

A CONTINUOUS FIRST-ORDER SLIDING MODE CONTROL LAW

Keyvan Mohammadi

School of Aerospace and Mechanical Engineering
 The University of Oklahoma
 Norman, OK 73019
 Email: keyvan.mohammadi@ou.edu

Andrea L’Afflitto*

School of Aerospace and Mechanical Engineering
 The University of Oklahoma
 Norman, OK 73019
 Email: a.lafflitto@ou.edu

ABSTRACT

Sliding mode control is a technique to design robust feedback control laws. In its classical formulation, this approach involves discontinuous controls that arise several theoretical and practical challenges, such as the existence of non-unique solutions of nonlinear differential equations and chattering. Numerous variations of the sliding mode control architecture, such as the higher-order sliding mode method, have been presented to overcome these issues. In this paper, we present an alternative sliding mode control architecture that involves Hölder continuous feedback control laws, is simpler to implement than other non-classical nonlinear robust control techniques, guarantees robustness and uniform asymptotic stability of the closed-loop system, and ensures both existence and uniqueness of the closed-loop system’s trajectory. Our results are applied to design a robust nonlinear observer in the same form as the Walcott and Źak observer. Moreover, a numerical example illustrates our theoretical results and compares the proposed control law to the classical sliding mode control, the second order sliding mode control, and the super-twisting control.

1 INTRODUCTION

In most cases of practical interest, it is difficult to accurately model dynamical systems and estimate the parameters characterizing such models. Consequently, designing feedback control laws that guarantee closed-loop asymptotic stability, satisfactory command following, and robustness to unmodeled dynamics may result in a daunting task. The robust control problem

for linear dynamical systems has been extensively studied in the 1970s and 1980s within the framework of \mathcal{H}_∞ and $\mathcal{H}_2/\mathcal{H}_\infty$ control theory [1–4], whereas notorious robust control techniques for nonlinear dynamical systems are adaptive control [5, 6] and sliding mode control [7, 8], [9, Ch. 7], [10, Ch. 14], [11].

The sliding mode control architecture, which was first devised in the 1960s by Emel’yanov and Barbashin [8], consists in steering in finite-time the system’s trajectory to a subspace of the state space, known as *sliding manifold*, which has been designed so that if the system’s trajectory reaches this manifold, then the system’s state asymptotically converges to zero. The control law that drives the system’s trajectory to the sliding manifold involves the signum function, and hence is discontinuous. It is well known that solutions of ordinary differential equations with discontinuous right-hand sides may not exist or may not be unique [12, Ch. 2], [13]. Furthermore, in most cases of practical interest discontinuous control inputs induce an undesired effect known as *chattering*, which consists in high-frequency oscillations of the system’s state about the sliding manifold [10, Ch. 14]; a detailed characterization of chattering is provided in [14].

In spite of the theoretical and practical challenges concerning sliding mode control, this technique has drawn considerable interest in aerospace [15], chemical [16], electrical [17], marine [18], and mechanical engineering [19] for its ease of implementation [20] and ability to compensate for disturbances and uncertainties [21]; for further details, see [22, 23] and the numerous references therein. Several approaches have been proposed to design sliding mode control laws that are not affected by chattering. The most popular of these chattering-free techniques consists in modifying the classical sliding mode control in an arbitrarily small neighborhood of the sliding manifold, known as *boundary layer*, where the signum function is approximated by the satura-

* Address all correspondence to this author.

tion, the sigmoid, or the arctangent functions [24, 25]. However, the presence of a boundary layer degrades the system's performance, since robustness is not guaranteed within this region of the state space. Considerable attention was risen by the *higher-order sliding mode control* [20, 26, 27], whereby the sliding mode architecture is applied to higher derivatives of the sliding variable to contrast chattering; the *super-twisting algorithm* [28] is a notable variation of the second-order sliding mode control [26]. It is also worth to recall the *chattering attenuation conventional sliding mode control*, whereby chattering is considerably reduced by employing a time-varying sliding manifold [29].

In this paper, we modify the classical sliding mode control technique and provide a *continuous* feedback control law that guarantees closed-loop asymptotic stability of a nonlinear dynamical system, whose dynamics is affected by time-varying matched and unmatched uncertainties, and failures of the control system. Specifically, we consider a sliding manifold such that the control input explicitly appears in the first-order time derivative of the sliding variable and design a feedback control law, which involves fractional powers of the sliding variable, to guarantee finite-time convergence of the closed-loop system trajectory to the sliding manifold. Closed-loop finite-time stability of the nonlinear dynamical system, which captures the system dynamics before it reaches the sliding manifold, is guaranteed by the sufficient conditions for strong uniform finite-time stability of time-varying dynamical systems recently developed in [30]. Our approach, which does *not* involve discontinuous functions, can be straightforwardly extended to tackle higher-order sliding mode control, that is, to address control problems wherein the control input explicitly appears in higher-order time derivatives of the sliding variable.

The advantages of the proposed control scheme are multi-fold. Specifically, the feedback control law presented in this paper is continuous and, in spite of the higher-order sliding mode control, does not involve differentiators, that is, real-time robust estimators of the higher-order time derivatives of the sliding variable [31]. Moreover, although the proposed control law is Hölder continuous, but not Lipschitz continuous, the existence and uniqueness of solutions of the closed-loop system on the infinite time horizon is proven. In addition, one can regulate the time needed by the closed-loop system to reach the sliding manifold by tuning several parameters. Finally, the feedback control law presented in this paper applies to affine in the control dynamical systems and mechanical, electromagnetic, and port-controlled Hamiltonian systems are affine in the control [12, Ch. 5].

The proposed approach resembles a form of sliding mode control, known as *terminal sliding mode control* [32–34], whereby control laws involve fractional powers to steer the system trajectory to the equilibrium point in finite-time. The terminal sliding mode control has been mostly applied to mechanical systems and is designed to guarantee finite-time converge of the system's trajectory to a manifold, known a *terminal sliding manifold*, and finite-time convergence of the system's trajectory to the equilibrium point along the terminal sliding manifold. However,

the classical terminal sliding mode control involves a discontinuous control law and hence is affected by the same limitations as the classical sliding mode control, which can be mitigated by introducing boundary layers or high-order sliding mode techniques [35].

Our theoretical results are applied to design a nonlinear robust observer in the same form as the Walcott and Žak observer. However, in spite of the classical Walcott and Žak observer, the proposed estimation architecture does not involve discontinuous functions. Finally, we provide a numerical example that clearly illustrates the main differences between our approach, the classical sliding mode control, the second-order sliding mode control, and the super-twisting sliding mode control.

2 NOTATION, DEFINITIONS, AND MATHEMATICAL PRELIMINARIES

In this section, we establish notation, definitions, and review some preliminary results. Let \mathbb{N} denote the set of positive integers, \mathbb{R} denote the set of real numbers, \mathbb{R}_+ denote the set of positive real numbers, \mathbb{R}^n denote the set of $n \times 1$ real column vectors, $\mathbb{R}^{n \times m}$ denote the set of $n \times m$ real matrices, and $\mathcal{B}_\varepsilon(x)$ denote the *open ball centered* at x with *radius* ε . We write $\|\cdot\|$ both for the Euclidean vector norm and the corresponding equi-induced matrix norm, $\|\cdot\|_\infty$ both for the infinity vector norm and the corresponding equi-induced matrix norm, $V'(x) \triangleq \frac{\partial V(x)}{\partial x}$ for the Fréchet derivative of V at x , I_n or I for the $n \times n$ identity matrix, $0_{n \times m}$ or 0 for the zero $n \times m$ matrix, and A^T for the transpose of the matrix A . For $b \in \mathbb{N}$ odd number and $x \in \mathbb{R}$, we denote by $x^{\frac{1}{b}}$ the b th *real root* of x .

