) = \frac{1}{6}k_{1p}(t - t_k)^3 + \frac{1}{2}k_{2p}(t - t_k)^2 + k_{3p}(t - t_k) + k_{4p}. \tag{20}$$

where k_{1p}, k_{2p}, k_{3p}, and k_{4p} are integration constants that depend on the task T1 – T4 to be accomplished.

The strengthen Legendre-Hadamard Condition holds because

$$\frac{\partial^2}{\partial x_{p*}^{"\,2}}\left(x_{p*}^{"T}(t) x_{p*}^{"}(t)\right) = 2I \succ 0 \tag{21}$$

where *I* is the identity matrix. Furthermore the (21) proves that the integrand of the (18) is a regular function.

The strengthen JNC can be proven to hold considering that $\frac{\partial^2}{\partial X_*^2} x_*^{"T}(t) x_*^{"}(t) = 2\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \in \mathbb{R}^{9\times 9}$ and the accessory cost index (12) is $\int_{t_k}^{t_{k+1}} \eta_p^{T"}(t) \eta_p^{"}(t) dt$, whose extremals are in the form

$$\eta_{p*}(t) = \frac{1}{6}\alpha_{1p}(t - t_k)^3 + \frac{1}{2}\alpha_{2p}(t - t_k)^2 + \alpha_{3p}(t - t_k) + \alpha_{4p} \tag{22}$$

where α_{1p}, α_{2p}, α_{3p}, and α_{4p} are integration constants. By imposing that $\eta_{p*}(t_k) = \eta_{p*}^{'}(t_k) = 0$, it follows that $\alpha_{4p} = \alpha_{3p} = 0$. Furthermore from $\eta_{p*}(t_c) = 0$ it follows that $\alpha_{1p} = -\frac{3\alpha_{2p}}{t_c - t_k}$ and $\eta_{p*}^{'}(t_c) = -\frac{1}{2}\alpha_{2p}(t_c - t_k)$. Lastly from the condition $\eta_{p*}^{'}(t_c) = 0$ it must follow that $\alpha_{2p} = \alpha_{1p} = 0$. This implies that $\eta_{p*}(t) \equiv 0$, which cannot occur, and therefore there not exist a secondary extremal for the given problem and the strengthen JNC holds on $\mathbb{R}$.

Being $x_{p*}(\cdot)$ a smooth function and as the EPNC, the LHC, and the JNC are always verified on any interval $[t_k, t_{k+1}]$, then the sufficient condition for weak local minima (S1) is met. In addition, as $a_p^T(\cdot) a_p(\cdot)$ has been proven to be regular, the sufficient condition for strong local minima (S2) holds. In conclusion, for $(z_0, z_1, z_2) \in [t_k, t_{k+1}] \times \mathbb{R}^{2n} \times \mathbb{R}^{2n}$, $z_2 = \left[z_{2_1}^T \; z_{2_2}^T\right]^T$, and $f(z_0, z_1, z_2) = z_{2_2}^T(t) z_{2_2}(t)$, $\frac{\partial}{\partial z_j} f(z_0, z_1, z_2)$ is continuous, $j \in \{0,1,2\}$, and $\dfrac{\partial^2 f(z_0, z_1, z_2)}{\partial\left(\left[z_1^T \ z_2^T\right]^T\right)^2} = 2\begin{bmatrix} 0 & 0 \\ 0 & \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \end{bmatrix} \succeq 0$. Thus the integrand of (18) is convex on $[t_k, t_{k+1}] \times \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ and by S3 $x_{p*}(\cdot)$ is a global minimum for $J(x(\cdot))$.

It is worth to stress that the LHC and the JNC, as well as the sufficient conditions for weak, strong, and global minima, are verified independently from the boundary conditions and the environmental conditions.

Accounting for (19), it holds that the propulsion system of the *p*-th vehicle needs to deliver adequate forces to ensure that the total acceleration is linear in *t*. By the superposition principle it holds that

$$a_{p*}(t) = a_{pc*}(t) + a_{pg*}(t) + a_{pa*}(t) \tag{23}$$

where $a_{pc*}(\cdot)$ is the optimal control acceleration delivered by the propulsion system, $a_{pg*}(\cdot)$ is the acceleration due to gravitational forces and $a_{pa*}(\cdot)$ is the acceleration generated by aerodynamic forces. For aircraft a simple model of aerodynamic force, $F_{pa}(\cdot)$, is given by

$$F_{pa}(t) = \frac{1}{2}\rho S v_p^T(t) v_p(t)\left(-C_D \hat{v}_p(t) + C_L \hat{v}_{p\perp}(t) + C_S \hat{v}_{p\times}(t)\right) \tag{24}$$

where ρ is the air density, *S* is the reference area, $C_{D/L/S}$ are the drag, lift, and side force coefficients, $\hat{v}_p(\cdot)$, $\hat{v}_{p\perp}(\cdot)$, and $\hat{v}_{p\times}(\cdot)$ are the unit vectors in the direction of the velocity, of the lift, and of the side force, respectively. From (24) it follows that the acceleration due to the aerodynamic force is

$$a_{pa}(t) = v_p^T(t) v_p(t)\left(-\tilde{k}_D \hat{v}_p(t) + \tilde{k}_L \hat{v}_{p\perp}(t) + \tilde{k}_S \hat{v}_{p\times}(t)\right) \tag{25}$$

where $\tilde{k}_{D/L/S} = \dfrac{\rho S C_{D/L/S}}{2m}$ and consequently

$$a_{pc*}(t) = k_{1p} t + v_{p*}^T(t) v_{p*}(t)\left(\tilde{k}_D \hat{v}_{p*}(t) - \tilde{k}_L \hat{v}_{p\perp*}(t) - \tilde{k}_S \hat{v}_{p\times*}(t)\right) + k_{2p} - g \tag{26}$$

where *g* is the constant gravitational acceleration.

