MRAC With Adaptive Uncertainty Bounds via Operator-Valued Reproducing Kernels

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***Abstract*—This letter presents three novel model reference adaptive control (MRAC) systems for nonlinear plants in which the matched, nonparametric uncertainty is only known to reside in a vector-valued reproducing kernel Hilbert space (RKHS). The first MRAC system is based on an extension of the classical projection operator for robust MRAC systems and assures ultimate boundedness of the trajectory tracking error. This MRAC system allows the user to design part of the control input, known as the compensator. The second MRAC method modifies the previous one and, based on an estimate of the largest admissible uncertainty in the RKHS norm, employs a particular choice of the compensator to assure asymptotic convergence of the tracking error to zero. The last MRAC method uses the previous one and, leveraging the error bounding method framework, employs an additional learning law to assure asymptotic convergence of the tracking error without any information on the largest admissible uncertainty. While the conventional approaches to adaptive error bounding methods usually leave it to an analyst to derive the required error bounds on a case-by-case basis, the proposed approaches work for any functional uncertainty that is contained in any native space of vector-valued functions defined in terms of an operator kernel.**

***Index Terms*—Model reference adaptive control, reproducing kernel Hilbert spaces, non-parametric uncertainties.**

I. Introduction

A. Motivation and Relevance of the Proposed Work

In CLASSICAL MRAC systems, nonlinear matched uncertainties are parameterized by an unknown matrix and a known regressor vector that is designed using some prior information about the plant dynamics or using some neural network or other learning mechanisms. In practice, these approaches are valid under the so-called *uniform approximation assumption*, which ensures the parameterization of the functional uncertainty is accurate [1], [2]. A plethora of publications assume the existence of a regressor vector and, more or less tacitly, considers the uniform approximation assumption as verified. Although the use of regressor vectors to span nonlinearities can be interpreted as a way to capture functional uncertainties, the ability to accurately capture uncertainties is limited by the richness of the regressor vector. Ultimately, these techniques are still mostly examples of a parametric adaptive control theory since the analysis of stability and convergence is performed for a fixed number of real parameters that are used to characterize the uncertainty.

This letter breaks away from employing some of the more common forms of the uniform approximation assumption, and obtains explicit, sharper controller performance bounds. These bounds are based on error characterizations that apply when the matched functional uncertainty lies in an infinite-dimensional reproducing kernel Hilbert space (RKHS) defined by an operator kernel. In particular, this letter proposes three MRAC systems of increasing complexity and performance level. The first result, which is summarized in Theorem 1, extends the continuous convex projection operator for classical MRAC systems to plants affected by uncertainties that lie in RKHSs. This result allows the user to choose, within some mild assumptions, a control term dubbed the compensator, which guarantees uniform boundedness of the trajectory tracking error and the adaptive gains and uniform ultimate boundedness of the trajectory tracking error. The second original result of this letter, given by Theorem 2, chooses a specific form for the compensator, which enforces uniform asymptotic convergence of the trajectory tracking error to zero, provided that the largest upper bound on the tracking error is known in the RKHS norm. Finally, introducing an additional adaptive law the final result of this letter, namely Theorem 3, proves uniform asymptotic convergence of the tracking error without requiring any knowledge of the largest admissible functional uncertainty.

The proposed results all can be understood as approximations of the same adaptive control law and the same adaptive law defined in terms of a partial differential equation (PDE). The PDE evolves in an infinite-dimensional RKHS and, its associated adaptive controllers are not implementable in practice. Therefore, this infinite-dimensional architecture, whose adaptive laws form a limiting distributed parameter system (DPS), is projected onto some user-defined space of functional uncertainties of dimension N. To our knowledge for nonlinear

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ODEs, only the adaptive control systems in [3], [4], [5] have been interpreted similarly. Some features of this strategy have also appeared in other references on RKHS methods in adaptive control, including notably [6], [7] and subsequent efforts that cite them.

This letter contributes to the ongoing goal suggested in research like that above of developing a nonparametric adaptive control theory that

(1) is well-defined for the vast collection of RKHS that are generated by operator-valued kernels;

(2) yields closed-form ultimate performance bounds on the tracking error that are explicit in N and hold for all the coordinate-centric implementations bundled together as N increases; and

(3) derive guarantees of performance over *nonparametric uncertainty classes* in the native space. In contrast to [3], [6], the proposed approach only uses deterministic Lyapunov analyses for the implementable controllers, which are the basis of classical texts such as [1], [2], [8], [9], [10]. While the development of additional stochastic elements of the approach in this letter would be beneficial, such as in the use of Gaussian process results in [3], here we restrict to the deterministic setting owing to the brevity of this letter.

This letter is inspired by the error bounding adaptive control strategies in Euclidean space presented in [2]. However, while the approach as summarized in [2] leaves it to the designer to find an error bounding function using case-by-case considerations, this letter derives and uses a general error bounding function that works whenever the functional uncertainty is contained in a vector-valued RKHS.

B. Problem Statement

In this letter, we consider the *plant model* given by the set of autonomous ordinary differential equations (ODEs)

$$\dot{x}(t) = Ax(t) + B(u(t) + f(x(t))), \quad x(t_0) = x_0, \quad t \geq t_0, \tag{1}$$

where $x : [t_0, \infty) \to \mathbb{X}$ denotes the *plant trajectory*, $\mathbb{X} = \mathbb{R}^n$, $u : [t_0, \infty) \to \mathbb{U}$ denotes the *control input*, $\mathbb{U} = \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}$ is unknown, $B \in \mathbb{R}^{n\times m}$ is known and such that the pair (A, B) is controllable, and $f : \mathbb{X} \to \mathbb{U}$ is unknown and assumed to be contained in the RKHS $\mathcal{H} \triangleq \overline{\text{span}\{\mathcal{K}_x\alpha \mid \alpha \in \mathbb{U},\ x \in \mathbb{X}\}}$ of $\mathbb{U}$-valued functions over $\mathbb{X}$, where $\mathcal{K}_x(\cdot) \triangleq \mathcal{K}(\cdot, x)$ is known as the *kernel function centered at* x, $\mathcal{K} : \mathbb{X} \times \mathbb{X} \to \mathcal{L}(\mathbb{U})$ is known as *operator kernel*, $\mathcal{L}(\mathbb{U})$ denotes the space of bounded linear operators on $\mathbb{U}$, and the closure is taken with respect to the candidate inner product $\langle \mathcal{K}_x(\cdot), \mathcal{K}_y(\cdot)\rangle_{\mathcal{H}}$ for all $x, y \in \mathbb{X}$; see [11], [12] for an axiomatic description of the closed linear span. Our goal is to design an adaptive control system so that $\lim_{t\to\infty} \|x(t) - x_r(t)\| = 0$ uniformly in $t_0 \geq 0$, where the *reference trajectory* $x_r : [t_0, \infty) \to \mathbb{R}^n$ is the trajectory of the *reference model*

$$\dot{x}_r(t) = A_r x_r(t) + B_r r(t), \quad x_r(t_0) = x_{r,0}, \quad t \geq t_0, \tag{2}$$