Consider the nonlinear dynamical system given by

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0, \quad t \in \mathcal{I}_{t_0, x_0}, \quad (1)$$

where, for every $t \in \mathcal{I}_{t_0, x_0}$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $\mathcal{I}_{t_0, x_0} \subseteq [t_0, \infty)$ is the maximal interval of existence of a solution $x(t)$ of (1), \mathcal{D} is an open set with $0 \in \mathcal{D}$, and $f: \mathcal{I}_{t_0, x_0} \times \mathcal{D} \rightarrow \mathbb{R}^n$ is such that, for every $(t, x) \in \mathcal{I}_{t_0, x_0} \times \mathcal{D}$, $f(t, 0) = 0$ and $f(\cdot, \cdot)$ is jointly continuous in t and x . A continuously differentiable function $x: \mathcal{I}_{t_0, x_0} \rightarrow \mathcal{D}$ is said to be the *solution* of (1) on the interval $\mathcal{I}_{t_0, x_0} \subset \mathbb{R}$ if $x(\cdot)$ satisfies (1) for all $t \in \mathcal{I}_{t_0, x_0}$. As shown in [30], it follows from Peano's theorem [12, Th. 2.24] that the joint continuity of $f(\cdot, \cdot)$ implies that, for every $x \in \mathcal{D}$, there exists $\tau_0 < t_0 < \tau_1$ and a solution $x(\cdot)$ of (1) defined on the open interval (τ_0, τ_1) such that $x(t_0) = x$. A solution $t \mapsto x(t)$ is said to be *right maximally* defined if x cannot be extended (either uniquely or nonuniquely) forward in time. We assume that all right maximal solutions to (1) exist on $[t_0, \infty)$, and hence, we assume that (1) is *forward complete*.

We assume that (1) possesses unique solutions in forward time for all initial conditions except possibly the origin in the following sense. For every $x \in \mathcal{D} \setminus \{0\}$ there exists $\tau_x > t_0$ such that, if $y_1: [t_0, \tau_1] \rightarrow \mathcal{D}$ and $y_2: [t_0, \tau_2] \rightarrow \mathcal{D}$ are two solutions of

(1) with $y_1(t_0) = y_2(t_0) = x$, then $\tau_x \leq \min\{\tau_1, \tau_2\}$ and $y_1(t) = y_2(t)$ for all $t \in [t_0, \tau_x]$. Without loss of generality, we assume that for each x , τ_x is chosen to be the largest such number in $[t_0, \infty)$. In this case, we denote by the continuously differentiable map $s^{t_0, x_0}(\cdot) \triangleq s(\cdot, t_0, x_0)$ the *trajectory* or the unique *solution curve* of (1) on \mathcal{I}_{t_0, x_0} satisfying $s(t_0, t_0, x_0) = x_0$. Sufficient conditions for forward uniqueness in the absence of Lipschitz continuity can be found in [36] [13, Section 10], [37], and [38, Section 1].

The following definition introduces the notion of finite-time stability for time-varying nonlinear dynamical systems, which plays a key role in this paper.

Definition 2.1 ([30]). *The nonlinear dynamical system (1) is finite-time stable if there exist an open neighborhood $\mathcal{D}_0 \subseteq \mathcal{D}$ of the origin and a function $T : [0, \infty) \times \mathcal{D}_0 \setminus \{0\} \rightarrow (t_0, \infty)$, called the settling-time function, such that the following statements hold:*

- i) *Finite-time convergence. For every $(t_0, x_0) \in [0, \infty) \times \mathcal{D}_0 \setminus \{0\}$, $s^{t_0, x_0}(t)$ is defined on $[t_0, T(t_0, x_0))$, $s^{t_0, x_0}(t) \in \mathcal{D}_0 \setminus \{0\}$ for all $t \in [t_0, T(t_0, x_0))$, and $s^{t_0, x_0}(t) \rightarrow 0$ as $t \rightarrow T(t_0, x_0)$.*
- ii) *Lyapunov stability. For every $\varepsilon > 0$ and $t_0 \in [0, \infty)$ there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that $\mathcal{B}_\delta(0) \subset \mathcal{D}_0$ and, for every $x_0 \in \mathcal{B}_\delta(0) \setminus \{0\}$, $s^{t_0, x_0}(t) \in \mathcal{B}_\varepsilon(0)$ for all $t \in [t_0, T(t_0, x_0))$.*

The nonlinear dynamical system (1) is uniformly finite-time stable if (1) is finite-time stable and the following statement holds:

- iii) *Uniform Lyapunov stability. For every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\mathcal{B}_\delta(0) \subset \mathcal{D}_0$ and, for every $x_0 \in \mathcal{B}_\delta(0) \setminus \{0\}$, $s^{t_0, x_0}(t) \in \mathcal{B}_\varepsilon(0)$ for all $t \in [t_0, T(t_0, x_0))$ and for all $t_0 \in [0, \infty)$.*

The nonlinear dynamical system (1) is strongly uniformly finite-time stable if (1) is uniformly finite-time stable and the following statement holds:

- iv) *Uniform finite-time convergence. For every $(t_0, x_0) \in [0, \infty) \times \mathcal{D}_0 \setminus \{0\}$, $s^{t_0, x_0}(t)$ is defined on $[t_0, T(t_0, x_0))$, $s^{t_0, x_0}(t) \in \mathcal{D}_0 \setminus \{0\}$ for all $t \in [t_0, T(t_0, x_0))$, and $s^{t_0, x_0}(t) \rightarrow 0$ as $t \rightarrow T(t_0, x_0)$ uniformly in t_0 for all $t_0 \in [0, \infty)$.*

The nonlinear dynamical system (1) is globally finite-time stable (respectively, globally uniformly finite-time stable or globally strongly uniformly finite-time stable) if it is finite-time stable (respectively, uniformly finite-time stable or strongly uniformly finite-time stable) with $\mathcal{D}_0 = \mathbb{R}^n$.

If (1) does not explicitly depend on time, then Definition 2.1 reduces to the definition of finite-time stability for autonomous dynamical systems provided in [39]. The following result shows that if the zero solution $x(t) \equiv 0$ to (1) is finite-time stable, then (1) has a unique solution $s(\cdot, \cdot, \cdot)$ defined on $[t_0, \infty) \times [0, \infty) \times \mathcal{D}_0$ for every initial condition in an open neighborhood of the origin, including the origin, and $s(t, t_0, x_0) = 0$ for all $t \geq T(t_0, x_0)$, $(t_0, x_0) \in [0, \infty) \times \mathcal{D}_0$, where $T(t_0, 0) \triangleq t_0$.

Proposition 2.1 ([30, 40]). *Consider the nonlinear dynamical system \mathcal{G} given by (1). Assume \mathcal{G} is finite-time stable and let*

$\mathcal{D}_0 \subseteq \mathcal{D}$ and $T : [0, \infty) \times \mathcal{D}_0 \setminus \{0\} \rightarrow (t_0, \infty)$ be defined as in Definition 2.1. Then, for every $(t_0, x_0) \in [0, \infty) \times \mathcal{D}_0$, there exists a unique solution $s(t, t_0, x_0)$, $t \geq t_0$, to (1) such that $s(t, t_0, x_0) \in \mathcal{D}_0$, $t \in [t_0, T(t_0, x_0))$, and such that $s(t, t_0, x_0) = 0$, $t \geq T(t_0, x_0)$, where $T(t_0, 0) \triangleq t_0$.

It follows from Proposition 2.1 that if the zero solution $x(t) \equiv 0$ to (1) is finite-time stable, then the solutions of (1) define a continuous global semiflow on \mathcal{D}_0 ; that is, $s : [t_0, \infty) \times [0, \infty) \times \mathcal{D}_0 \rightarrow \mathcal{D}_0$ is jointly continuous and satisfies the consistency property $s(t_0, t_0, x) = x$ and the semigroup property $s(t, \tau, s(\tau, t_0, x)) = s(t, t_0, x)$ for every $x \in \mathcal{D}_0$ and $t \geq \tau \geq t_0$. Furthermore, $s(\cdot, \cdot, \cdot)$ satisfies $s(T(t_0, x) + t, t_0, x) = 0$ for all $x \in \mathcal{D}_0$ and $t \in [0, \infty)$. In addition, it also follows from Proposition 2.1 that we can extend $T(t_0, \cdot)$ to all of \mathcal{D}_0 by defining $T(t_0, 0) \triangleq t_0$, for all $t_0 \in [0, \infty)$. Now, by uniqueness of solutions it follows that $s(T(t_0, x) + t, t_0, x) = 0$, $t \in [0, \infty)$, and hence, it is easy to see from Definition 2.1 that

$$T(t_0, x) = \inf\{t \in [t_0, \infty) : s(t, t_0, x) = 0\}, \quad (t_0, x) \in [0, \infty) \times \mathcal{D}_0. \quad (2)$$

Lastly, it follows from Definition 2.1 and Proposition 2.1 that if the zero solution $x(t) \equiv 0$ to (1) is finite-time stable, then it is asymptotically stable, and hence, finite-time stability is a stronger condition than asymptotic stability.

Next, we provide sufficient conditions for strong uniform finite-time stability of the nonlinear dynamical system given by (1). For the statement of the following result define $\dot{V}(t, x) \triangleq \frac{\partial V(t, x)}{\partial t} + \frac{\partial V(t, x)}{\partial x} f(t, x)$ for a continuously differentiable function $V : [t_0, \infty) \times \mathcal{D} \rightarrow \mathbb{R}$.