For spacecraft orbiting around a central massive body, an inverse distance square gravitational field is appropriate[16]. If we fix the origin of the inertial reference frame in the center of the massive body, then $a_{pg}(t) = -\dfrac{\mu}{\left\|r_p(t)\right\|_2^3} r_p(t)$ and the *globally* optimal trajectory requires a control acceleration

$$a_{pc*}(t) = k_{1p} t + k_{2p} + \frac{\mu}{\left\|r_{p*}(t)\right\|_2^3} r_{p*}(t) \tag{27}$$

where μ is the gravitational constant.

Satellites, especially in Low Earth Orbit (LEO), are subject to forces due to the impingement of molecules of air on their surfaces. This effect, known as *aerodynamic drag*, is modeled as in (24) with[16] $C_{L/S}=0$. Thus,

$$a_{pc*}(t) = k_{1p} t + \frac{\mu}{\left\|r_{p*}(t)\right\|_2^3} r_{p*}(t) + \tilde{k}_D v_{p*}^T(t) v_{p*}(t) \hat{v}_{p*}(t) + k_{2p}. \tag{28}$$

Solutions to eq. (26), (27), and (28) can be efficiently integrated numerically[19,20].

It is remarkable how the constants k_{1p} and k_{2p} depend on the boundary conditions only, i.e. on the task to perform, but not on the environmental conditions. In the phase T1, the *p*-th vehicle is supposed to leave an unspecified initial position at time t_1 at an unspecified velocity to reach the waypoint located in r_{p2} with velocity v_{p2} at time t_2. This can be modeled as a free endpoint problem and therefore from (13) it must hold that

$$x_{p*}(t_2) = r_{p2},\ x'_{p*}(t_2) = v_{p2},\ 2a_{p*}(t_1) = 2(k_{1p}t_1 + k_{2p}) = 0,\ \frac{d}{dt}(k_{1p}t + k_{2p})\Big|_{t=t_1} = 0$$

which lead to the following solution

$$k_{1p} = 0,\ k_{2p} = 0,\ k_{3p} = v_{p2},\ k_{4p} = r_{p2} - v_{p2}(t_2 - t_1). \tag{29}$$

Assuming without loss of generality that $m = 1$, task T2 can be modeled as a SPCV. Imposing $x_{p*}(t_3) = r_{p3}$ and $x'_{p*}(t_3) = v_{p3}$, the constants k_{1p}, k_{2p}, k_{3p}, and k_{4p} are obtained by applying the conditions E2 to eq. (20):

$$k_{1p} = 6\frac{v_{p2}+v_{p3}}{(t_3-t_2)^2} - 12\frac{r_{p3}-r_{p2}}{(t_3-t_2)^3},\ k_{2p} = 6\frac{r_{p3}-r_{p2}}{(t_3-t_2)^2} - 2\frac{2v_{p2}+v_{p3}}{t_3-t_2},\ k_{3p} = v_{p2},\ k_{4p} = r_{p2}. \tag{30}$$

Assume that the *p*-th vehicle leaves the waypoint r_{p3} to reach r_{p4} at time t_4 with velocity v_{p4}. Avoiding a static obstacle between these waypoints represents the task T3. Let this obstacle be an orthogonal cylinder with polynomial cross section and axis parallel to the third axis of the inertial reference frame. The cylinder constitutes an inequality constraint for the first two components of the optimal trajectory and therefore we need to impose that

$$X_{p*}(t) \le \Theta(t) \tag{31}$$

where the components of $X_{p*}(\cdot)$ are the first two components of $x_{p*}(\cdot)$ and

$$\Theta(t) = \left[\beta_{q+1}(t-t_3)\quad \sum_{j=0}^{q}\beta_j (t-t_3)^j\right]^T \tag{32}$$

with β_j, $j \in \{0,...,q+1\}$, constants and $\beta_{q+1} > 0$.

By the theory of holonomic pointwise inequality constraints, if the first two components of the unconstrained optimal solution violate (31) for some $t \in [t_3, t_4]$, then the candidate optimal trajectory can be written as

$$X_{p*}(t) = \begin{cases} \frac{1}{6}K_{1p}^{[1]}(t-t_3)^3 + \frac{1}{2}K_{2p}^{[1]}(t-t_3)^2 + K_{3p}^{[1]}(t-t_3) + K_{4p}^{[1]},\ t \in [t_3, t_{s1}] \\ \Theta(t),\ t \in [t_{s1}, t_{s2}] \\ \frac{1}{6}K_{1p}^{[2]}(t-t_{s2})^3 + \frac{1}{2}K_{2p}^{[2]}(t-t_{s2})^2 + K_{3p}^{[2]}(t-t_{s2}) + K_{4p}^{[2]},\ t \in [t_{s2}, t_4] \end{cases} \tag{33}$$

with $K_{ip}^{[j]}$, $(i,j) \in \{1,2,3,4\} \times \{1,2\}$, integration constants obtained by imposing both the boundary conditions at t_3 and t_4, and the continuity of $X_{p*}(\cdot)$ and $X'_{p*}(\cdot)$ at t_{s1} and t_{s2}, which can be determined by applying eq. (15). Thus $K_{4p}^{[1]} = R_{p3}$ and $K_{3p}^{[1]} = V_{p3}$, where R_{p3} and V_{p3} are the first two components of r_{p3} and v_{p3} respectively. Moreover