$A_r \in \mathbb{R}^{n\times n}$ is Hurwitz, $B_r \in \mathbb{R}^{n\times m}$, the pair (A_r, B_r) is controllable, and the *reference command input* $r : [t_0, \infty) \to \mathbb{U}$ is continous and bounded. To assure the existence of a control law that allows the plant trajectory to follow the reference trajectory, we assume that there exist $\alpha \in \mathbb{R}^{n\times m}$ and $\beta \in \mathbb{R}^{m\times m}$ such that the *matching conditions*

$$A_r = A + B\alpha^{\mathrm{T}}, \quad B_r = B\beta^{\mathrm{T}}, \tag{3}$$

are verified. The matching conditions signify that there exists at least a control input so that the plant can follow the reference model [1]. If the plant is not structured to follow the reference model, as it would occur for a car tasked with mimicking some aircraft dynamics, the matching conditions are not verified.

II. Fundamentals of Operator-Valued Kernels

In this section, we present a succinct compendium of RKHS theory needed for the scope of this letter. For more details, see [12, Ch. 6], [13]. The operator-valued reproducing kernel in this letter is a mapping $\mathcal{K} : \mathbb{X} \times \mathbb{X} \to \mathcal{L}(\mathbb{U})$ that takes values in the bounded linear operators on $\mathbb{U}$. On first reading, it can be useful to assume that the vector-valued native space is determined by $\mathcal{K}(x_1, x_2) \triangleq \mathfrak{K}(x_1, x_2)I_m$, where I_m is the identity matrix and $\mathfrak{K} : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ is a scalar-valued kernel that defines a native space of scalar-valued functions over $\mathbb{X}$ [11], [14]. Among classical scalar-valued kernels, we recall the Gaussian, inverse multiquadric, Sobolev-Matern, and Wendland kernels [15]. In this letter, we refer to vector-valued native space for "native space of vector-valued functions" and, similarly, for scalar-valued native spaces.

A. Some Critical Identities

For all $x \in \mathbb{X}$, $\alpha \in \mathbb{U}$, and $h \in \mathcal{H}$, it holds that

$$\langle \mathcal{K}_x\alpha, h\rangle_{\mathcal{H}} = \langle \alpha, E_x h\rangle_{\mathbb{U}} = \langle \alpha, h(x)\rangle_{\mathbb{U}} = \alpha^{\mathrm{T}} h(x), \tag{4}$$

where the *evaluation operator* $E_x : h \in \mathcal{H} \mapsto h(x) \in \mathbb{U}$ is bounded and linear for each $x \in \mathbb{X}$ and such that $E_x = (\mathcal{K}_x)^* \triangleq \mathcal{K}_x^*$ for each $x \in \mathbb{X}$, and $(\cdot)^*$ denotes the *adjoint operator* the is defined with respect to $\mathcal{H}$ and $\mathbb{R}$.

For all $x, z \in \mathbb{X}$, it holds that

$$\mathcal{K}(z, x) = E_z\mathcal{K}(\cdot, x) = E_z\mathcal{K}_x = \mathcal{K}_z^*\mathcal{K}_x = E_z E_x^*. \tag{5}$$

Both (4) and (5) are of paramount importance in this letter for showing the relationship between the kernel function, evaluation operator, and inner products on $\mathbb{U}$ and $\mathcal{H}$.

B. Approximation in Vector-Valued Native Spaces

In this letter, control schemes are realized on specific types of approximations of infinite-dimensional RKHSs that are defined in terms of scattered bases [12], [15]. To this goal, we define the *set of centers* $\Xi_N \triangleq \{\xi_i \in \mathbb{X} \mid 1 \leq i \leq N\}$. We suppose that the collection of all centers $\Xi \triangleq \bigcup_{N=N_0}^{\infty} \Xi_N$ is dense in a compact *set of interest* $\Omega \subseteq \overline{\bigcup_{N\geq 0} \Xi_N} \subset \mathbb{X}$; such a set contains the plant's controlled trajectory.

We assume that each operator kernel $\mathcal{K}$ is *strictly positive definite*, that is, the *generalized Grammian matrix* $\mathbb{K}_N \triangleq [\mathcal{K}(\xi_i, \xi_j)]_{(i,j)} \in \mathbb{R}^{mN\times mN}$ is positive definite for any choice of N distinct centers in Ξ_N, where $[\,\cdot\,]_{(i,j)}$ denotes the element on the i-th row and j-th column of its matrix argument. This assumption on the Grammian matrix is not overly restrictive in our applications. For instance, it is verified whenever $\mathcal{K}(\cdot, \cdot) \triangleq I_m\mathfrak{K}(\cdot, \cdot)$ and $\mathfrak{K}$ is one of the scalar-valued kernels mentioned above. This assumption ensures that the family of functions $\{\mathcal{K}_{\xi_i} e^j \mid 1 \leq i \leq N, 1 \leq j \leq m\}$ is linearly independent and constitutes a basis for $\mathcal{H}_N$, where $\{e^j\}_{j=1}^m$ denotes the *set of canonical basis vectors* for $\mathbb{U}$. Note that the number of real parameters $p \triangleq \dim(\mathcal{H}_N) = mN$ is the dimension of the space of approximants when using N centers for approximation.

To produce adaptive laws in finite-dimensional spaces, we introduce the *space of approximants*

$$\boldsymbol{\mathcal{H}}_N \triangleq \operatorname{span}\left\{\mathcal{K}_{\xi_i} e^j \mid \xi_i \in \Xi_N, 1 \leq i \leq N, 1 \leq j \leq m\right\}. \quad (6)$$

Thus, we define $\mathbf{\Pi}_N : \boldsymbol{\mathcal{H}} \to \boldsymbol{\mathcal{H}}_N$ as the *$\boldsymbol{\mathcal{H}}$-orthogonal projection onto $\boldsymbol{\mathcal{H}}_N \subseteq \boldsymbol{\mathcal{H}}$*, and we use $\mathbf{\Pi}_N$ to build approximations of infinite-dimensional representations of functional uncertainties as well as control and adaptive laws designed based on infinite-dimensional terms.