Theorem 2.1 ([30]). *Consider the nonlinear dynamical system \mathcal{G} given by (1). Then the following statements hold:*

- i) *If there exist a continuously differentiable function $V : [t_0, \infty) \times \mathcal{D} \rightarrow \mathbb{R}$, class \mathcal{K} functions $\alpha(\cdot)$ and $\beta(\cdot)$, real numbers $\theta \in (0, 1)$ and $k > 0$, and an open neighborhood $\mathcal{M} \subseteq \mathcal{D}$ of the origin such that*

$$\alpha(\|x\|) \leq V(t, x) \leq \beta(\|x\|), \quad (t, x) \in [t_0, \infty) \times \mathcal{M}, \quad (3)$$

$$\dot{V}(t, x) \leq -k(V(t, x))^\theta, \quad (t, x) \in [t_0, \infty) \times \mathcal{M}, \quad (4)$$

then \mathcal{G} is strongly uniformly finite-time stable. Moreover, there exist a neighborhood \mathcal{D}_0 of the origin and a settling-time function $T : [0, \infty) \times \mathcal{D}_0 \rightarrow [t_0, \infty)$ such that

$$T(t_0, x_0) \leq \frac{(V(t_0, x_0))^{1-\theta}}{k(1-\theta)}, \quad (t_0, x_0) \in [0, \infty) \times \mathcal{D}_0, \quad (5)$$

and $T(\cdot, \cdot)$ is jointly continuous on $[0, \infty) \times \mathcal{D}_0$.

ii) If $\mathcal{M} = \mathcal{D} = \mathbb{R}^n$ and there exist a continuously differentiable function $V : [t_0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}$, class \mathcal{K}_∞ functions $\alpha(\cdot)$ and $\beta(\cdot)$, and real numbers $\theta \in (0, 1)$ and $k > 0$ such that (3) and (4) hold, then \mathcal{G} is globally strongly uniformly finite-time stable. Moreover, there exists a settling-time function $T : [0, \infty) \times \mathbb{R}^n \rightarrow [t_0, \infty)$ such that (5) holds with $\mathcal{D}_0 = \mathbb{R}^n$ and $T(\cdot, \cdot)$ is jointly continuous on $[0, \infty) \times \mathbb{R}^n$.

3 A CONTINUOUS SLIDING MODE CONTROL FOR ASYMPTOTIC STABILIZATION

In this section, we apply the results of Section 2 to design a continuous robust control law that guarantees uniform asymptotic stability of the nonlinear time-varying dynamical system

$$\begin{aligned}\dot{x}(t) &= f(x(t)) + B(x(t)) [G(x(t))E(x(t))u(t) \\ &\quad + \delta_1(t, x(t), u(t))] + \delta_2(x(t)), \\ x(t_0) &= x_0, \quad t \geq t_0,\end{aligned}\quad (6)$$

where $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $t \geq t_0$, $u(t) \in \mathbb{R}^m$, $f : \mathcal{D} \rightarrow \mathbb{R}^n$, $B : \mathcal{D} \rightarrow \mathbb{R}^{n \times m}$, $G : \mathcal{D} \rightarrow \mathbb{R}^{m \times m}$, and $E : \mathcal{D} \rightarrow \mathbb{R}^{m \times m}$ are continuous in x , $f(0) = 0$, $E(x)$ is invertible for all $x \in \mathcal{D}$, $G(\cdot)$ is an unknown positive-definite diagonal matrix, that is, $G(x) \geq g_0 I_m$ for some $g_0 > 0$, $\delta_1 : [t_0, \infty) \times \mathcal{D} \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ is continuous in t , x , and u , and $\delta_2 : \mathcal{D} \rightarrow \mathbb{R}^m$ is continuous in x . The terms $\delta_1(\cdot, \cdot, \cdot)$ and $\delta_2(\cdot)$ in (6) are *unknown* and capture the matched and unmatched uncertainties, respectively. Moreover, $\delta_1(\cdot, \cdot, \cdot)$ and $\delta_2(\cdot)$ capture parametric uncertainties. Assuming that the elements of $G(\cdot)$ are unknown allows accounting for failures in the control mechanism. As in the classical sliding-mode control, we make the following assumptions, for the statement of which we recall that if $\mathcal{T} : \mathcal{D} \rightarrow \mathbb{R}^n$ is continuously differentiable, invertible, and $\mathcal{T}^{-1}(\cdot)$ is continuously differentiable, then $\mathcal{T}(\cdot)$ is a *diffeomorphism* [10, p. 508].

Assumption 3.1. Consider the nonlinear dynamical system (6). There exists a diffeomorphism $\mathcal{T} : \mathcal{D} \rightarrow \mathbb{R}^n$ such that $\mathcal{T}(0) = 0$ and

$$\frac{\partial \mathcal{T}(x)}{\partial x} B(x) = \begin{bmatrix} 0_{(n-m) \times m} \\ I_m \end{bmatrix}, \quad x \in \mathcal{D}. \quad (7)$$

Let $\mathcal{T} : \mathcal{D} \rightarrow \mathbb{R}^n$ be a diffeomorphism such that Assumption 3.1 is satisfied and $\mathcal{T}(x) = [\eta^\top, \xi^\top]^\top$, where $\eta \in \mathbb{R}^{n-m}$ and $\xi \in \mathbb{R}^m$. Then (6) is equivalent to

$$\begin{aligned}\dot{\eta}(t) &= f_\eta(\eta(t), \xi(t)) + \delta_\eta(\eta(t), \xi(t)), \\ \eta(t_0) &= [I_{n-m}, 0_{(n-m) \times m}] \mathcal{T}(x_0), \quad t \geq t_0,\end{aligned}\quad (8)$$

$$\begin{aligned}\dot{\xi}(t) &= f_\xi(\eta(t), \xi(t)) + G(\eta(t), \xi(t))E(\eta(t), \xi(t))u(t) \\ &\quad + \delta_\xi(t, \eta(t), \xi(t), u(t)), \\ \xi(t_0) &= [0_{m \times (n-m)}, I_m] \mathcal{T}(x_0).\end{aligned}\quad (9)$$

Assumption 3.2. Consider the nonlinear dynamical system (6) and let $\mathcal{T} : \mathcal{D} \rightarrow \mathbb{R}^n$ be a diffeomorphism such that Assumption 3.1 is satisfied. There exists a continuous function $\phi : \mathbb{R}^{n-m} \rightarrow \mathbb{R}^m$ such that $\phi(0) = 0$ and the equilibrium point $\eta(t) \equiv 0$, $t \geq t_0$, of the time-invariant closed-loop system

$$\begin{aligned}\dot{\eta}(t) &= f_\eta(\eta(t), \phi(\eta(t))) + \delta_\eta(\eta(t), \phi(\eta(t))), \\ \eta(t_0) &= [I_{n-m}, 0_{(n-m) \times m}] \mathcal{T}(x_0), \quad t \geq t_0,\end{aligned}\quad (10)$$

is asymptotically stable.

Assumptions 3.1 and 3.2 are common in the classical sliding mode control law design; for details, see [10, p. 564]. The classical sliding mode control architecture provides a *discontinuous* feedback control law $u(\cdot)$ such that $\|\xi(t) - \phi(\eta(t))\| \rightarrow 0$ as $t \rightarrow T$, for some finite-time $T > 0$, in spite of the uncertainties captured by $G(\eta, \xi, u)$, $\delta_\xi(\eta, \xi, u)$, and $\delta_\eta(\eta, \xi)$. Hence, Assumption 3.2 is key to guarantee closed-loop asymptotic stability of (6).