$$
\begin{cases}
\begin{bmatrix} \frac{1}{6} I T_{s1}^3 & \frac{1}{2} I T_{s1}^2 \\ \frac{1}{2} I T_{s1}^2 & I T_{s1} \end{bmatrix} \begin{bmatrix} K_{1p}^{[1]} \\ K_{2p}^{[1]} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \beta_{q+1} T_{s1} & \sum_{j=0}^{q} \beta_j T_{s1}^j \end{bmatrix}^T & \begin{bmatrix} \beta_{q+1} & \sum_{j=1}^{q} j \beta_j T_{s1}^{j-1} \end{bmatrix}^T \end{bmatrix}^T - \begin{bmatrix} V_{p3} T_{s1} + R_{p3} \\ V_{p3} \end{bmatrix} \\
\left(K_{1p}^{[1]} T_{s1} + K_{2p}^{[1]} \right)^T \left(K_{1p}^{[1]} T_{s1} + K_{2p}^{[1]} \right) - \left[\sum_{j=2}^{q} j(j-1) \beta_j T_{s1}^{j-2} \right]^2 = 2 \left(\frac{1}{2} K_{1p}^{[1]} T_{s1}^2 + K_{2p}^{[1]} T_{s1} + K_{3p}^{[1]} - \begin{bmatrix} \beta_{q+1} \\ \sum_{j=1}^{q} j \beta_j T_{s1}^{j-1} \end{bmatrix} \right)^T \left(K_{1p}^{[1]} T_{s1} + K_{2p}^{[1]} \right)
\end{cases} \tag{34}
$$

where $T_{s1} := t_{s1} - t_3$. From the first of (34) it is possible to deduce $K_{1p}^{[1]}$ and $K_{2p}^{[1]}$ as functions of T_{s1} and then t_{s1} can be computed from the second of (34), preferably by applying numerical algorithms. Similarly it holds that

$$
\begin{cases}
\frac{1}{6} K_{1p}^{[2]} \left(t_4 - t_{s2} \right)^3 + \frac{1}{2} K_{2p}^{[2]} \left(t_4 - t_{s2} \right)^2 + K_{3p}^{[2]} \left(t_4 - t_{s2} \right) + K_{4p}^{[2]} = R_{p4} \\
\frac{1}{2} K_{1p}^{[2]} \left(t_4 - t_{s2} \right)^2 + K_{2p}^{[2]} \left(t_4 - t_{s2} \right) + K_{3p}^{[2]} = V_{p4} \\
K_{2p}^{[2]T} K_{2p}^{[2]} = \left[\sum_{j=2}^{q} j(j-1) \beta_j \left(t_{s2} - t_3 \right)^{j-2} \right]^2
\end{cases} \tag{35}
$$

and $K_{4p}^{[2]} = \begin{bmatrix} \beta_{q+1} \left(t_{s2} - t_3 \right) \\ \sum_{j=0}^{q} \beta_j \left(t_{s2} - t_3 \right)^j \end{bmatrix}$, $K_{3p}^{[2]} = \begin{bmatrix} \beta_{q+1} \\ \sum_{j=1}^{q} j \beta_j \left(t_{s2} - t_3 \right)^{j-1} \end{bmatrix}$. It is assumed that R_{p4} and V_{p4} are the first two components of r_{p4} and v_{p4} respectively. Numerical solutions to (35) are recommended.

The advantage of the approach presented consists in the fact that the switching between dynamical systems in (33) is guaranteed to be optimal and smooth. Furthermore, static obstacles such as no fly-zones can be modeled as orthogonal cylinders whose border can be approximated by interpolating polynomials, e.g. Hermite ones.

The final task T4 can be tackled as a point to surface problem: the *p*-th vehicle leaves r_{p4} at time t_4 with velocity v_{p4} to reach an unspecified point on the surface $S(\cdot)$ at time t_5^*. Let $S(\cdot)$ be a quadric defined as

$$
S(z) := z^T \Lambda z + \psi_1^T z + \psi_2 = 0 \tag{36}
$$

with $(z, \psi_1, \psi_2) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$ and $\Lambda \in \mathbb{R}^{3\times3}$. As $\Upsilon = -2k_{1p}$, by imposing (14) it holds that $k_{4p} = r_{p4}$, $k_{3p} = v_{p4}$, and

$$
\begin{cases}
k_{1p} = -\frac{1}{2} \left(\Lambda + \Lambda^T \right) \left(\lambda x'_{p*} \left(t_5^* \right) + \mu x_{p*} \left(t_5^* \right) \right) \\
2 \left[k_{1p} \left(t_5^* - t_4 \right) + k_{2p} \right] = \lambda \left(\Lambda + \Lambda^T \right) x_{p*} \left(t_5^* \right) \\
k_{2p}^T k_{2p} = 2 k_{1p}^T v_{p4} \\
x_{p*}^T \left(t_5^* \right) \Lambda x_{p*} \left(t_5^* \right) + \psi_1^T x_{p*} \left(t_5^* \right) + \psi_2 = 0
\end{cases}
$$

thus

$$\begin{cases}\begin{bmatrix}\left[2I+\frac{1}{2}\left(\Lambda+\Lambda^T\right)\left(\lambda+\frac{1}{3}\mu T\right)T_5^2\right] & \left(\Lambda+\Lambda^T\right)\left(\lambda+\frac{1}{2}\mu T_5\right)T_5 \\ \left[2I-\frac{1}{6}\lambda T_5^2\left(\Lambda+\Lambda^T\right)\right]T_5 & 2I-\frac{1}{2}\lambda\left(\Lambda+\Lambda^T\right)T_5^2\end{bmatrix}\begin{bmatrix}k_{1p}\\ k_{2p}\end{bmatrix}=\begin{bmatrix}-\left(\Lambda+\Lambda^T\right)\left(\lambda+\mu T_5\right)v_{p4}-\left(\Lambda+\Lambda^T\right)\mu r_{p4}-\mu\psi_1 \\ \lambda T_5\left(\Lambda+\Lambda^T\right)v_{p4}+\lambda\left(\Lambda+\Lambda^T\right)r_{p4}+\lambda\psi_1\end{bmatrix} \\ k_{2p}^T k_{2p}=2k_{1p}^T v_{p4} \\ x_{p*}^T\left(t_5^*\right)\Lambda x_{p*}\left(t_5^*\right)+\psi_1^T x_{p*}\left(t_5^*\right)+\psi_2=0\end{cases} \tag{37}$$