The accuracy of approximations in a RKHS can be captured by the power function [15]. In this letter, we define the *vector-valued power function of the subspace $\boldsymbol{\mathcal{H}}_N \subseteq \boldsymbol{\mathcal{H}}$ in the direction $\alpha \in \mathbb{U}$* as

$$\mathcal{P}_N^\alpha(x) \triangleq \sqrt{\langle(\mathcal{K}(x,x) - \mathcal{K}_N(x,x))\alpha, \alpha\rangle_{\mathbb{U}}} \quad (7)$$

for all $x \in \mathbb{X}$, where $\mathcal{K}_N(\cdot,\cdot)$ denotes the known operator kernel that defines $\boldsymbol{\mathcal{H}}_N$; for details, see [16]. This is a generalization of the power function for scalar-valued native spaces [15], [17], and it enables an error bound in the vector-valued native space. Specifically, according to Corollary 2.11 of [16], for each $x \in \mathbb{X}, \alpha \in \mathbb{U}, h \in \boldsymbol{\mathcal{H}}$, it holds that

$$\begin{aligned} |\langle E_x(I - \mathbf{\Pi}_N)h, \alpha\rangle_{\mathbb{U}}| &\leq \mathcal{P}_N^\alpha(x)\|(I - \mathbf{\Pi}_N)h\|_{\boldsymbol{\mathcal{H}}} \\ &\leq \mathcal{P}_N^\alpha(x)\|h\|_{\boldsymbol{\mathcal{H}}}. \end{aligned} \quad (8)$$

If $\boldsymbol{\mathcal{H}} = \mathcal{H}^m$, that is, if the vector-valued RKHS is given as the Cartesian product of a scalar-valued RKHS N times, then

$$\|E_x(I - \mathbf{\Pi}_N)f\|_{\boldsymbol{\mathcal{H}}} \leq \mathcal{P}_N(x)\|f\|_{\boldsymbol{\mathcal{H}}}, \quad x \in \mathbb{X}, \quad (9)$$

where

$$\mathcal{P}_N(x) \triangleq \sqrt{\mathfrak{K}(x,x) - \mathfrak{K}_N(x,x)} \quad (10)$$

denotes the *power function of* the scalar-valued native space $\mathcal{H}$ and $\mathfrak{K}(\cdot,\cdot)$ denotes the kernel function underlying $\mathcal{H}$.

III. Robust MRAC in a Native Space

A. General Framework

To steer the trajectories of (1) toward the trajectories of (2), in this letter, we employ the control input

$$\begin{aligned} u_N(t) = {}& \hat{\alpha}_N^{\mathrm{T}}(t)x_N(t) + \hat{\beta}_N^{\mathrm{T}}(t)r(t) \\ & - E_{x_N(t)}\left(\hat{f}_N(t,\cdot) + c_N(\cdot, e_N(t))\right), \quad t \geq t_0, \end{aligned} \quad (11)$$

where the *matrix adaptive gains* $\hat{\alpha} : [t_0,\infty) \to \mathbb{R}^{n\times m}$ and $\hat{\beta} : [t_0,\infty) \to \mathbb{R}^{m\times m}$ verify the *adaptive laws*

$$\dot{\hat{\alpha}}_N(t) = \begin{cases} \operatorname{proj}\left(\hat{\alpha}_N(t), -\Gamma_\alpha x_N(t)e_N^{\mathrm{T}}(t)PB\right), \\ -\operatorname{dead}\left(\Gamma_\alpha x_N(t)e_N^{\mathrm{T}}(t)PB\right), \end{cases} \hat{\alpha}(t_0) = \hat{\alpha}_0, \quad (12)$$

$$\dot{\hat{\beta}}_N(t) = \begin{cases} \operatorname{proj}\left(\hat{\beta}_N(t), -\Gamma_\beta r(t)e_N^{\mathrm{T}}(t)PB\right), \\ -\operatorname{dead}\left(\Gamma_\beta r(t)e_N^{\mathrm{T}}(t)PB\right), \end{cases} \hat{\beta}(t_0) = \hat{\beta}_0, \quad (13)$$

the *adaptive rate matrices* $\Gamma_\alpha \in \mathbb{R}^{n\times n}$ and $\Gamma_\alpha \in \mathbb{R}^{m\times m}$ are symmetric and positive-definite, $P \in \mathbb{R}^{n\times n}$ denotes the symmetric, positive-definite solution of the *algebraic Lyapunov equation* $A_r^{\mathrm{T}}P + PA_r = -Q$, $Q \in \mathbb{R}^{n\times n}$ is user-defined, symmetric, and positive-definite, the *functional adaptive gain* $\hat{f}_N(\cdot,\cdot) \in \boldsymbol{\mathcal{H}}_N$ verifies the *adaptive law*

$$\frac{\partial \hat{f}_N(t,\cdot)}{\partial t} = \begin{cases} \mathfrak{Proj}\left(\hat{f}_N(t,\cdot), \mathbf{\Pi}_N\Gamma_f E^*_{x_N(t)}B^{\mathrm{T}}Pe_N(t)\right), \\ \operatorname{dead}\left(\mathbf{\Pi}_N\Gamma_f E^*_{x_N(t)}B^{\mathrm{T}}Pe_N(t)\right), \end{cases}$$
$$\hat{f}(t_0,\cdot) = \hat{f}_0, \quad (14)$$

the *adaptive rate matrix* $\Gamma_f \in \mathbb{R}^{m\times m}$ is symmetric and positive-definite, $x_N : [t_0,\infty) \to \mathbb{X}$ denotes the solution of (1) with control input (11), $e_N(t) \triangleq x_N(t) - x_r(t)$, and the compensator $c_N : \mathbb{X}\times\mathbb{R}^n \to \mathbb{U}$ is user-defined. Note that (14) is a partial differential equation taking values in $\boldsymbol{\mathcal{H}}$. This equation can be viewed as an approximation of the adaptive law in the limiting system discussed in Section III-B below.