The next theorem, which is the main result of this section, provides a *continuous* robust feedback control law, such that $\|\xi(t) - \phi(\eta(t))\| \rightarrow 0$ as $t \rightarrow T$ and the equilibrium point $x(t) \equiv 0$, $t \geq t_0$, of (6) is uniformly asymptotically stable. For the statement of this result, let the invertible matrix function $\hat{G} : \mathbb{R}^{n-m} \times \mathbb{R}^m \rightarrow \mathbb{R}^{m \times m}$ denote an *estimate* of $G(\cdot)$, define

$$\begin{aligned}\psi(\eta, \xi, w) &\triangleq -E^{-1}(\eta, \xi)\hat{G}^{-1}(\eta, \xi) \\ &\quad \cdot \left[f_\xi(\eta, \xi) - \frac{\partial \phi(\eta)}{\partial \eta} f_\eta(\eta, \xi) \right] + E^{-1}(\eta, \xi)w, \\ (t, \eta, \xi, w) &\in [t_0, \infty) \times \mathbb{R}^{n-m} \times \mathbb{R}^m \times \mathbb{R}^m,\end{aligned}\quad (11)$$

$$\delta(t, \eta, \xi, w) \triangleq \delta_\xi(t, \eta, \xi, w) - \frac{\partial \phi(\eta)}{\partial \eta} \delta_\eta(\eta, \xi), \quad (12)$$

$$\begin{aligned}\Delta(t, \eta, \xi, w) &\triangleq \delta(t, \eta, \xi, \psi(\eta, \xi, w)) + [I - G(\eta, \xi)\hat{G}^{-1}(\eta, \xi)] \\ &\quad \cdot \left[f_\xi(\eta, \xi) - \frac{\partial \phi(\eta)}{\partial \eta} f_\eta(\eta, \xi) \right],\end{aligned}\quad (13)$$

$$z(\eta, \xi) \triangleq \xi - \phi(\eta), \quad (14)$$

where $\phi(\cdot)$ satisfies Assumption 3.2, and let x_i denote the i th component of $x \in \mathbb{R}^n$.

Theorem 3.1. Consider the nonlinear dynamical system (6) and suppose that Assumptions 3.1 and 3.2 are verified. If there exist continuous functions $\rho_i : \mathbb{R}^{n-m} \times \mathbb{R}^m \rightarrow \mathbb{R}_+$, $i = 1, \dots, m$, such that

$$\begin{aligned}[G^{-1}(\eta, \xi)\Delta(t, \eta, \xi, \psi(\eta, \xi, w))]_i &\leq \rho_i(\eta, \xi)|z_i(\eta, \xi)|, \\ (t, \eta, \xi, w) &\in [t_0, \infty) \times \mathbb{R}^{n-m} \times \mathbb{R}^m \times \mathbb{R}^m,\end{aligned}\quad (15)$$

then (6) is uniformly asymptotically stable with feedback control law

$$u = \psi(\eta, \xi, \gamma(\eta, \xi)), \quad (\eta, \xi) \in \mathbb{R}^{n-m} \times \mathbb{R}^m, \quad (16)$$

where

$$\gamma_i(\eta, \xi) = -\left[\rho_i(\eta, \xi)z_i(\eta, \xi) + cz_i^{2\theta-1}(\eta, \xi)\right], \quad i = 1, \dots, m, \quad (17)$$

where $c > 0$, $\theta = \theta_1/\theta_2$, $\theta_1 \in \mathbb{N}$, $\theta_2 \in \mathbb{N}$ is an odd number, and $\theta \in (\frac{1}{2}, 1)$. Moreover, if $\mathcal{D} = \mathbb{R}^n$ and the feedback control $\phi(\cdot)$ guarantees global asymptotic stability of (10), then (6) with $u = \psi(\eta, \xi, \gamma(\eta, \xi))$ is globally uniformly asymptotically stable.

Proof: It follows from Assumption 3.1 that there exists a diffeomorphism $\mathcal{T}(x) = [\eta^T, \xi^T]^T$ such that $\mathcal{T}(0) = 0$, (7) is satisfied, and (6) is equivalent to (8) and (9). Moreover, it follows from Assumption 3.2 that there exists a feedback control law $\phi : \mathbb{R}^{n-m} \rightarrow \mathbb{R}^m$ such that the equilibrium point $\eta(t) \equiv 0$, $t \geq t_0$, of the closed-loop system (10) is asymptotically stable.

Next, it follows from (8) and (9) that

$$\begin{aligned} \dot{z}(t) &= f_\xi(\eta(t), \xi(t)) - \frac{\partial \phi(\eta(t))}{\partial \eta} f_\eta(\eta(t), \xi(t)) \\ &\quad + G(\eta(t), \xi(t))E(\eta(t), \xi(t))u(t) + \delta(t, \eta(t), \xi(t), u(t)), \\ z(t_0) &= \xi(t_0) - \phi(\eta(t_0)), \quad t \geq t_0, \end{aligned} \quad (18)$$

and (18) with $u = \psi(\eta, \xi, w)$, where $\psi(\cdot, \cdot, \cdot)$ is given by (11), implies that

$$\begin{aligned} \dot{z}(t) &= G(\eta(t), \xi(t))w(t) + \Delta(t, \eta(t), \xi(t), w(t)), \\ z(t_0) &= \xi(t_0) - \phi(\eta(t_0)), \quad t \geq t_0, \end{aligned} \quad (19)$$

where $\Delta(\cdot, \cdot, \cdot, \cdot)$ is given by (13). To verify that (19) with feedback control law $w = \gamma(\eta, \xi)$ is globally strongly uniformly finite-time stable, and hence $\|\xi(t) - \phi(\eta(t))\| \rightarrow 0$ as $t \rightarrow T$, for some finite-time $T \geq t_0$, let

$$V(z) = \|z\|^2, \quad z \in \mathbb{R}^m, \quad (20)$$

and note that it follows from (20), (19), (15), and (17) that

$$\begin{aligned} \dot{V}(z) &= 2 \sum_{i=1}^m z_i [g_i(\eta, \xi)\gamma_i(\eta, \xi) + \Delta_i(t, \eta, \xi, \gamma(\eta, \xi))] \\ &\leq 2 \sum_{i=1}^m z_i g_i(\eta, \xi) [\gamma_i(\eta, \xi) + \rho_i(\eta, \xi)z_i] \\ &\leq -2cg_0(V(z))^\theta, \end{aligned} \quad (21)$$

where $g_i(\cdot, \cdot)$ denotes the element on the i th row and i th column of $G(\cdot, \cdot)$ and g_0 is such that $G(\eta, \xi) \geq g_0 I_m > 0$, $(\eta, \xi) \in \mathbb{R}^{n-m} \times \mathbb{R}^m$. Since (20) implies that $V(\cdot)$ satisfies (3) with $\alpha(\|z\|) = \beta(\|z\|) = V(z)$ and (21) implies that $\dot{V}(\cdot)$ satisfies (4) with $k = 2cg_0$, it follows from Theorem 2.1 that the zero solution $z(t) \equiv 0$, $t \geq t_0$, to (19) with $w = \gamma(\eta, \xi)$ is globally strongly finite-time stable and it follows from and Proposition 2.1 that the solution of (19) with $w = \gamma(\eta, \xi)$ exists and is unique. Moreover, there exists a settling-time function $T_\xi : [0, \infty) \times \mathbb{R}^n \rightarrow [t_0, \infty)$ such that

$$T_\xi(t_0, z(t_0)) \leq \frac{1}{2cg_0(1-\theta)}[V(z(t_0))]^{1-\theta}, \quad (22)$$

for all $z(t_0) \in \mathbb{R}^m$.

The nonlinear dynamical system (6) is equivalent to (8) and (9) or, alternatively, (8) and (18). Since (18) with $u(\cdot)$ given by (16) is globally strongly finite-time stable with settling-time function $T_\xi(t_0, z(t_0))$, it holds that $\xi(t) = \phi(\eta(t))$ for all $t \geq T_\xi(t_0, z(t_0))$ and it follows from Definition 2.1 and Proposition 2.1 that (18) with $u(\cdot)$ given by (16) is globally uniformly asymptotically stable. Moreover, it follows from Assumption 3.2 that the time-invariant dynamical system (8) with $\xi = \phi(\eta)$ is asymptotically stable. Consequently, the nonlinear dynamical system given by (8) and (18) with $u(\cdot)$ given by (16) is uniformly asymptotically stable. Hence, (6) with $u(\cdot)$ given by (16) is uniformly asymptotically stable. Lastly, global uniform asymptotic stability of (6) with $u(\cdot)$ given by (16) follows if $\mathcal{D} = \mathbb{R}^n$ and (8) with $\xi = \phi(\eta)$ is globally asymptotically stable. ■

Theorem 3.1 provides a feedback control law that guarantees uniform asymptotic stability of the nonlinear dynamical system (6), whose dynamics is affected by matched and unmatched uncertainties, and failures of the control system. In the classical sliding mode control, if the conditions of Theorem 3.1 are satisfied, then (6) is uniformly asymptotically stable with feedback control law (16) and

$$\begin{aligned} \gamma_i(\eta, \xi) &= -[\rho_i(\eta, \xi)|z_i(\eta, \xi)| + c]\text{sign}(z_i(\eta, \xi)), \\ i &= 1, \dots, m, \quad (\eta, \xi) \in \mathbb{R}^{n-m} \times \mathbb{R}^m, \end{aligned} \quad (23)$$

where $\text{sign}(\cdot)$ denotes the signum function. The control law (23) involves discontinuities of the first kind, that is, jump discontinuities, and hence existence and uniqueness of solutions of the closed-loop system is not guaranteed [13] and chattering will occur [10, Ch. 14], [14]. An important feature of our approach to the problem of designing nonlinear robust controls is that the robust feedback control law (16) is Hölder continuous and hence continuous [41, p. 42]. Although (6) with $u(\cdot)$ given by (16) is not Lipschitz continuous, it follows from the proof of Theorem 3.1 that the solution of the closed-loop system exists and is unique for all $t \geq t_0$.