where $T_5 := t_5^* - t_4$. It is recommended to solve the first of (37) for k_{1p} and k_{2p} as functions of T_5 and then to deduce t_5^* from the second one. Consequently k_{1p}, k_{2p}, and t_5^* are functions of λ and μ that can be chosen to minimize for instance $\left\|x_{p*}\left(t_5^*\right)-r_{p4}\right\|_2$ or t_5^* and such that the last of (37) holds. In general numerical solutions are preferable. If $\Lambda=0$, then $S(\cdot)$ degenerates in a plane and from (37) it follows that

$$\begin{cases}k_{1p}=-\frac{\mu}{2}\psi_1,\ k_{2p}=\frac{1}{2}\left(\lambda+\mu T_5\right)\psi_1 \\ \mu^2\psi_1^T\psi_1T_5^2+2\lambda\mu\psi_1^T\psi_1T_5+\lambda^2\psi_1^T\psi_1+4\mu\psi_1^Tv_{p4}=0 \\ -2\mu\psi_1^T\psi_1T_5^3-3\lambda\psi_1^T\psi_1T_5^2-12\psi_1^Tv_{p4}T_5-12\psi_1^T\left(r_{p4}-\psi_2\right)=0.\end{cases} \tag{38}$$

The last two equations of (38) imply that for $\Pi_2^2 \ge 4\Pi_1\Pi_3$

$$T_5=\frac{-\Pi_2\pm\left(\Pi_2^2-4\Pi_1\Pi_3\right)^{1/2}}{2\Pi_1} \tag{39}$$

where $\Pi_1=\lambda\mu\psi_1^T\psi_1$, $\Pi_2=\lambda^2\psi_1^T\psi_1-2\mu\psi_1^Tv_{p4}$, and $\Pi_3=-12\mu\psi_1^T\left(r_{p4}-\psi_2\right)$. Eq. (39) yields for $T_5>0$.

If $\mu=0$, then $\lambda=0$, which cannot occur. Instead, if $\lambda=0$, then $T_5=-\frac{3\psi_1^Tr_{p4}-\psi_2}{\psi_1^Tv_{p4}}$, which is remarkably independent from μ, and $x_{p*}\left(t_5^*\right)=\frac{1}{6}\mu AT_5^3+v_{p4}T_5+r_{p4}$. In this case $\left\|x_{p*}\left(t_5^*\right)-r_{p4}\right\|_2$ is minimized for $\mu=-\frac{6\psi_1^Tv_{4p}}{T_5^2\psi_1^T\psi_1}$.

This concludes the study of the path planning problem for a single vehicle and the assigned mission scenario. It is relevant that, because of the approach chosen, tasks can be recombined in a different order by tuning the integration constant already herein computed. This is advantageous for building libraries of optimal solutions to implement onboard aerospace vehicles for fast trajectory optimization. Optimizing $x_{p*}(\cdot)$ for the entire mission is possible but the computational cost increases and the modularity of the solution found is not guaranteed.

B. Consumed Energy Optimization – Formation of Vehicles

Consider the path planning problem for a formation of aerospace vehicles. The trajectory of the *p*-th vehicle needs to optimize the energy consumption, to account for the collision avoidance constraint and to let the formation be as tight as possible. This can be achieved by imposing that the *p*-th element of the formation always maintains a given distance from some other vehicles. Specifically, by applying the theorem of multipliers, (18) becomes

$$I_p=\int_{t_k}^{t_{k+1}}\left[\lambda_{0_p}a_p^T(t)a_p(t)+\lambda_i(t)\left(\left\|x_p(t)-x_i(t)\right\|_2^2-r_{pi}^2(t)\right)+\lambda_l(t)\left(\left\|x_p(t)-x_l(t)\right\|_2^2-r_{pl}^2(t)\right)\right]dt \tag{40}$$

where $r_{pj}(\cdot): t \in \mathbb{R} \to \mathbb{R}_{++}$, $j \in \{i,l\}^2 \subset (\{1,....,N\} \setminus \{p\})^2$. It is worth to stress how, having modeled the vehicles as point masses (i.e. $n=3$), only two equality constraints on $x_{p*}(\cdot)$ can be imposed.

The normality of (40) can be proven considering that the constraint is

$$\varphi(t, x_p(t)) = \left[\|x_p(t) - x_i(t)\|_2^2 - r_{pi}^2(t) \quad \|x_p(t) - x_l(t)\|_2^2 - r_{pl}^2(t) \right]^T \tag{41}$$

and it can be verified that $\frac{\partial \varphi(t, x_p(t))}{\partial x_p} = 2\left[x_p(t) - x_i(t) \quad x_p(t) - x_l(t) \right]$, which is always full rank assuming that for any t $x_j(t) \neq x_h(t)$, $(j,h) \in \{p,i,l\}^2$ and $j \neq h$. Consequently it holds that $\lambda_{0_p} \neq 0$.

According to the EPNC any extremal for (41) satisfies the following equation:

$$\frac{d^4}{dt^4} x_{p*}(t) + (\lambda_i(t) + \lambda_l(t)) x_{p*}(t) = \lambda_i(t) x_{i*}(t) + \lambda_l(t) x_{l*}(t) \tag{42}$$

subject to the boundary conditions of the problem to solve, i.e. of the task to accomplish.

In general, as $x_{i*}(\cdot)$ and $x_{l*}(\cdot)$ are unknown, eq. (42) represents the *p*-th equation of a coupled autonomous*** system of *N* non-linear ordinary differential equations, whose analytical integrals, if any, are difficult to find[2]. Eq. (21) still applies and therefore the strict LHC is verified and the integrand in eq. (40) is proven to be regular. For (40) it holds that $\Phi\left(f\left(t, \tilde{x}_*(t), \tilde{x}_*^{'}(t)\right)\right) = 2\begin{bmatrix} (\lambda_i(t) + \lambda_l(t)) I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \in \mathbb{R}^{9\times 9}$ and the accessory cost index (12) is $\int_{t_k}^{t_{k+1}} \left[\eta_p^{T''}(t) \eta_p^{''}(t) + (\lambda_i(t) + \lambda_l(t)) \eta_p^T(t) \eta_p(t) \right] dt$, whose extremals are solutions of the following problem

$$\begin{cases} \frac{d^2 \eta_p^{''}(t)}{dt^2} + (\lambda_i(t) + \lambda_l(t)) \eta_p(t) = 0 \\ \eta_p(t_k) = \eta_p^{'}(t_k) = \eta_p(t_c) = \eta_p^{'}(t_c) = 0 \end{cases} \tag{43}$$

if t_c exists. Numerical solutions to (43), are recommended accounting for the fact that the imaginary part of $\eta_p(\cdot)$ is identically zero over the given time interval[9], i.e. $\mathrm{Im}(\eta_p(t)) \equiv 0$.