Let $\mathcal{D} \subseteq \mathbb{R}^m$ be convex and let $h : \mathcal{D} \to \mathbb{R}$ be convex and such that $\inf_{\theta\in\mathcal{D}} h(\theta) < 0$. The *continuous vector projection operator* is defined as

$$\operatorname{proj}(\theta,\theta_d) \triangleq \begin{cases} \mathfrak{P}(\theta,\theta_d) & \text{if } h(\theta) > 0, \\ & \text{and } \left\langle h'(\theta), \theta_d\right\rangle_{\mathbb{R}^d} > 0, \\ \theta_d & \text{otherwise,} \end{cases} \quad (15)$$

where

$$\mathfrak{P}(\theta,\theta_d) \triangleq \theta_d - h(\theta)\left(\frac{\partial h(\theta)}{\partial\theta}\right)^{\mathrm{T}}\left(\frac{\partial h(\theta)}{\partial\theta}\right)\left\|\frac{\partial h(\theta)}{\partial\theta}\right\|^2\theta_d \quad (16)$$

for all $(\theta,\theta_d) \in \mathcal{D}\times\mathbb{R}^m$. Applying the vector projection operator to each column of matrices $\Theta, \Theta_d \in \mathbb{R}^{m\times N}$, we obtain the continous matrix projection operator employed in (12) and (13); we employ the same symbol for both operators.

Let $\mathcal{D} \subseteq \boldsymbol{\mathcal{H}}$ be convex, and $h : \mathcal{D} \to \mathbb{R}$ be convex, Fréchet differentiable, and so that $\inf_{f\in\mathcal{D}} h(f) < 0$. The Fréchet derivative of $h \in \boldsymbol{\mathcal{H}}$ at $f \in \boldsymbol{\mathcal{H}}$ is denoted by $Dh(f)$. Define the *convex projection operator over $\boldsymbol{\mathcal{H}}$* $\mathfrak{P} : \mathcal{D}\times\boldsymbol{\mathcal{H}} \to \boldsymbol{\mathcal{H}}$ as

$$\mathfrak{Proj}(f,f_d) \triangleq \begin{cases} \mathfrak{P}(f,f_d) & \text{if } h(f) > 0, \\ & \text{and } \langle Dh(f), f_d\rangle_{\boldsymbol{\mathcal{H}}} > 0, \\ f_d & \text{otherwise,} \end{cases} \quad (17)$$

where

$$\mathfrak{P}(f,f_d) \triangleq \left[I - h(f)\left(\cdot, \frac{Dh(f)}{\|Dh(f)\|_{\boldsymbol{\mathcal{H}}}}\right)_{\boldsymbol{\mathcal{H}}}\frac{Dh(f)}{\|Dh(f)\|_{\boldsymbol{\mathcal{H}}}}\right]f_d \quad (18)$$

for all $(f,f_d) \in \mathcal{D}\times\boldsymbol{\mathcal{H}}$. We purposely employed the same symbol in (16) and (18) to remark on their analogies. The *deadzone operator* $\operatorname{dead} : \mathbb{Y} \to \mathbb{Y}$ used in (12) and (13), where $\mathbb{Y} = \mathbb{R}^{n\times m}$ in (12) and $\mathbb{Y} = \mathbb{R}^{m\times m}$ in (13), and $\operatorname{dead} : \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ used in (14) is defined in [1, p. 319]. The adaptive laws in (12)–(14) are mutually exclusive.

Theorem 1: Suppose that $\boldsymbol{\mathcal{H}} \triangleq \mathcal{H}^m$. Assume that the DPS given by (1) with control input (11) and adaptive laws (12)–(14) have complete solutions on $[t_0,\infty)$. Furthermore, suppose that the compensator $c_N(\cdot,\cdot)$ is such that $\left\langle E_{x_N(t)}c_N(\cdot,e_N(t)), B^{\mathrm{T}}Pe_N(t)\right\rangle_{\mathbb{R}^m} \leq 0$ for all $t \geq t_0$. Then, the trajectories of the DPS are uniformly bounded in $\mathbb{X}\times\mathbb{R}^{n\times m}\times\mathbb{R}^{m\times m}\times\boldsymbol{\mathcal{H}}_N$. Additionally, for any arbitrarily small constant $\eta > 0$, there exists $T \triangleq T(\eta) > t_0$ such that $\|e_N(t)\|_{\mathbb{X}} \leq \mathcal{S}$ for all $t \geq T$, where

$$\mathcal{S} \triangleq (1+\eta)\frac{2\|B^{\mathrm{T}}P\|}{\lambda_{\min}(Q)}\overline{\epsilon}_N \sup_{\xi\in\Omega}\mathcal{P}_N(\xi), \quad (19)$$

$\overline{\epsilon}_N \triangleq \sup_{x\in\Omega}\|E_x(I - \mathbf{\Pi}_N f))\|_{\boldsymbol{\mathcal{H}}}$, and $\Omega \subset \mathbb{X}$ is compact and such that $x_N(t) \in \Omega$ for all $t \geq t_0$.

Proof: For brevity, we prove the result using employing the projection operators in (12)–(14). Employing the deadzone operator, the proof follows identically. Let

$$\begin{aligned} V\left(e_N, \tilde{\alpha}_N, \tilde{\beta}_N, \tilde{f}_N\right) \triangleq {}& \langle Pe_N, e_N\rangle_{\mathbb{R}^n} + \left\langle\tilde{f}_N, \Gamma_f^{-1}\tilde{f}_N\right\rangle_{\boldsymbol{\mathcal{H}}} \\ & + \left\langle\tilde{\alpha}_N(t), \Gamma_\alpha^{-1}\tilde{\alpha}_N\right\rangle_{\mathrm{tr}} + \Gamma_\beta^{-1}\left\langle\tilde{\beta}_N, \Gamma_\beta^{-1}\tilde{\beta}_N\right\rangle_{\mathrm{tr}}, \end{aligned} \quad (20)$$

where $\tilde{\alpha}_N(t) \triangleq \hat{\alpha}_N(t) - \alpha$, $t \geq t_0$, $\tilde{\beta}_N(t) \triangleq \hat{\beta}_N(t) - \beta$, and $\tilde{f}_N(t,\cdot) \triangleq f - \hat{f}_N(t,\cdot)$. Thus, taking the derivative of (20) along the trajectories of (1) with control input (11) and (12)–(14), we obtain that, for all $t \geq t_0$,

$$\begin{aligned}\dot{V}(t) = & \underbrace{-\langle e_N(t), Qe_N(t)\rangle_{\mathbb{R}^n}}_{\text{Term 0}} \\ & \underbrace{-2\Big\langle \tilde{\alpha}_N(t), x_N(t)e_N^{\mathrm{T}}(t)PB + \Gamma_\alpha^{-1}\dot{\hat{\alpha}}_N(t)\Big\rangle_{\mathrm{tr}}}_{\text{Term 1}} \\ & \underbrace{-2\Big\langle \tilde{\beta}_N(t), r(t)e_N^{\mathrm{T}}(t)PB + \Gamma_\beta^{-1}\dot{\hat{\beta}}_N(t)\Big\rangle_{\mathrm{tr}}}_{\text{Term 2}} \\ & \underbrace{+2\Big\langle \tilde{f}_N(t,\cdot), \mathbf{\Pi}_N E^*_{x_N(t)}B^{\mathrm{T}}Pe_N(t) + \Gamma_f^{-1}\dot{\tilde{f}}_N(t,\cdot)\Big\rangle_{\boldsymbol{\mathcal{H}}}}_{\text{Term 3}} \\ & + \underbrace{2\big\langle E_{x_N(t)}(I-\mathbf{\Pi}_N)f, B^{\mathrm{T}}Pe_N(t)\big\rangle_{\mathbb{U}}}_{\text{Term 4}} \\ & + \underbrace{2\big\langle E_{x_N(t)}c_N(\cdot, e_N(t)), B^{\mathrm{T}}Pe_N(t)\big\rangle_{\mathbb{U}}}_{\text{Term 5}}, \end{aligned} \quad (21)$$