Since both $G(\cdot, \cdot)$ and $E(\cdot, \cdot)$ are invertible for all $(\eta, \xi) \in \mathbb{R}^{n-m} \times \mathbb{R}^m$, the control input u explicitly appears in the first

derivative of the sliding variable $z = \xi - \phi(\eta)$. Therefore, (16) is a first-order sliding mode controller. Theorem 3.1 can be easily extended to address higher-order sliding mode control, that is, to tackle problems wherein the control input explicitly appears in higher-order time derivatives of the sliding variable.

It follows from (22) that the settling-time function $T_\xi(t_0, z(t_0))$, that is, the time for (18) with $u = \psi(\eta, \xi, \gamma(\eta, \xi))$ to reach the *sliding manifold* $\mathcal{S}_\phi \triangleq \{(\eta, \xi) \in \mathbb{R}^{n-m} \times \mathbb{R}^m : \xi = \phi(\eta)\}$, can be regulated by varying the parameters c and θ in (17). As shown in the proof of Theorem 3.1, the requirement that $\theta < 1$ is a direct consequence of Theorem 2.1. Moreover, assuming that $\theta > \frac{1}{2}$ guarantees the existence of the feedback control law (16) for $(\eta, \xi) \in \mathcal{S}_\phi$, that is, $z = 0$. Lastly, one needs to assume that θ is a rational number and b is an odd number, since fractional powers of real numbers have real roots if the exponent is rational and the denominator of the exponent is odd.

If the matched uncertainty $\delta(\cdot, \cdot, \cdot)$ in (6) does *not* explicitly depend on time, then (18) is a time-invariant dynamical system. In this case, by proceeding as in the proof of Theorem 3.1, one can prove that the feedback control law (16) guarantees closed-loop asymptotic stability of (6).

4 A SLIDING MODE OBSERVER

In this section, we apply the results developed in Section 3 to design a robust observer for the uncertain nonlinear dynamical system

$$\dot{x}(t) = Ax(t) + B[u(t) + \hat{\delta}_1(t, x(t), u(t))], \quad x(t_0) = x_0, \quad t \geq t_0, \quad (24)$$

$$y(t) = Cx(t), \quad (25)$$

where $x(t) \in \mathbb{R}^n$, $t \geq 0$, $u(t) \in \mathbb{R}^m$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $\hat{\delta}_1 : [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ is unknown and jointly continuous in t , x , and u , and $C \in \mathbb{R}^{l \times n}$ has full row rank; the rank condition on the matrix C implies that there is no redundancy amongst the measurements captured by the *output* $y(\cdot)$.

The Walcott and Žak observer [42] and its several variations [43–46] address the state observation problem for nonlinear dynamical systems in the same form as (24) and (25). In practice, we consider nonlinear dynamical systems, whose linearized dynamics is known. This is not a considerably limiting assumption, since in most problems of practical interest, several system identification techniques can be applied to compute linear models that provide satisfactory first-order approximations of nonlinear dynamical systems [47, 48]. Moreover, it is common practice to consider some linear function of the measured state vector as the system output; this design choice greatly simplifies the output calibration.

Consider the *nonlinear observer*

$$\dot{\hat{x}}(t) = A\hat{x}(t) + B[u(t) - v(t)] + K[y(t) - C\hat{x}(t)],$$

$$\hat{x}(t_0) = \hat{x}_0, \quad t \geq t_0, \quad (26)$$

where $K \in \mathbb{R}^{n \times l}$ is the the *observer gain* and $y(t)$, $t \geq t_0$, satisfies (24) and (25). In the following, we apply Theorem 3.1 and provide an expression for $v(\cdot)$ in (26), so that the *estimated state* $\hat{x}(\cdot)$ converges to the plant state $x(\cdot)$ exponentially in time.

Let $e(t) \triangleq x(t) - \hat{x}(t)$, $t \geq t_0$, denote the *estimation error*, and note that (24)–(26) imply that

$$\begin{aligned} \dot{e}(t) &= A_e e(t) + B[v(t) + \delta_1(t, e(t))], \\ e(t_0) &= x_0 - \hat{x}_0, \quad t \geq t_0, \end{aligned} \quad (27)$$

where $A_e \triangleq A - KC$ and $\delta_1(t, e) \triangleq \hat{\delta}_1(t, x, u(t))$. The estimation error dynamics (27) is in the same form as (6) with $f(e) = A_e e$, $e \in \mathbb{R}^n$, $B(e) = B$, $G(e) = E(e) = I_m$, and $\delta_2(e) = 0$, and if there exists an invertible matrix $T \in \mathbb{R}^{n \times n}$ such that

$$TB = \begin{bmatrix} 0_{(n-m) \times m} \\ I_m \end{bmatrix}, \quad (28)$$

then (27) is equivalent to

$$\begin{aligned} \begin{bmatrix} \dot{\eta}(t) \\ \dot{\xi}(t) \end{bmatrix} &= \begin{bmatrix} A_{1e} & A_{12e} \\ A_{21e} & A_{2e} \end{bmatrix} \begin{bmatrix} \eta(t) \\ \xi(t) \end{bmatrix} \\ &\quad + \begin{bmatrix} 0_{(n-m) \times m} \\ I_m \end{bmatrix} [v(t) + \delta_1(t, \eta(t), \xi(t))], \\ \begin{bmatrix} \eta(t_0) \\ \xi(t_0) \end{bmatrix} &= T(x_0 - \hat{x}_0), \quad t \geq t_0, \end{aligned} \quad (29)$$

where $[\eta^T, \xi^T]^T = Te$, $e \in \mathbb{R}^n$, and $TA_e T^{-1} = \begin{bmatrix} A_{1e} & A_{12e} \\ A_{21e} & A_{2e} \end{bmatrix}$. In this case, Assumption 3.1 is satisfied by the diffeomorphism $\mathcal{T}(e) = Te$, $e \in \mathbb{R}^n$. In order to verify Assumption 3.2, we present the following result.

Proposition 4.1. Consider the nonlinear dynamical system (27), which captures the estimation error dynamics and suppose there exists an invertible matrix $T \in \mathbb{R}^{n \times n}$ such that (28) is satisfied. If there exists $\Phi \in \mathbb{R}^{m \times (n-m)}$ such that the equilibrium point $\eta(t) \equiv 0$, $t \geq t_0$, of the closed-loop system

$$\begin{aligned} \dot{\eta}(t) &= (A_{1e} + A_{12e}\Phi)\eta(t), \\ \eta(t_0) &= [I_{n-m} \ 0_{(n-m) \times m}] T(x_0 - \hat{x}_0), \quad t \geq t_0, \end{aligned} \quad (30)$$

is globally asymptotically stable, then Assumption 3.2 is satisfied with $\phi(\eta) = \Phi\eta$.

Proof: The proof of this result is immediate and omitted for brevity. ■

If the linear dynamical system (30) is globally asymptotically stable, then (30) is globally exponentially stable [49]. Hence Proposition 4.1 provides sufficient conditions for $\eta(t) \rightarrow 0$ as $t \rightarrow \infty$ exponentially in time, that is, $\|\eta(t)\| < e^{\lambda_{\min}(A_{1e} + A_{12e}\Phi)t} \|\eta(t_0)\|$, $t \geq t_0$, where $\lambda_{\min}(\cdot)$ denotes the minimum eigenvalue of its argument. The next theorem is the main result of this section and provides an explicit expression for $v(\cdot)$ such that (26) is a robust observer for the uncertain nonlinear dynamical system (24) and (25). For the statement of this result, let $z = \xi - \Phi\eta$, $(\eta, \xi) \in \mathbb{R}^{n-m} \times \mathbb{R}^m$, and let x_i denote the i th component of $x \in \mathbb{R}^n$.