The strengthen LHC has been proven to hold and by (40) it implies the regularity of its integrand, hence the smoothness of $x_{p*}(\cdot)$. Consequently, assuming that the strengthen JNC is verified, the sufficient conditions for weak and strong local minima (S1-S2) hold. Furthermore, for $(z_0, z_1, z_2) \in [t_k, t_{k+1}] \times \mathbb{R}^{2n} \times \mathbb{R}^{2n}$, $z_1 = \left[z_{1_1}^T \; z_{1_2}^T \right]^T$, $z_2 = \left[z_{2_1}^T \; z_{2_2}^T \right]^T$, and $f(z_0, z_1, z_2) = z_{2_2}^T(t) z_{2_2}(t) + \lambda_i(t)\left(\|z_{1_1}(t) - x_i(t)\|_2^2 - r_{pi}^2(t)\right) + \lambda_l(t)\left(\|z_{1_1}(t) - x_l(t)\|_2^2 - r_{pl}^2(t)\right)$, because $\lambda_i(\cdot)$ and $\lambda_l(\cdot)$ are continuous by the theorem of multipliers, and because $\tilde{x}_{k*}(t) \in PWS(t_k, t_{k+1})$, then $\frac{\partial}{\partial z_j} f(z_0, z_1, z_2)$ is continuous, $j \in \{0,1,2\}$, and $\frac{\partial^2 f(z_0, z_1, z_2)}{\partial \left(\left[z_1^T \; z_2^T \right]^T \right)^2} = 2\begin{bmatrix} \begin{bmatrix} (\lambda_i(t) + \lambda_l(t)) I & 0 \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \end{bmatrix}$ which is

*** A system of differential equations is autonomous if it does not explicitly depend on the independent variable.

convex if any only if $\lambda_i(t)+\lambda_l(t)\geq 0$. Thus, if the integrand of (40) is convex on $[t_k, t_{k+1}]\times\mathbb{R}^{2n}\times\mathbb{R}^{2n}$ and if the strengthen JNC applies, then the sufficient condition S3 holds and $x_{p*}(\cdot)$ is the global minimizer for the cost index (40) on Ω and satisfies the collision avoidance constraint (41) for any $t\in[t_k, t_{k+1}]$. The integration constants for $x_{p*}(\cdot)$ to accomplish tasks T1-T4 can be computed by applying the boundary conditions as for the single vehicle case. Because of the coupling among the vehicles' optimal trajectories, numerical solutions are recommended[2].

C. Fuel Consumption Optimization – Single Vehicle

The fuel consumption for the *p*-th vehicle of the formation, without accounting for collision avoidance constraints, can be expressed as[18]

$$J_p = \int_{t_k}^{t_{k+1}} \sqrt{a_p^T(t)\,a_p(t)}\,dt. \tag{44}$$

Instead of (44) it is often preferred to minimize the integral of the *1*-norm of $a_p(\cdot)$. This prevents singularities in the optimization process but offers conservative results. By applying the EPNC to (44) it results that

$$\frac{a_{p*}(t)}{\left\|a_{p*}(t)\right\|_2} = k_{5p}t + k_{6p} \tag{45}$$

which implies that $\left\|k_{5p}t + k_{6p}\right\|_2^2 = 1$, thus $k_{5p}=0$. The relevance of this result consists in the fact that from eq. (9) $x_{p*}(\cdot)$ satisfies a fourth order ordinary differential equation in *t* and therefore it depends on four integration constants[8,22], among which k_{5p} and k_{6p}, defined by the boundary conditions. From (45), instead, it follows that $k_{5p}=0$ and $\left\|k_{6p}\right\|_2 = 1$ independently from the boundary conditions and therefore some problems might not have any optimal solution. Alternatively it can be inferred that some of the boundary conditions for the SPCV, FEP, PSP, and the inequality constrained problem, cannot be arbitrarily chosen but might be defined to be compatible with the remaining ones and to ensure the existence of extremals for the cost index (44) on Ω. Furthermore, *any* function $a_{p*}(t) := \Xi_{p1}(t)k_6$, such that $\Xi_{p1}(\cdot):[t_k, t_{k+1}]\to\mathbb{R}_{++}$ is bounded and twice integrable, satisfies eq. (45). As not only one class of functions is an extremal for (44), then the fuel consumption cost index is said to have *null Lagrangian*.