with $V(t) = V(e_N(t), \tilde{\alpha}_N(t), \tilde{\beta}_N(t), \tilde{f}_N(t,\cdot))$ for brevity. Note that Term 3 in (21) is obtained passing the time derivative through the inner product on $\boldsymbol{\mathcal{H}}$. This is possible owing to the fact that $\tilde{f}_N(t,\cdot) = f - \hat{f}_N(t,\cdot)$, f is not a function of time, and $\hat{f}_N(t,\cdot)$ is finite-dimensional, and, hence $\Big\langle \tilde{f}_N(t,\cdot), \Gamma_f^{-1}\tilde{f}_N(t,\cdot)\Big\rangle_{\boldsymbol{\mathcal{H}}}$ can be expanded along each component of $\boldsymbol{\mathcal{H}}_N$, the limit of the underlying incremental ratio can be taken component-wise, and $-\Big\langle \dot{\hat{f}}_N(t,\cdot), \Gamma_f^{-1}\tilde{f}_N(t,\cdot)\Big\rangle_{\boldsymbol{\mathcal{H}}} - \Big\langle \tilde{f}_N(t,\cdot), \Gamma_f^{-1}\dot{\hat{f}}_N(t,\cdot)\Big\rangle_{\boldsymbol{\mathcal{H}}}$ is obtained. Applying [1, Lemma 11.3], we deduce that Terms 1 and 2 are nonpositive for all $t \geq t_0$. Furthermore, for all $t \geq t_0$, it holds that

$$\begin{aligned}\text{Term } 3 = 2\Big\langle & \mathbf{\Pi}_N f - \hat{f}_N(t,\cdot), \\ & \mathbf{\Pi}_N E^*_{x_N(t)}B^{\mathrm{T}}Pe_N(t) - \mathfrak{Proj}\Big(\hat{f}_N(t), \mathbf{\Pi}_N\Gamma_f E^*_{x_N(t)}B^{\mathrm{T}}Pe(t)\Big)\Big\rangle_{\boldsymbol{\mathcal{H}}}.\end{aligned}$$

Now, let $\delta \in (-\infty, 1]$ and $\Theta_\delta \triangleq \{f \in \mathcal{D} : \|\mathbf{\Pi}_N f\|_{\boldsymbol{\mathcal{H}}} < \delta\}$. If $f \in \Theta_\delta$, then $\mathbf{\Pi}_N f \in \Theta_\delta$. Thus, it follows from Theorem 4 in the Appendix that Term 3 in (21) is nonpositive.

By assumption, Term 5 is nonpositive. Finally, recalling that $\boldsymbol{\mathcal{H}}_N$ can be expressed as the Cartesian product of scalar-valued RKHS, it follows from (4) and (9) that

$$\begin{aligned}&\text{Terms 0 \&4} \\ &= -\langle e_N(t), Qe_N(t)\rangle_{\mathbb{R}^n} + 2\big\langle E_{x_N(t)}(I-\mathbf{\Pi}_N)f, B^{\mathrm{T}}Pe_N(t)\big\rangle_{\mathbb{U}} \\ &= -\langle e_N(t), Qe_N(t)\rangle_{\mathbb{R}^n} \\ &\quad + 2\Big\langle \tilde{f}_N(t,\cdot), (I-\mathbf{\Pi}_N)E^*_{x_N(t)}B^{\mathrm{T}}Pe_N(t)\Big\rangle_{\boldsymbol{\mathcal{H}}} \\ &\leq -\lambda_{\min}(Q)\|e_N(t)\| \\ &\quad \cdot \left(\|e_N(t)\| - \frac{2\|B^{\mathrm{T}}P\|}{\lambda_{\min}(Q)}\|\epsilon_N\|_{\boldsymbol{\mathcal{H}}} \sup_{x\in\Omega}\mathcal{P}_N(x)\right)\end{aligned} \quad (22)$$

for all $t \geq t_0$, where $\Omega \subset \mathbb{X}$ is a compact set such that $x(t) \in \Omega$ at all times. ∎

Theorem 1 proves that, given a compensator $c_N(\cdot,\cdot)$ such that $\big\langle E_{x_N(t)}c_N(\cdot, e_N(t)), B^{\mathrm{T}}Pe_N(t)\big\rangle_{\mathbb{U}} \leq 0$ for all $t \geq t_0$, the trajectory tracking error is uniformly ultimately bounded. Note that if we define an uncertainty class $\mathcal{C}_R \triangleq \{f \in \boldsymbol{\mathcal{H}} \mid \|(I-\mathbf{\Pi}_N)f\|_{\boldsymbol{\mathcal{H}}} \leq \|f\|_{\boldsymbol{\mathcal{H}}} \leq R\}$, then the above proof can be used to define a performance bound that holds over all functions $f \in \mathcal{C}_R$, which is an infinite dimensional functional uncertainty class contained in $\boldsymbol{\mathcal{H}}$. This theorem, though generic, relies on the user's ability to design $c_N(\cdot,\cdot)$; this limitation will be lifted in the following.

Theorem 1 leverages the existence of a compact set $\Omega \subset \mathbb{X}$ such that $x_N(t) \in \Omega$ for all $t \geq t_0$. A way to construct the set Ω can be the following. From the proof of Theorem 1, we deduce that $\dot{V}(t) \leq 0$ for all $t \in [t_0, T(\eta))$. Thus, we consider the level set $\{z \in \mathcal{Z} : V(z) \leq V(z_0)\}$, where $z \triangleq (e_N, \tilde{\alpha}, \tilde{\beta}, \tilde{f})$ and $\mathcal{Z} \triangleq \mathbb{R}^n \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times m} \times \boldsymbol{\mathcal{H}}_N$. Since this level set can expressed as a Cartesian product of sets, whose finite-dimensional elements are compact, we can deduce a compact set $\mathcal{E} \subset \mathbb{R}^n$ such that $e_N(t) \in \mathcal{E}$ for all $t \geq t_0$. Finally, since $x_r(t)$ is bounded for all $t \geq t_0$ due to the boundedness of $r(t)$ and the Hurwitz nature of A_r, we can choose $\Omega = \bigcup_{t\geq t_0}\{\mathcal{E} + x_r(t)\}$. This approach may lead to conservative results. The results in the next section relieve the user from such a procedure.