Theorem 4.1. Consider the uncertain nonlinear dynamical system (24) and (25) and the nonlinear observer (26), suppose there exists an invertible matrix $T \in \mathbb{R}^{n \times n}$ such that (28) is verified, and assume that the conditions of Proposition 4.1 are satisfied. If there exist a continuous function $\rho : \mathbb{R}^{n-m} \times \mathbb{R}^m \rightarrow \mathbb{R}_+$ such that

$$\|\delta_1(t, \eta, \xi)\|_\infty \leq \rho(\eta, \xi) \|z(\eta, \xi)\|_\infty, \\ (t, \eta, \xi) \in [t_0, \infty) \times \mathbb{R}^{n-m} \times \mathbb{R}^m, \quad (31)$$

where $\delta_1(t, \eta, \xi) = \hat{\delta}_1(t, x, u(t))$, then the equilibrium point $[\eta^T(t), \xi^T(t)]^T \equiv 0$, $t \geq t_0$, of the nonlinear dynamical system (29), which captures the estimation error dynamics, is globally uniformly exponentially stable with feedback control law

$$v = (\Phi A_{1e} - A_{21e})\eta + (\Phi A_{12e} - A_{2e})\xi + \gamma(\eta, \xi), \\ (\eta, \xi) \in \mathbb{R}^{n-m} \times \mathbb{R}^m, \quad (32)$$

where

$$\gamma_i(\eta, \xi) = -[\rho(\eta, \xi)z_i(\eta, \xi) + cz_i^{2\theta-1}(\eta, \xi)], \quad i = 1, \dots, m, \quad (33)$$

$z_i(\eta, \xi)$ denotes the i th component of $z(\eta, \xi) = \xi - \Phi\eta$, $c > 0$, $\theta = \theta_1/\theta_2$, $\theta_1 \in \mathbb{N}$, $\theta_2 \in \mathbb{N}$ is an odd number, and $\theta \in (\frac{1}{2}, 1)$.

Proof: It follows from (24)–(26) that the estimation error dynamics is given by (27), and since there exists an invertible matrix T such that (28) is satisfied, (27) is equivalent to (29) and Assumption 3.1 is verified by the diffeomorphism $\mathcal{T}(e) = Te$, $e \in \mathbb{R}^n$. Moreover, it follows from Proposition 4.1 that Assumption 3.2 is satisfied. The result now follow as in the proof of Theorem 3.1 with $f_\eta(\eta, \xi) = A_{1e}\eta + A_{12e}\xi$, $(\eta, \xi) \in \mathbb{R}^{n-m} \times \mathbb{R}^m$, $f_\xi(\eta, \xi) = A_{21e}\eta + A_{2e}\xi$, $u = v$, $\delta_\eta(\eta, \xi) = 0$, $\delta_\xi(t, \eta, \xi, w) = \delta_1(t, \eta, \xi)$, $\phi(\eta) = \Phi\eta$, $E(\eta, \xi) = G(\eta, \xi) = \hat{G}(\eta, \xi) = I_m$, $k_i = 0$, $i = 1, \dots, m$, $\rho_i(\eta, \xi) = \rho(\eta, \xi)$, and $z = \xi - \Phi\eta$. ■

Theorem 4.1 provides an explicit expression for $v(\cdot)$ so that (26) is a robust observer for the uncertain dynamical system (24)

and (25). Similarly to the Walcott and Źak observer, the observer (26) with feedback control law (32) guarantees exponential convergence of the estimation error $e(\cdot)$ to zero. However, the Walcott and Źak observer requires to choose the observer gain K in (26) such that $A_e = A - KC$ is asymptotically stable. Although it is desirable that the uncontrolled and undisturbed estimation error dynamics (27) is asymptotically stable, this assumption is not needed for the observer proposed in this paper. Lastly, the Walcott and Źak observer requires to satisfy some matrix equality and inequality constraints that usually restrict the applicability of this estimator to lower-order systems [50, 51], whereas these constraints do not need to be verified to apply our robust observer.

5 ILLUSTRATIVE NUMERICAL EXAMPLE

In order to illustrate the applicability of the sliding mode control law presented in Section 3, consider the nonlinear dynamical system

$$\dot{x}_1(t) = [x_1^2(t) + 1]x_2(t), \quad x_1(0) = x_{10}, \quad t \geq 0, \quad (34)$$

$$\dot{x}_2(t) = a[x_1(t) + x_2(t)] \sin x_1(t) + b[x_1^2(t) - x_2^2(t)] + u(t), \\ x_2(0) = x_{20}, \quad (35)$$

where $a \in (\underline{a}, \bar{a})$ and $b \in (\underline{b}, \bar{b})$ are positive uncertain parameters. Denoting the estimates of a and b by \hat{a} and \hat{b} , respectively, (34) and (35) is equivalent to

$$\dot{x}_1(t) = [x_1^2(t) + 1]x_2(t), \quad x_1(0) = x_{10}, \quad t \geq 0, \quad (36)$$

$$\dot{x}_2(t) = \hat{a}[x_1(t) + x_2(t)] \sin x_1(t) + \hat{b}[x_1^2(t) - x_2^2(t)] \\ + \delta(x_1(t), x_2(t)) + u(t), \quad x_2(0) = x_{20}, \quad (37)$$

where $\delta(x_1, x_2) = (a - \hat{a})[x_1 + x_2] \sin x_1 + (b - \hat{b})[x_1^2 - x_2^2]$, $(x_1, x_2) \in \mathbb{R} \times \mathbb{R}$, captures the parametric uncertainties. The nonlinear dynamical system (36) and (37) is in the same form as (8) and (9) with $\eta = x_1$, $\xi = x_2$, $f_\eta(\eta, \xi) = (\eta^2 + 1)\xi$, $f_\xi(\eta, \xi) = \hat{a}[\eta + \xi] \sin \eta + \hat{b}[\eta^2 - \xi^2]$, $G(\eta, \xi) = 1$, $E(\eta, \xi) = 1$, $\delta_\eta(\eta, \xi) = 0$, and $\delta_\xi(t, \eta, \xi, u) = \delta(x_1, x_2)$. Our objective is to apply Theorem 3.1 and design a feedback control law such that $\lim_{t \rightarrow \infty} x(t) = 0$, in spite of the uncertainties captured by $\delta(x_1, x_2)$.

Since (36) and (37) is in the same form as (8) and (9), Assumption 3.1 is verified with $\mathcal{T}(x) = x$, $x = [x_1, x_2]^T \in \mathbb{R}^2$. Furthermore, note that if $x_2 = \phi(x_1)$, where $\phi(x_1) = -x_1$, then (34) is globally asymptotically stable. Indeed, let

$$V(x_1) = \frac{1}{2}x_1^2, \quad x_1 \in \mathbb{R}, \quad (38)$$

and since

$$\dot{V}(x_1) = -(x_1^4 + x_1^2) < 0, \quad x_1 \neq 0, \quad (39)$$

along the trajectory of (34) with $x_2 = \phi(x_1)$, it follows from Lyapunov's theorem [12, Th. 3.1] that (34) with $x_2 = \phi(x_1)$ is globally asymptotically stable and Assumption 3.2 is satisfied. Lastly, note that $G(\eta, \xi) = \hat{G}(\eta, \xi) = 1$, $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}$, is known and it follows from (11)–(13) that

$$\begin{aligned} \varphi(\eta, \xi, w) &= -(\eta^2 + 1)\xi - \hat{a}(\eta + \xi)\sin\eta - \hat{b}(\eta^2 - \xi^2), \\ (\eta, \xi) &\in \mathbb{R} \times \mathbb{R}, \end{aligned} \quad (40)$$

$$\Delta(\eta, \xi) = (a - \hat{a})[\eta + \xi]\sin\eta + (b - \hat{b})[\eta^2 - \xi^2], \quad (41)$$

which implies that (15) is satisfied with $m = 1$ and

$$\rho(\eta, \xi) = |\eta + \xi|[(\bar{a} - \hat{a}) + (\bar{b} - \hat{b})|\eta - \xi|], \quad (\eta, \xi) \in \mathbb{R} \times \mathbb{R}. \quad (42)$$

In this case, the feedback control law (16) specializes to

$$\begin{aligned} u(t, x_1, x_2) &= -x_2(1 + x_1^2) - \hat{a}(x_1 + x_2)\sin x_1 - \hat{b}(x_1^2 - x_2^2) \\ &\quad - \gamma(t, x_1, x_2), \quad (t, x_1, x_2) \in [0, \infty) \times \mathbb{R} \times \mathbb{R}, \end{aligned} \quad (43)$$

where

$$\begin{aligned} \gamma(t, x_1, x_2) &= (x_1 + x_2)[(\bar{a} - \hat{a}) + (\bar{b} - \hat{b})|x_1 - x_2|] \\ &\quad + c(x_1 + x_2)^{2\theta-1}, \end{aligned} \quad (44)$$

$c > 0$, and $\theta \in (\frac{1}{2}, 1)$. Since all the assumptions of Theorem 3.1 are satisfied, (34) and (35) with feedback control law (43) and (44) is globally uniformly asymptotically stable.