If $\Xi_{p1}(\cdot)$ is twice integrable and $\Xi_{p(j+1)}(t) = \int_{t_k}^{t} \Xi_{pj}(s)\,ds + x^{(j+1)}(t_k)$, $j\in\{1,2\}$, then any candidate optimal solution is

$$x_{p*}(t) = r_{p*}(t) = \left(\Xi_{p3}(t) - \Xi_{p3}(t_k)\right)k_{6p} + \left(v_{p*}(t_k) - k_{6p}\Xi_{p2}(t_k)\right)(t-t_k) + r_{p*}(t_k). \tag{46}$$

It holds that

$$\frac{\partial^2}{\partial a_p^2}\sqrt{a_p^T(t)\,a_p(t)} = \frac{1}{\left\|a_{p*}(t)\right\|_2} I - \frac{a_{p*}(t)\,a_{p*}^T(t)}{\left\|a_{p*}(t)\right\|_2^3}. \tag{47}$$

The matrix $\frac{\partial^2}{\partial a_p^2}\sqrt{a_p^T(t)\,a_p(t)}$ has zero determinant, its eigenvalues are $\frac{1}{\left\|a_{p*}(t)\right\|_2}(0,1,1)$, and its eigenvectors are $\begin{bmatrix}\frac{a_{p*_1}(t)}{a_{p*_3}(t)} & \frac{a_{p*_2}(t)}{a_{p*_3}(t)} & 1\end{bmatrix}^T$, $\begin{bmatrix}-\frac{a_{p*_3}(t)}{a_{p*_1}(t)} & 0 & 1\end{bmatrix}^T$, and $\begin{bmatrix}-\frac{a_{p*_2}(t)}{a_{p*_1}(t)} & 1 & 0\end{bmatrix}^T$, with $a_{p_j}(\cdot)$, $j\in\{1,2,3\}$, the *j*-th component of $a_p(\cdot)$. The matrix (47) is therefore positive definite and the LHC holds but not in its strengthen formulation. This implies another relevant result: because all the sufficient conditions for weak, strong, and global minima need the strengthen LHC to hold, any function (46) can be proven to be a candidate optimal solution only.

For $(z_0, z_1, z_2) \in [t_k, t_{k+1}] \times \mathbb{R}^{2n} \times \mathbb{R}^{2n}$, $z_2 = \begin{bmatrix} z_{2_1}^T & z_{2_2}^T \end{bmatrix}^T$, and $f(z_0, z_1, z_2) = \sqrt{z_{2_2}^T(t) z_{2_2}(t)}$, it holds that $\dfrac{\partial f(z_0, z_1, z_2)}{\partial z_0} = 0$, $\dfrac{\partial f(z_0, z_1, z_2)}{\partial z_1} = 0$ and $\dfrac{\partial f(z_0, z_1, z_2)}{\partial z_2} = \begin{bmatrix} 0^T & \dfrac{z_{2_2}^T}{\|z_{2_2}\|_2} \end{bmatrix}^T$, which has a discontinuity in the origin. Furthermore, $\dfrac{\partial^2 f(z_0, z_1, z_2)}{\partial\left(\begin{bmatrix} z_1^T, z_2^T \end{bmatrix}\right)^{T2}} = \begin{bmatrix} 0 & 0 \\ 0 & \begin{bmatrix} 0 & 0 \\ 0 & M \end{bmatrix} \end{bmatrix} \succeq 0$, with $M = \dfrac{1}{\|z_{2_2}(t)\|_2} I - \dfrac{z_{2_2}(t) z_{2_2}^T(t)}{\|z_{2_2}(t)\|_2^3}$. Thus the integrand of (44) is not convex on $[t_k, t_{k+1}] \times \mathbb{R}^{2n} \times \mathbb{R}^{2n}$. As for the energy consumption optimization, accounting for (24), by the superposition principle it follows that the candidate optimal control acceleration provided by the propulsion system of an aircraft is

$$a_{pc*}(t) = \Xi_{p1}(t) k_{6p} + v_{p*}^T(t) v_{p*}(t)\left(\tilde{k}_D \hat{v}_{p*}(t) - \tilde{k}_L \hat{v}_{p\perp*}(t) - \tilde{k}_S \hat{v}_{p\times*}(t)\right) - g \tag{48}$$

while for a spacecraft subject to aerodynamic drag we obtain

$$a_{pc*}(t) = \Xi_{p1}(t) k_{6p} + \frac{\mu}{\|r_{p*}(t)\|_2^3} r_{p*}(t) + \tilde{k}_D v_{p*}^T(t) v_{p*}(t) \hat{v}_{p*}(t). \tag{49}$$

Proceeding as for the energy consumption optimization, the constant k_{6p}, $v_{p*}(t_k)$, and $r_{p*}(t_k)$ are computed hereafter. Concerning task T1, from the second of (13) it follows that $\dfrac{a_{p*}(t)}{\|a_{p*}(t)\|_2} = \dfrac{\Xi_{p1}(t) \mathrm{k}_{6p}}{\|\Xi_{p1}(t) \mathrm{k}_{6p}\|_2} = 0$ which cannot occur. Therefore, the FEP does not have any solution for the cost index (44).

Consider the candidate optimal solution (46). Imposing the condition E2 for task T2 it follows that $x_{p*}(t_2) = r_{p2}$ and $x'_{p*}(t_2) = v_{p2}$ and from the condition $x_{p*}(t_3) = r_{p3}$ we obtain

$$k_{6p} = \frac{r_{p3} - v_{p2}(t_3 - t_2) - r_{p2}}{\Xi_{p3}(t_3) - \Xi_{p3}(t_2) - \Xi_{p2}(t_2)(t_3 - t_2)}. \tag{50}$$

The last condition to impose, i.e. $v_{p*}(t_3) = v_{p3}$, needs to be dealt as a compatibility condition:

$$v_{p*}(t_3) = \left(\Xi_{p2}(t_3) - \Xi_{p2}(t_2)\right) k_{6p} + v_{p2}. \tag{51}$$

As for the consumed energy optimization, if the first two components of the unconstrained optimal solution violate (31) for some $t \in [t_3, t_4]$, then the candidate optimal trajectory for task T3 is

$$X_{p*}(t) = \begin{cases} K_{6p}^{[1]} \Xi_{p3}(t) - K_{6p}^{[1]} \Xi_{p3}(t_3) + \left(V_{p*}(t_3) - K_{6p}^{[1]} \Xi_{p2}(t_3)\right)(t - t_3) + X_{p*}(t_3), \ t \in [t_3, t_{s1}] \\ \Theta(t), \ t \in [t_{s1}, t_{s2}] \\ K_{6p}^{[2]} \Xi_{p3}(t) - K_{6p}^{[2]} \Xi_{p3}(t_{s2}) + \left(V_{p*}(t_{s2}) - K_{6p}^{[2]} \Xi_{p2}(t_{s2})\right)(t - t_{s2}) + X_{p*}(t_{s2}), \ t \in [t_{s2}, t_4]. \end{cases} \tag{52}$$