Remark 1: Theorem 1 holds for $\boldsymbol{\mathcal{H}} = \mathcal{H}^m$. If $\boldsymbol{\mathcal{H}}$ is instead induced by general the operator-valued kernel $\mathcal{K}(\cdot,\cdot)$, then (19) holds replacing $\sup_{\xi\in\Omega}\mathcal{P}_N(\xi)$ with $\sup_{x\in\Omega}\sqrt{\|\mathcal{K}(x,x) - \mathcal{K}_N(x,x)\|}$. For brevity, we do not include this proof; see Section II-B.

B. The Limiting DPS

If, instead of (14), we employed an adaptive law that does not include any approximation due to the projection $\mathbf{\Pi}_N$, such as

$$\frac{\partial\hat{f}(t,\cdot)}{\partial t} = -\Gamma_f E^*_{x(t)}B^{\mathrm{T}}Pe(t),\ \hat{f}(t_0,\cdot) = \hat{f}_0,\ \ t \geq t_0, \quad (23)$$

then we would obtain the adaptive gain $\hat{f}(t,\cdot) \in \boldsymbol{\mathcal{H}}$ instead of $\hat{f}_N(t,\cdot) \in \boldsymbol{\mathcal{H}}_N \subsetneq \boldsymbol{\mathcal{H}}$. This equation is a PDE. Furthermore, the adaptive gain $\hat{f}(t,\cdot)$ would match the vector-valued RKHS $\boldsymbol{\mathcal{H}}$ for all $t \geq t_0$, and would not be guaranteed to lie in some finite-dimensional space. If $\boldsymbol{\mathcal{H}}$ is infinite-dimensional, then such adaptive gain would not be implementable in practice. The orthonormal projection $\mathbf{\Pi}_N$ in (14) allows us to produce adaptive gains that reside in the user-defined finite-dimensional RKHS $\boldsymbol{\mathcal{H}}_N$ designed to approximate $\boldsymbol{\mathcal{H}}$. Applying Gronwall's inequality, it can be shown that, for each $t \in [t_0, T)$, $\lim_{N\to\infty}\hat{f}_N(t,\cdot) = \hat{f}(t,\cdot)$ [18]. The DPS given by (1) with control input (11) and adaptive laws (12), (13), and (23) is called the *limiting DPS*.

C. Error Bounding Adaptive Control

To present the results of this section, it is worthwhile noting that if $\alpha = e^i$, then for all $x \in \mathbb{X}$, (8) reduces to

$$\begin{aligned}\big|\langle E_x(I-\mathbf{\Pi}_N)f, e^i\rangle_{\mathbb{R}}\big| &= \big|E_x((I-\mathbf{\Pi}_N)f)_i\big| \\ &\leq \mathcal{P}_N^{e^i}(x)\|(I-\mathbf{\Pi}_N)f\|_{\boldsymbol{\mathcal{H}}} \leq \mathcal{P}_N^{e^i}(x)\|f\|_{\boldsymbol{\mathcal{H}}},\end{aligned}$$

where $(\cdot)_i$ denotes the i-th component of its argument and e^i denotes the i-th element of the canonical basis in $\mathbb{U}$. Now, it follows from (7) that

$$\big|E_x((I-\mathbf{\Pi}_N)f)_i\big| \leq \sqrt{\Delta\mathcal{K}_{N,ii}(x,x)}\,\|(I-\mathbf{\Pi}_N)f\|_{\boldsymbol{\mathcal{H}}}, \quad (24)$$

where $\Delta\mathcal{K}_{N,ii}(x,x) \triangleq [\mathcal{K}(x,x)]_{(i,i)} - [\mathcal{K}_N(x,x)]_{(i,i)}$. Let

$$\Delta\mathcal{K}_N(x,x) \triangleq \big[\Delta\mathcal{K}_{N,11}(x,x), \ldots, \Delta\mathcal{K}_{N,mm}(x,x)\big]^{\mathrm{T}} \in \mathbb{U},$$

and define the compensator $v_N : \mathbb{X} \times \mathbb{R}^n \to \mathbb{U}$ whose i-th component is

$$v_{N,i}(x, e_N) \triangleq -\mathrm{sign}\big(B^{\mathrm{T}} Pe\big)_i \sqrt{\Delta\mathcal{K}_{N,ii}(x,x)}\|f\|_{\mathcal{H}}, \quad (25)$$

for $i = 1, \ldots, m$, where sign : $\mathbb{R} \to \{-1, 0, 1\}$ denotes the *signum function*. Finally, we consider the control input (11) with $E_{x_N(t)} c_N(\cdot, e_N(t)) = v_N(x(t), e_N(t))$ for all $t \geq t_0$. The next result characterizes the performance of this control input and of the adaptive laws (12)–(14).

Theorem 2: Assume that the DPS given by (1) with control input (11) and adaptive laws (12)–(14) have complete solutions on $[t_0, \infty)$. Furthermore, let $E_{x_N(t)} c_N(\cdot, e_N(t)) = v_N(x_N(t), e_N(t))$, where $v_{N,i}(\cdot,\cdot)$ is given by (25). Then, the trajectories of the DPS are uniformly bounded in $\mathbb{X} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times m} \times \mathcal{H}$, and $\lim_{t\to\infty} \|x_N(t) - x_r(t)\|_{\mathbb{X}} = 0$ uniformly in $t_0 \geq 0$.