Let $a = 2, b = 3, \bar{a} = 6, \bar{b} = 8, \hat{a} = 4, \hat{b} = 5, c = 5$, and $\theta = \frac{8}{9}$. Figure 1 shows the trajectory of the closed-loop system (34) and (35) with feedback control law (43) and (44) computed using the fourth-order Runge-Kutta integration method and an absolute tolerance of 10^{-4} ; note that $x(t) \rightarrow 0$ as $t \rightarrow \infty$. Figure 2 shows the feedback control law (43) and (44), the classical sliding mode control law [10, Ch. 14] given by (43) and

$$\begin{aligned} \gamma(t, x_1, x_2) &= [1 + |x_1 + x_2|(\bar{a} - \hat{a}) \\ &\quad + (\bar{b} - \hat{b})|x_1 + x_2||x_1 - x_2|] \operatorname{sign}(x_1 + x_2), \\ (t, x_1, x_2) &\in [0, \infty) \times \mathbb{R} \times \mathbb{R}, \end{aligned} \quad (45)$$

the second-order sliding mode control [20] given by (43) and

$$\gamma(t, x_1, x_2) = -v(t), \quad (t, x_1, x_2) \in [0, \infty) \times \mathbb{R} \times \mathbb{R}, \quad (46)$$

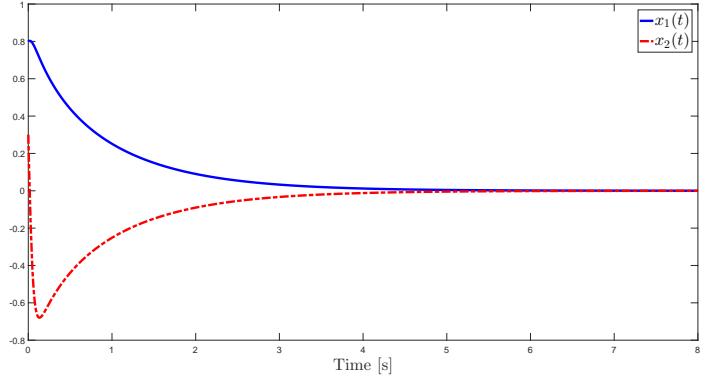


FIGURE 1. Closed-loop system trajectories versus time

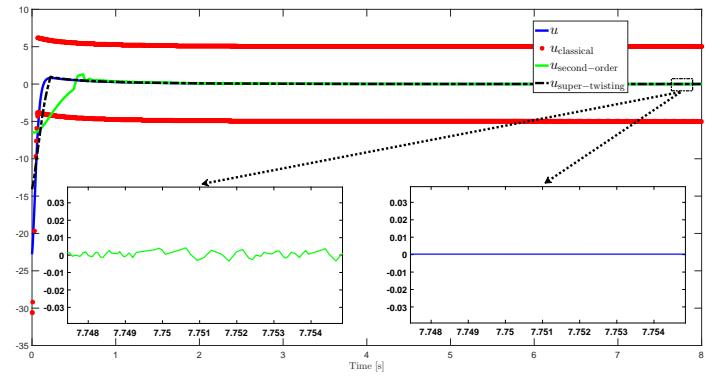


FIGURE 2. Sliding mode control laws. The subfigure on the left shows that the second-order sliding mode control is affected by chattering. The subfigure on the right shows that neither the proposed feedback control nor the super-twisting control suffer from chattering

where $v(\cdot)$ verifies

$$\begin{aligned} \dot{v}(t) &= -k_1(x_1(t) + x_2(t)) - k_2 \frac{d}{dt}(x_1(t) + x_2(t)) \\ &\quad - k_3 \operatorname{sign}(x_1(t) + x_2(t)) - k_4 \operatorname{sign}\left[\frac{d}{dt}(x_1(t) + x_2(t))\right], \\ v(0) &= v_0, \quad t \geq 0, \end{aligned} \quad (47)$$

and $x_1(t)$, $t \geq 0$, and $x_2(t)$ verify (34) and (35), and the super-twisting sliding mode control [52] given by (43) and

$$\begin{aligned} \gamma(t, x_1, x_2) &= k_5(x_1 + x_2)^{\frac{1}{2}} \operatorname{sign}(x_1 + x_2) - v(t) \\ (t, x_1, x_2) &\in [0, \infty) \times \mathbb{R} \times \mathbb{R}, \end{aligned} \quad (48)$$

where $v(\cdot)$ verifies

$$\dot{v}(t) = -k_6 \operatorname{sign}(x_1(t) + x_2(t)), \quad v(0) = v_0, \quad t \geq 0, \quad (49)$$

and $x_1(t)$, $t \geq 0$, and $x_2(t)$ verify (34) and (35). Conditions on $k_i \in \mathbb{R}$, $i = 1, \dots, 6$, and v_0 in (47)–(49), can be found in [20] and [52]; results shown in Figure 2 have been obtained by setting $k_1 = k_2 = 4$, $k_3 = 20$, $k_4 = 10$, and $k_5 = k_6 = 9$.

It is apparent from Figure 2 that both the classical sliding mode control and the second-order sliding mode control are affected by chattering. However, chattering is considerably less evident in the second-order sliding mode control. The proposed control law given by (43) and (44) and the super-twisting sliding mode control given by (43), (48), and (49), which coincide for $t \geq 0.3$, are not affected by chattering. However, the proposed control law is easier to implement than the super-twisting sliding mode control, since it does not require to integrate a discontinuous nonlinear differential equation. It is also worth to note that in this example the control effort for the proposed control law, super-twisting control, and second-order sliding mode control are comparable; the classical sliding mode control, instead, involves higher control effort.

6 CONCLUSION

In this paper, we presented a form of sliding mode control that guarantees robustness and uniform asymptotic stability of the closed-loop system, ensures both existence and uniqueness of the closed-loop system's trajectory, does not involve discontinuous functions, and does not require differentiators. The proposed approach, which is enabled by recent advances in the stability theory of finite-time stable time-varying dynamical systems, has been applied to design a nonlinear robust observer in the same form as the Walcott and Žak observer. A numerical example illustrates the feasibility of our framework and compares the proposed control law with the classical sliding mode control, the higher-order sliding mode control, and the super-twisting sliding mode control.

The author in [53] provides some counter-examples that question the higher-order sliding mode control and the super-twisting control in feedback control laws to mitigate chattering. These problems will be further investigated in future.

ACKNOWLEDGMENT

This work was supported in part by the NOAA/Office of Oceanic and Atmospheric Research under NOAA-University of Oklahoma Cooperative Agreement #NA16OAR4320115, the U.S. Department of Commerce.