In order to find t_{s1} and $K_{6p}^{[1]}$ for the first of (52), impose the boundary conditions at t_3, the continuity of $X_{p*}(\cdot)$ and $X'_{p*}(\cdot)$ at t_{s1}, and eq. (15). The boundary conditions at t_3 are $X_{p*}(t_3)=R_{p3}$, $V_{p*}(t_3)=V_{p3}$, and moreover

$$\begin{cases} K_{6p}^{[1]}\Xi_{p3}(t_{s1})-K_{6p}^{[1]}\Xi_{p3}(t_3)+\left(V_{p3}-K_{6p}^{[1]}\Xi_{p2}(t_3)\right)(t_{s1}-t_3)+R_{p3}=\begin{bmatrix}\beta_{q+1}(t_{s1}-t_3)\\ \sum_{j=0}^{q}\beta_j(t_{s1}-t_3)^j\end{bmatrix} \\ K_{6p}^{[1]}\Xi_{p2}(t_{s1})-K_{6p}^{[1]}\Xi_{p2}(t_3)+V_{p3}=\begin{bmatrix}\beta_{q+1}\\ \sum_{j=1}^{q}j\beta_j(t_{s1}-t_3)^{j-1}\end{bmatrix} \\ \sqrt{\left(\Xi_{p1}(t_{s1})K_{6p}^{[1]}\right)^T\left(\Xi_{p1}(t_{s1})K_{6p}^{[1]}\right)}-\begin{bmatrix}0\\ \sum_{j=2}^{q}j(j-1)\beta_j(t_{s1}-t_3)^{j-2}\end{bmatrix}^T\begin{bmatrix}0\\ \sum_{j=2}^{q}j(j-1)\beta_j(t_{s1}-t_3)^{j-2}\end{bmatrix}= \\ \quad =2\left(K_{6p}^{[1]}\Xi_{p2}(t_{s1})-K_{6p}^{[1]}\Xi_{p2}(t_3)+V_{p*}(t_3)-\begin{bmatrix}\beta_{q+1}\\ \sum_{j=1}^{q}j\beta_j(t_{s1}-t_3)^{j-1}\end{bmatrix}\right)^T\left(\Xi_{p1}(t_{s1})K_{6p}^{[1]}\right) \end{cases}$$

which is a system of five scalar equations in three scalar unknowns, namely the components of $K_{6p}^{[1]}$ and t_{s1}. Thus, considering the second equation as a compatibility condition, the remaining ones become

$$\begin{cases} K_{6p}^{[1]}=\dfrac{1}{\Xi_{p3}(t_{s1})-\Xi_{p3}(t_3)-\Xi_{p2}(t_3)(t_{s1}-t_3)}\left[\begin{bmatrix}\beta_{q+1}(t_{s1}-t_3)\\ \sum_{j=0}^{q}\beta_j(t_{s1}-t_3)^j\end{bmatrix}-R_{p3}-V_{p3}(t_{s1}-t_3)\right] \\ \Xi_{p1}(t_{s1})-\left[\sum_{j=2}^{q}j(j-1)\beta_j(t_{s1}-t_3)^{j-2}\right]^2= \\ \quad =2\left(\Xi_{p1}(t_{s1})\Xi_{p2}(t_{s1})-\Xi_{p1}(t_{s1})\Xi_{p2}(t_3)+\Xi_{p1}(t_{s1})K_{6p}^{[1]T}V_{p3}-\Xi_{p1}(t_{s1})K_{6p}^{[1]T}\begin{bmatrix}\beta_{q+1}\\ \sum_{j=1}^{q}j\beta_j(t_{s1}-t_3)^{j-1}\end{bmatrix}\right). \end{cases} \tag{53}$$

Once $K_{6p}^{[1]}$ is determined in terms of t_{s1}, the optimal switching time can be found by solving the last of (53).

The constant $K_{6p}^{[2]}$ and t_{s2} can be deduced with a similar approach: by imposing eq. (15), the boundary conditions at t_4, and the continuity of $X_{p*}(\cdot)$ and $X'_{p*}(\cdot)$ at t_{s2}, then accounting for (32) it holds that $X_{p*}(t_{s2})=\begin{bmatrix}\beta_{q+1}(t_{s2}-t_3)\\ \sum_{j=0}^{q}\beta_j(t_{s2}-t_3)^j\end{bmatrix}$, $V_{p*}(t_{s2})=\begin{bmatrix}\beta_{q+1}\\ \sum_{j=1}^{q}j\beta_j(t_{s2}-t_3)^{j-1}\end{bmatrix}$, and moreover at t_4 it holds that

$$\begin{cases} R_{p4}=K_{6p}^{[2]}\Xi_{p3*}(t_4)-K_{6p}^{[2]}\Xi_{p3}(t_{s2})+\left(V_{p*}(t_{s2})-K_{6p}^{[2]}\Xi_{p2}(t_{s2})\right)(t_4-t_{s2})+X_{p*}(t_{s2}) \\ V_{p4}=K_{6p}^{[2]}\Xi_{p2}(t_4)-K_{6p}^{[2]}\Xi_{p2}(t_{s2})+V_{p*}(t_{s2}) \\ \Xi_{p1}(t_{s2})-\left[\sum_{j=2}^{q} j(j-1)\beta_j(t_{s2}-t_3)^{j-2}\right]^2= \\ \quad =2\left(\Xi_{p1}(t_{s2})\Xi_{p2}(t_{s2})-\Xi_{p1}(t_{s2})\Xi_{p2}(t_{s2})+\Xi_{p1}(t_{s2})K_{6p}^{[2]T}V_{p*}(t_{s2})-\Xi_{p1}(t_{s2})K_{6p}^{[2]T}\begin{bmatrix}\beta_{q+1}\\ \sum_{j=1}^{q} j\beta_j(t_{s2}-t_3)^{j-1}\end{bmatrix}\right). \end{cases}$$