Proof: By proceeding as in Theorem 1, we obtain that

$$\begin{aligned}\dot V(t) \leq{}& -\langle e_N(t), Qe_N(t)\rangle_{\mathbb{R}^n}\\ &+ 2\big\langle E_{x_N(t)}(I - \mathbf{\Pi}_N)f + v_N(x_N(t), e_N(t)), B^{\mathrm{T}} Pe_N(t)\big\rangle_{\mathbb{U}} \quad (26)\end{aligned}$$

for all $t \geq t_0$. Expanding the second term on the right-hand side of (26), we obtain that

$$\begin{aligned}&\big\langle E_{x_N(t)}(I - \mathbf{\Pi}_N)f + v_N(x_N(t), e_N(t)), B^{\mathrm{T}} Pe_N(t)\big\rangle_{\mathbb{U}}\\ &= \sum_{i=1}^{m} \big(B_P^{\mathrm{T}} e_N(t)\big)_i \big(E_{x_N(t)}((I - \mathbf{\Pi}_N)f)_i + (v_N(x_N(t), e_N(t))_i\big)\\ &= \sum_{i=1}^{m} \big|(B^{\mathrm{T}} Pe_N(t))_i\big| \Big(\mathrm{sign}(B^{\mathrm{T}} Pe_N(t))_i E_{x_N(t)}((I - \mathbf{\Pi}_N)f)_i\\ &\quad - \sqrt{\mathcal{K}_{ii}(x_N(t), x_N(t)) - \mathcal{K}_{N,ii}(x_N(t), x_N(t))}\|f\|_{\mathcal{H}}\Big) \leq 0,\end{aligned}$$

by virtue of the entry-wise bound in (24). The result now follows by applying Barbalat's lemma. ■

Theorem 2 provides a stronger result than Theorem 1 because it guarantees asymptotic convergence to zero of the trajectory tracking error. A challenge in the implementation of (25) lies in the need to know $\|f\|_{\mathcal{H}}$ or an upper bound thereof. This problem is addressed in Section III-D below. An additional challenge in the application of Theorem 2 lies in the discontinuities due to the signum function in (25). Thus, the implementation of the control input (11) with compensator $E_{x_N(t)} c_N(\cdot, e_N(t)) = v_N(x(t), e_N(t))$, $t \geq t_0$, may induce chattering. Future work directions involve the use of higher-order structures that, similarly to higher-order sliding mode control, will ease the implementation of this result.

D. Adaptive Upper Error Bounding Control

In this section, we introduce an additional adaptive mechanism that allows us to overcome the need to estimate $\|f\|_{\mathcal{H}}$. To this goal, we consider the compensator $v_N : \mathbb{X} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{U}$, with i-th component

$$v_{N,i}(x_N, e, \hat\lambda_N) \triangleq -\mathrm{sign}\big(B^{\mathrm{T}} Pe\big)_i \sqrt{\Delta\mathcal{K}_{N,ii}(x_N, x_N)}\hat\lambda_N, \quad (27)$$

where $\hat\lambda_N : [t_0, \infty) \to \mathbb{R}$ verifies

$$\begin{aligned}\dot{\hat\lambda}_N(t) &= -\Gamma_\lambda \sum_{i=1}^{m} \sqrt{\Delta\mathcal{K}_{N,ii}(x_N(t), x_N(t))}\big|\big(B^{\mathrm{T}} Pe_N(t)\big)_i\big|,\\ \hat\lambda_N(t_0) &= \hat\lambda_{N,0}, \quad t \geq t_0, \quad (28)\end{aligned}$$

and $\Gamma_\lambda > 0$ denotes an *adaptive rate*. We employ the same notation for the compensator in (25) and (27) to remark on their tight relationship. Indeed, if $\hat\lambda_N = \|f\|_{\mathcal{H}}$, then (27) is equivalent to (25).

Theorem 3: Assume that the DPS given by (1) with control input (11) and adaptive laws (12)–(14), and (28) have complete solutions on $[t_0, \infty)$. Furthermore, let $E_{x_N(t)} c_N(\cdot, e_N(t)) = v_N(x_N(t), e_N(t), \hat\lambda_N(t))$, where $v_{N,i}(\cdot,\cdot,\cdot)$ is given by (27). Then, the trajectories of the DPS are uniformly bounded in $\mathbb{X} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times m} \times \mathcal{H} \times \mathbb{R}$, and $\lim_{t\to\infty} \|x_N(t) - x_r(t)\|_{\mathbb{X}} = 0$, uniformly in $t_0 \geq 0$.

Proof: To prove this result, let $\tilde\lambda(t) \triangleq \|f\|_{\mathcal{H}} - \hat\lambda(t)$, and consider the Lyapunov function candidate $\mathcal{V}(e_N, \tilde\alpha, \tilde\beta, \tilde\lambda) = V(e_N, \tilde\alpha, \tilde\beta) + \big\langle \tilde\lambda, \Gamma_\lambda^{-1}\tilde\lambda\big\rangle_{\mathbb{R}}$, where $V(\cdot)$ is given by (20). Thus, by proceeding as in the proof of Theorem 1, we obtain that

$$\begin{aligned}\dot{\mathcal{V}}(t) ={}& -\langle e_N(t), Qe_N(t)\rangle_{\mathbb{R}^n}\\ &+ 2\big\langle B^{\mathrm{T}} Pe_N(t), E_{x_N(t)}(I - \mathbf{\Pi}_N)f\big\rangle_{\mathbb{R}}\\ &+ 2\big\langle B^{\mathrm{T}} Pe_N(t), v_N\big(x_N(t), e_N(t), \|f\|_{\mathcal{H}}\big)\big\rangle_{\mathbb{R}}\\ &- 2\Big\langle \sum_{i=1}^{m} \Delta\mathcal{K}_{N,ii}(x_N(t), x_N(t))\big|\big(B^{\mathrm{T}} Pe_N(t)\big)_i\big|, \tilde\lambda_N(t)\Big\rangle_{\mathbb{R}}\\ &- 2\big\langle \tilde\lambda_N(t), \Gamma_\lambda^{-1}\dot{\tilde\lambda}_N(t)\big\rangle_{\mathbb{R}}, \quad t \geq t_0 \quad (29)\end{aligned}$$

where $\mathcal{V}(t) = \mathcal{V}(e_N(t), \tilde\alpha(t), \tilde\beta(t), \tilde\lambda(t))$ for brevity. Thus, by proceeding as in Theorem 2, we deduce that $\dot{\mathcal{V}}(t) \leq -\langle e_N(t), Qe_N(t)\rangle_{\mathbb{R}^n}$, $t \geq t_0$, and the result follows applying Barbalat's lemma. ■

The adaptive laws (12)–(14) and (28) do not assure convergence of the adaptive gains to their unknown counterparts. For example, $\hat\lambda_N(\cdot)$ is not designed to converge to $\|f\|_{\mathcal{H}}$. This result can be attained by imposing some form of persistently exciting input, which is left for future works.