REFERENCES

- [1] Doyle, J. C., Glover, K., Khargonekar, P. P., and Francis, B. A., 1989. “State-space solutions to standard H_2 and H_∞ control problems”. *IEEE Transactions on Automatic Control*, **34**(8), pp. 831–847.
- [2] Bernstein, D., and Haddad, W., 1989. “LQG control with an \mathcal{H}_∞ performance bound: a Riccati equation approach”. *IEEE Transactions on Automatic Control*, **34**(3), pp. 293–305.
- [3] Francis, B. A., 1987. *A course in \mathcal{H}_∞ control theory*. Springer, Berlin, Germany.
- [4] Green, M., and Limebeer, D., 1995. *Linear robust control*. Prentice Hall, Mineola, NY.
- [5] Hovakimyan, N., and Cao, C., 2010. *\mathcal{L}_1 Adaptive Control Theory: Guaranteed Robustness with Fast Adaptation*. Society for Industrial and Applied Mathematics, Boston, MA.
- [6] Ioannou, P., and Fidan, B., 2006. *Adaptive Control Tutorial*. Society for Industrial and Applied Mathematics, Boston, MA.
- [7] Itkis, U., 1976. *Control systems of variable structure*. Wiley, New York, NY.
- [8] Utkin, V., 1977. “Variable structure systems with sliding modes”. *IEEE Transactions on Automatic Control*, **22**(2), pp. 212–222.
- [9] Slotine, J.-J., and Li, W., 1991. *Applied Nonlinear Control*. Prentice Hall, Princeton, NJ.
- [10] Khalil, H. K., 2002. *Nonlinear Systems*. Pearson Education. Prentice Hall, Princeton, NJ.
- [11] Perruquetti, W., and Barbot, J. P., 2002. *Sliding Mode Control In Engineering*. CRC Press, New York, NY.
- [12] Haddad, W. M., and Chellaboina, V., 2008. *Nonlinear Dynamical Systems and Control: A Lyapunov-Based Approach*. Princeton Univ. Press, Princeton, NJ.
- [13] Filippov, A., 1988. *Differential Equations with Discontinuous Right-Hand Sides*. Mathematics and its Applications (Soviet Series). Kluwer Academic Publishers, Dordrecht, The Netherlands.
- [14] Levant, A., 2010. “Chattering analysis”. *IEEE Transactions on Automatic Control*, **55**(6), pp. 1380–1389.
- [15] Shtessel, Y., Buffington, J., and Banda, S., 2002. “Tailless aircraft flight control using multiple time scale reconfigurable sliding modes”. *IEEE Transactions on Control Systems Technology*, **10**(2), pp. 288–296.
- [16] de Andrade, G. A., Pagano, D. J., Guzman, J. L., Berenguel, M., Fernndez, I., and Acin, F. G., 2016. “Distributed sliding mode control of pH in tubular photobioreactors”. *IEEE Transactions on Control Systems Technology*, **24**(4), pp. 1160–1173.
- [17] Betin, F., Pinchon, D., and Capolino, G. A., 2002. “A time-varying sliding surface for robust position control of a DC motor drive”. *IEEE Transactions on Industrial Electronics*, **49**(2), pp. 462–473.
- [18] Soylu, S., Buckham, B. J., and Podhorodeski, R. P., 2008. “A chattering-free sliding-mode controller for underwater vehicles with fault-tolerant infinity-norm thrust allocation”. *Ocean Engineering*, **35**(16), pp. 1647 – 1659.
- [19] Jin, Y., Chang, P. H., Jin, M., and Gweon, D. G., 2013. “Stability guaranteed time-delay control of manipulators using nonlinear damping and terminal sliding mode”. *IEEE Transactions on Industrial Electronics*, **60**(8), pp. 3304–3317.
- [20] Levant, A., 2002. “Introduction to high-order sliding

- modes". In *Sliding Mode Control in Engineering*, W. Perruquetti and J. P. Barbot, eds. Marcel Dekker, Inc., Basel, Switzerland, ch. 1.
- [21] Levant, A., 2003. "Higher-order sliding modes, differentiation and output-feedback control". *International Journal of Control*, **76**(9-10), pp. 924–941.
- [22] Pisano, A., and Usai, E., 2011. "Sliding mode control: A survey with applications in math". *Mathematics and Computers in Simulation*, **81**(5), pp. 954–979.
- [23] Hung, J. Y., Gao, W., and Hung, J. C., 1993. "Variable structure control: A survey". *IEEE Transactions on Industrial Electronics*, **40**(1), pp. 2–22.
- [24] Slotine, J.-J., and Sastry, S. S., 1983. "Tracking control of non-linear systems using sliding surfaces, with application to robot manipulators". *International Journal of Control*, **38**(2), pp. 465–492.
- [25] Burton, J. A., and Zinober, A. S. I., 1986. "Continuous approximation of variable structure control". *International Journal of Systems Science*, **17**(6), pp. 875–885.
- [26] Friedman, L., and Levant, A., 2002. "Higher order sliding modes". In *Sliding Mode Control in Engineering*, W. Perruquetti and J. P. Barbot, eds. Marcel Dekker, Inc., Basel, Switzerland, ch. 3.
- [27] Levant, A., 1993. "Sliding order and sliding accuracy in sliding mode control". *International Journal of Control*, **58**(6), pp. 1247–1263.
- [28] Levant, A., 1993. "Sliding order and sliding accuracy in sliding mode control". *International Journal of Control*, **58**(6), pp. 1247–1263.
- [29] Nemati, H., Bando, M., and Hokamoto, S., 2017. "Chattering attenuation sliding mode approach for nonlinear systems". *Asian Journal of Control*, pp. 1–13.
- [30] Haddad, W. M., and L'Afflitto, A., 2015. "Finite-time partial stability and stabilization, and optimal feedback control". *Journal of the Franklin Institute*, **352**(6), pp. 2329 – 2357.
- [31] Levant, A., and Livne, M., 2012. "Exact differentiation of signals with unbounded higher derivatives". *IEEE Transactions on Automatic Control*, **57**(4), pp. 1076–1080.
- [32] Venkataraman, S. T., and Gulati, S., 1993. "Control of nonlinear systems using terminal sliding modes". *Journal of Dynamic Systems, Measurement, and Control*, **115**(3), pp. 554–560.
- [33] Man, Z., Paplinski, A. P., and Wu, H. R., 1994. "A robust MIMO terminal sliding mode control scheme for rigid robotic manipulators". *IEEE Transactions on Automatic Control*, **39**(12), pp. 2464–2469.
- [34] Man, Z., and Yu, X. H., 1997. "Terminal sliding mode control of MIMO linear systems". *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, **44**(11), pp. 1065–1070.
- [35] Wu, Y., Yu, X., and Man, Z., 1998. "Terminal sliding mode control design for uncertain dynamic systems". *System & Control Letters*, **34**(5), pp. 281–287.
- [36] Agarwal, R., and Lakshminikantham, V., 1993. *Uniqueness and Nonuniqueness Criteria for Ordinary Differential Equations*. World Scientific, Singapore.
- [37] Kawsaki, M., 1989. "Stabilization of nonlinear systems in the plane". *Systems and Control Letters*, **12**, pp. 169–175.
- [38] Yoshizawa, T., 1966. *Stability Theory by Liapunov's Second Methods*. Publications of the Mathematical society of Japan. Mathematical society of Japan, Tokyo, Japan.
- [39] Bhat, S. P., and Bernstein, D. S., 2000. "Finite-time stability of continuous autonomous systems". *SIAM Journal on Control and Optimization*, **38**(3), pp. 751–766.
- [40] Haddad, W. M., Nersesov, S. G., and Du, L., 2009. "Finite-time stability for time-varying nonlinear dynamical systems". In *Advances in Nonlinear Analysis: Theory Methods and Application*, S. Sivasundaram, J. V. Devi, Z. Drici, and F. Mcrae, eds. Cambridge Scientific Publishers, Cambridge, UK, pp. 139–150.
- [41] Lang, S., 1983. *Real analysis*. Addison-Wesley, Reading, MA.
- [42] Walcott, B., and Źak, S. H., 1987. "State observation of nonlinear uncertain dynamical systems". *IEEE Transactions on Automatic Control*, **32**(2), pp. 166–170.
- [43] Źak, S. H., and Hui, S., 1993. "Output feedback variable structure controllers and state estimators for uncertain/nonlinear dynamic systems". *IEE Proceedings D - Control Theory and Applications*, **140**(1), pp. 41–50.
- [44] Edwards, C., Spurgeon, S. K., Tan, C. P., and Patel, N., 2007. *Sliding-Mode Observers*. Springer, London, UK, pp. 221–242.
- [45] Hui, S., and Źak, S. H., 2005. "Observer design for systems with unknown inputs". *International Journal of Applied Mathematics and Computer Science*, **15**(4), pp. 431–446.
- [46] Spurgeon, S. K., 2008. "Sliding mode observers: A survey". *International Journal of Systems Science*, **39**(8), pp. 751–764.
- [47] Keesman, K., 2011. *System Identification: An Introduction*. Advanced Textbooks in Control and Signal Processing. Springer London.
- [48] Pintelon, R., and Schoukens, J., 2012. *System Identification: A Frequency Domain Approach*. Wiley, New York, NY.
- [49] Kalman, R. E., 1960. "Contributions to the theory of optimal control". *Bol. Soc. Mat. Mexicana*, **5**, pp. 102–119.
- [50] Edwards, C., Spurgeon, S. K., and Patton, R. J., 2000. "Sliding mode observers for fault detection and isolation". *Automatica*, **36**(4), pp. 541–553.
- [51] Steinberg, A., and Corless, M., 1985. "Output feedback stabilization of uncertain dynamical systems". *IEEE Transactions on Automatic Control*, **30**(10), pp. 1025–1027.
- [52] Khan, M. K., Goh, K. B., and Spurgeon, S. K., 2003. "Second order sliding mode control of a diesel engine". *Asian Journal of Control*, **5**(4), pp. 614–619.
- [53] Utkin, V., 2016. "Discussion aspects of high-order sliding mode control". *IEEE Transactions on Automatic Control*, **61**(3), pp. 829–833.