Addressing the second of these equations as compatibility conditions, then

$$\begin{cases} K_{6p}^{[2]}=\dfrac{R_{p4}-V_{p*}(t_{s2})(t_4-t_{s2})-X_{p*}(t_{s2})}{\Xi_{p3}(t_4)-\Xi_{p3}(t_{s2})-\Xi_{p2}(t_{s2})(t_4-t_{s2})} \\ \Xi_{p1}(t_{s2})-\left[\sum_{j=2}^{q} j(j-1)\beta_j(t_{s2}-t_3)^{j-2}\right]^2= \\ \quad =2\left(\Xi_{p1}(t_{s2})\Xi_{p2}(t_{s2})-\Xi_{p1}(t_{s2})\Xi_{p2}(t_{s2})+\Xi_{p1}(t_{s2})K_{6p}^{[2]T}V_{p*}(t_{s2})-\Xi_{p1}(t_{s2})K_{6p}^{[2]T}\begin{bmatrix}\beta_{q+1}\\ \sum_{j=1}^{q} j\beta_j(t_{s2}-t_3)^{j-1}\end{bmatrix}\right). \end{cases} \tag{54}$$

In conclusion task T4 is optimized as follows: the first of (14) leads to $x_{p*}(t_4)=r_{p4}$ and $x'_{p*}(t_4)=v_{p4}$. As $\Upsilon=0$, it holds that the fourth of (14) is a an identity and finally

$$\begin{cases} \left\{I-\lambda\left[\Xi_{p3}(t_5^*)+\Xi_{p3}(t_4)+(t_5^*-t_4)\Xi_{p2}(t_4)\right](\Lambda+\Lambda^T)\right\}k_{6p}=\lambda(\Lambda+\Lambda^T)\left[(t_5^*-t_4)v_{p4}+r_{p4}\right]+\lambda\psi_1 \\ (\Lambda+\Lambda^T)\left(\lambda x'_{p*}(t_5^*)+\mu x_{p*}(t_5^*)\right)+\mu\psi_1=0 \\ x_{p*}^T(t_5^*)\Lambda x_{p*}(t_5^*)+\psi_1^T x_{p*}(t_5^*)+\psi_2=0. \end{cases} \tag{55}$$

This is a system of seven scalar equations in six unknowns, namely the three components of k_{6p}, λ, μ, and t_5^*. Thus, the last of (55) needs to be imposed as compatibility condition. Assume the quadric $S(\cdot)$ degenerates in a plane, i.e. $\Lambda=0$, then from the second of (55) it follows that $\mu=0$, from the first of (55) it follows that k_6 is perpendicular to $S(\cdot)$ and that $\lambda=\pm\dfrac{1}{\|\psi_1\|_2}$, where the sign depends on the orientation of the normal vector identifying $S(\cdot)$, and finally t_5^* is determined by solving $\psi_1^T x_{p*}(t_5^*)=-\psi_2$.

D. Fuel Consumption Optimization – Formation of Vehicles

Accounting for collision avoidance and imposing the formation to remain as tight as possible, as for the consumed energy optimization problem, the cost index to optimize the fuel consumption for the *p*-th vehicle is

$$I_p=\int_{t_k}^{t_{k+1}}\left[\lambda_{0_p}\sqrt{a_p^T(t)a_p(t)}+\lambda_i(t)\left(\|x_p(t)-x_i(t)\|_2^2-r_{pi}^2(t)\right)+\lambda_l(t)\left(\|x_p(t)-x_l(t)\|_2^2-r_{pl}^2(t)\right)\right]dt. \tag{56}$$

The equality constraint accounted in (56) is given by eq. (41) and therefore the problem has already been shown to be normal, i.e. $\lambda_{0_p}\neq 0$. By applying the EPNC, any extremal for (56) is solution of a system of *N* coupled ordinary nonlinear differential equations, whose *p*-th equation is:

$$\frac{d^2}{dt^2}\frac{x''_{p*}(t)}{\left\|x''_{p*}(t)\right\|_2}+\left(\lambda_i(t)+\lambda_l(t)\right)x_{p*}(t)=\lambda_i(t)x_{i*}(t)+\lambda_l(t)x_{l*}(t). \tag{57}$$

The second variation of the integrand of the (56) with respect to $a_p(\cdot)$ is given by eq. (47). Thus, for the cost index (56), as for (44), the LHC holds but not in its strengthen formulation and any $x_{p*}(\cdot)$ satisfying (57) is a candidate optimal trajectory only. As for eq. (42), numerical solutions of (57), if any, are preferable[2].

V. Conclusions

This work rigorously presents theoretical results in the field of aircraft and spacecraft formation flying path planning and is contextualized in a systematic study developed by the authors using an analytical approach[1,3]. Results exposed have been achieved by mean of Calculus of Variations, the branch of mathematics aimed at finding extremals of functionals, and their optimality is guaranteed by necessary and sufficient conditions.

A complex mission scenario has been analyzed: while constantly keeping an assigned distance from each other, the vehicles of the formation leave from an unspecified initial position with an unknown velocity, pass through assigned waypoints, avoid a static obstacle represented by an orthogonal cylinder, and finally reach a given surface constituted by quadric. The four phases of the mission have been developed modularly to be easily reconfigurable by varying a finite number of parameters.

By applying the necessary conditions of Euler-Poisson, Legendre-Hadamard, and Jacobi, as well as the associated sufficient conditions, trajectories calculated are shown to optimize the consumed energy and the fuel consumption. Collision avoidance and compactness of the formations have been ensured by applying the Theorem of Multipliers.

While schematizing the aircraft and the spacecraft as point masses, realistic environmental conditions have been considered, accounting for constant and radial gravitational fields as well as for aerodynamic forces.

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