IV. Numerical Example

The numerical simulations presented in this section apply the adaptive control system presented by Theorem 3 with the deadzone operator in the adaptive laws (12)–(14). These simulations concern the plant model

$$\begin{aligned}\begin{bmatrix}\dot x_1(t)\\ \dot x_2(t)\end{bmatrix} &= \begin{bmatrix}0 & 1\\ -1 & 3\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}\big[(u(t) + 3x_1^2(t)x_2(t)\big],\\ x(0) &= [2.75, -0.75]^{\mathrm{T}}, \qquad t \geq 0, \quad (30)\end{aligned}$$

which captures a forced Van der Pol oscillator. Note that since the nonlinearity is polynomial, its restriction to any compact set contained in $\mathbb{R}^2$ is contained in the vector-valued native spaces used in this letter when they are restricted to the same set. The reference model (2) is characterized by

$$A_r = \begin{bmatrix}0 & 1\\ -100 & -13\end{bmatrix}, \quad B_r = \begin{bmatrix}0\\ 100\end{bmatrix},$$

$r(t) = 2\cos t$, and $x_r(0) = [2, 0]^{\mathrm{T}}$. The algebraic Lyapunov equation is solved for $Q = I_2$. The adaptive rates in (12)–(14) and (28) are $\Gamma_\alpha = 10^2 \cdot I_2$ $\Gamma_\beta = 10^2$, $\Gamma_f = 10^5$, and $\Gamma_\lambda = 10^6$, and the initial conditions are $\hat\alpha(0) = 0$, $\hat\beta(0) = 0$, $\hat f_N(0) = 0$, and $\hat\lambda_N(0) = 1000$. The RKHS is generated by the exponential kernel function $\mathfrak{K}(x, y) = \exp(-(x-y)^2/2l)$, $(x, y) \in \mathbb{R}$, where the hyperparameter $l = 1$ sets the width of the kernel function. We set $N = 16$, and evenly distributed the 16 basis centers along the circle formed by the reference trajectory $x_{\mathrm{ref}}(\cdot)$ at steady-state.

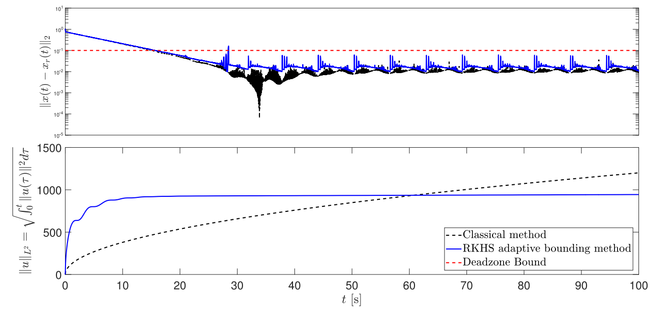

Fig. 1. A combined plot that shows the difference in the norm of the tracking error and the controller effort between the classical and RKHS adaptive bounding controller. Note that the red line in the tracking error plot is the deadzone bound used for both controllers. The value of the deadzone is 0.1.

We compare the proposed approach to the classical adaptive controller with an error bounding term discussed in [2, Ch. 4, 6], for which

$$u_c(t) = \hat{\alpha}_c^T(t)x(t) + \hat{\beta}_c^T(t)r(t) + v_c(e, t), \quad t \geq t_0, \tag{31}$$

where $\hat{\alpha}_c(\cdot)$ and $\hat{\beta}_c(\cdot)$ verify (12) and (13). The bounding term is defined as $v_c(e, t) = -\text{sign}(B^T Pe)\,\bar{f}$, where $\bar{f} \geq \|\hat{f}\|_{L^\infty(\Omega)}$ captures a user-defined upper bound on the infinity-norm on the uncertainty f. For this example, $\|f\|_{L^\infty(\Omega)} \leq 12$, and, to consider conservative assumptions, we set $\bar{f} = 120$.

Figure 1 shows the norm of the tracking error and the L_2-norm of the control effort obtained employing the results of Theorem 3 and those in [2, Ch. 4, 6]. The transient performance of both methods is indistinguishable. In the steady-state regime, the proposed RKHS adaptive bounding control technique provides a trajectory tracking error comparable to that of classical bounding control. However, the L_2-norm of the control input shows that the proposed system requires less effort. The classical approach in [2, Ch. 4, 6] shows an increasing control effort. The controller effort at each time t is almost an order of magnitude smaller than that of simple error bounding control during the stead-state regime. Using both methods, the trajectory tracking error shows chattering due to the signum function.

As the upper bound $\bar{f}$ on the functional uncertainty decreases, chattering becomes more enhanced for the adaptive method shown in [2, Ch. 4, 6] since the control law becomes more aggressive for the same levels of performance. The proposed adaptive control system does not require information on the functional uncertainty and does not show this effect. It can be shown that the tracking error decreases as N increases, in accordance with the theorems. This has been reported elsewhere by the authors, see [4], [18]. However, the computational time increases with N as well.

Thus, the proposed RKHS adaptive bounding control requires less information about the system, guarantees convergence over a nonparametric uncertainty class in $\mathcal{H}$, and does not require upper bounds on the uncertainty. Future work involve studying the trade-offs between the size of the uncertainty class, the selection of $\mathcal{K}$ and $\mathcal{H}$, the number of centers N, numerical conditioning and sensitivity due to factorization of $\mathbb{K}_N$, and the noise on the controller performance.

V. Conclusion

This letter presented three MRAC systems for plants affected by matched and parametric uncertainties, and whose nonlinear functional uncertainties are known to lie in an RKHS of vector-valued functions defined in terms of an operator-kernel. All the implementable control schemes can be understood as approximations of the limiting DPS, which contains a learning law that is a PDE for the functional gains. These adaptive control systems give performance guarantees that hold over classes of the functional uncertainties that generally lie in infinite-dimensional spaces. Future works involve higher-order variable structure systems to design the compensator and prevent chattering in the control input. Furthermore, we will investigate the merge of these results to RKHS-based methods for nonlinearity reconstruction [7].

Appendix

To state the next result, let $\delta \in (-\infty, 1]$, $\Omega_\delta \triangleq \{f \in \mathcal{D} : \|\mathbf{\Pi}_N f\|_{\mathcal{H}} < \delta\}$ and $\partial\Omega_\delta \triangleq \{f \in \mathcal{D} : \|\mathbf{\Pi}_N f\|_{\mathcal{H}} = \delta\}$.

Theorem 4: Consider the convex projection operator $\mathfrak{Proj} : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$. Then,

$$(f_b - f_i, \mathfrak{Proj}(f_b, f_d) - f_d)_{\mathcal{H}} \leq 0 \tag{32}$$

for all $f_i \in \Omega_\delta$, $f_b \in \partial\Omega_\delta$, and $f_d \in \mathcal{H}$.

Proof: This result follows by extending [1, Lemma 11.3] to infinite-dimensional Hilbert spaces, and is omitted for brevity. ∎

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